PART A: TOPOLOGICAL VECTOR SPACES

1. (Folland's Exercise 5.47) Suppose that X and Y are Banach spaces.

   (a) If $\{T_n\}_{n=1}^{\infty} \subset L(X,Y)$ and $T_n \to T$ weakly (or strongly), then $\sup_n \|T_n\| < \infty$.
   Proof. Since strong convergence of operators implies weak convergence, it is enough to consider the case that $T_n \to T$ weakly; *i.e.* that $f(T_n x) \to f(Tx)$ for every $x \in X$ and every $f \in Y^*$. Recalling the isometric embedding $Y \to Y^{**} \ : \ y \mapsto \hat{y}$ given by $\hat{y}(f) = f(y)$, we see that $\widehat{T_n x}(f) \to \widehat{Tx}(f)$ for every $f \in Y^*$ and so, in particular, $\{\widehat{T_n x}(f)\}_{n=1}^{\infty}$ is bounded. Thus, $\{\widehat{T_n x}\}_{n=1}^{\infty} \subset Y^{**}$ is pointwise bounded and so, by the uniform boundedness principle, is uniformly bounded: $\sup_n \|\widehat{T_n x}\| < \infty$. Since $\|\widehat{T_n x}\| = \|T_n x\|$, we conclude that $\sup_n \|T_n x\| < \infty$ (for each $x \in X$); *i.e.* that $\{T_n\}_{n=1}^{\infty}$ is pointwise bounded. Another application of the uniform boundedness principle shows that $\{T_n\}_{n=1}^{\infty}$ is uniformly bounded, as required. ∎

   (b) Every weakly convergent sequence in X, and every weak*-convergent sequence in X^*, is bounded (with respect to the norm).
   Proof. Let $\{x_n\}_{n=1}^{\infty} \subset X$ converge weakly to $x \in X$; *i.e.* $f(x_n) \to f(x)$ for every $f \in X^*$. Then, if we define as before, $\hat{x}(f) = f(x)$, we have that $\widehat{x_n}(f) \to \hat{x}(f)$, so that $\{\widehat{x_n}\}_{n=1}^{\infty}$ is pointwise bounded. By the uniform boundedness principle, $\{\widehat{x_n}\}$, and so also $\{x_n\}$, is uniformly bounded.
   For the second part, if $\{f_n\}_{n=1}^{\infty} \subset X^*$ is weak*-convergent to $f \in X^*$, then $f_n(x) \to f(x)$ for every $x \in X$, and so $\{f_n\}_{n=1}^{\infty}$ is pointwise bounded. By the uniform bounedness principle, it is also uniformly bounded. ∎

2. (Folland's Exercise 5.48) Suppose that X is a Banach space.

   (a) The norm-closed unit ball $B = \{x \in X \ : \ \|x\| \le 1\}$ is also weakly closed.
   Proof. Let $x \in X$ be an element of the weak closure of B. Then, by Folland's Proposition 4.18, there is a net $\{x_\alpha\}_{\alpha \in A}$ in B which converges weakly to x; *i.e.* $f(x_\alpha) \to f(x)$ for every $f \in X^*$. Applying the usual embedding of X into X^{**}, and using $\|\widehat{x_\alpha}\| = \|x_\alpha\| \le 1$, we see that

   $$|\hat{x}(f)| = \lim_{\alpha} |\widehat{x_\alpha}(f)| \le \sup_{\alpha} \|\widehat{x_\alpha}\| \, \|f\| \le \|f\|. \tag{1}$$

   By the definition of the operator norm, this implies that $\|x\| = \|\hat{x}\| \le 1$, so that $x \in B$. Thus, B is weakly closed, as required. ∎

   (b) If $E \subset X$ is bounded (with respect to the norm), so is its weak closure.
   Proof. We begin by observing that scalar multiplication is continuous in any topological vector space and so, in particular, in the weak and weak* topologies. (This fact does require proof, of course, but we will not use it in any really essential way.) Since, for $r \neq 0$, multiplication by r^{-1} is inverse to multiplication by r, multiplication by $r \neq 0$ is actually a homeomorphism. The point of this remark is that we may now assume any ball centred on the origin is weakly (or weak*) closed r. (Alternatively,

you can prove that fact directly, by repeating the proof for the unit ball in part (a) of this exercise.)

Now, if $E \subset X$ is bounded, there is some ball $B_r = \{x \in X \ : \ \|x\| \leq r\}$ such that $E \subset B$. Since B is weakly closed, the weak closure of E is contained in B, and is therefore bounded. ■

(c) If $F \subset X^*$ is bounded (with respect to the norm), so is its weak* closure.

Proof. The same proof as in (b) works, using the fact that closed balls in X^* are weak*-closed. ■

(d) Every weak*-Cauchy sequence in X^* converges.

Proof. Recall the definition of a Cauchy sequence in a topological vector space: $\{x_n\}_{n=1}^{\infty} \subset X$ is **Cauchy** if, for every neighbourhood U of the origin in X, there is some $N \in \mathbb{N}$ such that $x_n - x_m \in U$ whenever $m, n > N$. If we choose $\epsilon > 0$ and $x \in X$, and let U be the weak* neighbourhood $\{f \in X^* \ : \ |f(x)| < \epsilon\} \subset X^*$, we see that $|f_n(x) - f_m(x)| < \epsilon$ for $m, n > N$; *i.e.* that $\{f_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{K}$, and so converges, to $f(x)$, say.

It is straightforward to verify that $f : X \to \mathbb{K}$ is linear, and since the sequence $\{f_n\}_{n=1}^{\infty} \subset X^*$ is pointwise bounded, the uniform boundedness principle implies that $\sup_n \|f_n\| < \infty$. Taking pointwise limits, we see that $\|f\| < \infty$, so that $f \in X^*$; moreover, $f_n \to f$ pointwise, and so in the weak* topology, as required. ■

3. (Folland's Exercise 5.49) Suppose that X is an infinite-dimensional Banach space.

(a) Every nonempty weakly open set in X, and every nonempty weak*-open set in X^*, is unbounded (with respect to the norm).

Proof. Let $U \subset X$ be a nonempty weakly open set. Since translation is a homeomorphism in a topological vector space, we may assume that $0 \in U$. By the definition of the weak topology, we may also assume that U is of the form

$$U = \bigcap_{n=1}^{N} \{x \in X \ : \ |f_n(x)| < \epsilon\}, \tag{2}$$

for some $\epsilon > 0$ and $f_1, \ldots, f_N \in X^*$. We claim there is some $x \neq 0$ in X such that $f_n(x) = 0$ for all $1 \leq n \leq N$. Indeed, if there were no such x, the map $x \mapsto (f_1(x), \ldots, f_N(x)) \in \mathbb{R}^N$ would be injective, so that X would be finite-0dimensional. Therefore, the 1-dimensional subspace $\langle x \rangle = \{\lambda x \ : \ \lambda \in \mathbb{K}\} \subset U$. Since $\langle x \rangle$ is unbounded, so is U.

For the second part, it is sufficient to observe that any weak* open set in X^* is weakly open, and therefore unbounded by the first part. ■

(b) Every bounded subset of X is nowhere dense in the weak topology, and every bounded subset of X^* is nowhere dense in the weak* topology.

Proof. If $E \subset X$ is bounded, then so is its weak closure, by Exercise 2(b). The interior of that weak closure, if nonempty, would be unbounded by part (a) of the present exercise; hence the interior of the weak closure of E must be empty, and E is nowhere dense in the weak topology.

The proof of the second part is identical, starting with Exercise 2(c). ■

(c) X is meager in itself with respect to the weak topology, and X^* is meager in itself with respect to the weak* topology.

Proof. Let $B_r = \{x \in X \ : \ \|x\| \leq r\} \subset X$, a bounded set in X. Since $X = \cup_{n=1}^{\infty} B_n$, X is a countable union of nowhere-dense sets, and so is meager in itself.

The proof of the second part is the same, starting with the balls $\{f \in X^* \ : \ \|f\| \leq r\} \subset X^*$. ■

(d) The weak* topology on X^* is not defined by any translation-invariant metric.

Proof. It is easy to see, using the definition of Cauchy sequence in a topological vector space given in the proof of Exercise 2(d), that if the topology in a topological vector space is given by a translation-invariant metric, a sequence is Cauchy in the topological sense if and only if it is Cauchy in the metric. (This is not necessarily true if the metric is not translation-invariant: consider the bounded metric $d_1(x,y) = |x-y|/(1+|x-y|)$ on $\mathbb{R}$.) Thus, if the weak* topology were defined by such a metric, any Cauchy sequence in the metric would be Cauchy in the weak$*$ topology, and so would converge (in the weak$*$ topology, and so also in the metric), by Exercise 2(d). Thus, X^* would be a complete metric space, and so non-meager in itself by the Baire Category Theorem. But this contradicts the previous part of this exercise. ■

Note. In fact, a stronger result is true: if X is an infinite-dimensional Banach space, neither the weak topology on X nor the weak$*$ topology on X^* are given by any metric—translation-invariant or not.

PART B: WEAK DERIVATIVES

(This part of the exam is taken from Chapter 7 of *Elliptic Partial Differential Equations of Second Order* by D. Gilbarg and N. S. Trudinger.)

Throughout this part, let Ω be a connected, open set in $\mathbb{R}^n$. Since we will be talking about partial derivatives, we will use the multi-index notation (see, for example, page 236 in Folland).

A **mollifier** is a non-negative function ρ in $C_c^\infty(\mathbb{R}^n)$ with support contained in the (open) unit ball, and satisfying

$$\int \rho(x)\,dx = 1. \tag{3}$$

For $u \in L^1_{loc}(\Omega)$ and $h > 0$, the **regularization** of u is defined by

$$u_h(x) = h^{-n} \int_\Omega \rho\left(\frac{x-y}{h}\right) u(y)\,dy, \tag{4}$$

provided $h < \operatorname{dist}(x, \partial\Omega)$. (In some sense, the function $h^{-n}\rho((x-\cdot)/h)$ converges to the Dirac measure δ_x as $h \to 0$; as Folland says on page 242, "Draw a picture if this isn't clear." In fact, although you do NOT need to prove this, $h^{-n}\rho((x-\cdot)/h)$ converges to δ_x in the vague (weak-$*$) topology on $M(\mathbb{R}^n)$.) You may assume the following facts, which we will discuss shortly in class (they are related to Folland's Theorem 8.14):

Fact 1 Let $u \in C(\Omega)$. Then u_h converges to u locally uniformly in Ω.

Fact 2 Let $u \in C_c(\Omega)$. Then u_h converges to u uniformly in Ω.

Fact 3 Let $u \in L^p(\Omega)$ (resp. $L^p_{loc}(\Omega)$). Then u_h converges to u in $L^p(\Omega)$ (resp. $L^p_{loc}(\Omega)$).

Let u be locally integrable in Ω and α any multi-index. A locally-integrable function v is called the α^{th} **weak derivative** of u if it satisfies

$$\int_\Omega v\phi\,dx = (-1)^{|\alpha|}\int_\Omega uD^\alpha\phi\,dx \text{ for all } \phi\in C_0^\infty(\Omega). \tag{5}$$

Note that v is determined uniquely up to sets of measure zero, so it makes sense to write $v = D^\alpha u$.

Now complete the following exercises.

1. Let $u \in L^1_{loc}(\Omega)$, and α a multi-index. Suppose that $D^\alpha u$ exists. Then if $\mathrm{dist}(x, \partial\Omega) > h$, we have

$$D^\alpha u_h(x) = (D^\alpha u)_h(x). \tag{6}$$

(So differentiation commutes with regularization.)

Proof. Differentiation under the integral sign shows that

$$\begin{aligned} D^\alpha u_h(x) &= h^{-n}\int_\Omega D_x^\alpha\rho\left(\frac{x-y}{h}\right)u(y)\,dy && (7)\\ &= (-1)^{|\alpha|}h^{-n}\int_\Omega D_y^\alpha\rho\left(\frac{x-y}{h}\right)u(y)\,dy && (8)\\ &= h^{-n}\int_\Omega \rho\left(\frac{x-y}{h}\right)D^\alpha u(y)\,dy && (9)\\ &= (D^\alpha)_h(x). && (10)\end{aligned}$$

Note that the derivatives in the first two lines may be considered as classical derivatives. On the other hand, the derivative on u in the third line must be considered as a weak derivative, and the "integration by parts" is only justified if $\rho((x-\cdot)/h) \in C_c^\infty(\Omega)$, which will be true if $h < \mathrm{dist}(x, \partial\Omega)$. (Recall that the support of ρ is contained in the unit ball in $\mathbb{R}^n$.) ■

2. Let $u, v \in L^1_{loc}(\Omega)$. Then $v = D^\alpha u$ if and only if there exists a sequence of $C^\infty(\Omega)$ functions $\{u_m\}_{m=1}^\infty$ converging to u in $L^1_{loc}(\Omega)$, whose derivatives $D^\alpha u_m$ converge to v in $L^1_{loc}(\Omega)$.

Proof. Let $v = D^\alpha u$. Then, by Exercise 1 and Fact 3, the regularizations $\{u_{1/m}\}_{m=1}^\infty$ converge to u in $L^1_{loc}(\Omega)$, and their derivatives $D^\alpha u_{1/m}$ converge to v in $L^1_{loc}(\Omega)$. (This uses the fact that for any compact set $K \subset \Omega$, $\mathrm{dist}(K, \partial\Omega) > 0$, so that $Du_h = (Du)_h$ on K whenever $h < \mathrm{dist}(K, \partial\Omega)$.)

Conversely, suppose that $\{u_m\}_{m=1}^\infty$ is a sequence of functions as described in the problem. Then for any $\phi \in C_c^\infty(\Omega)$, classical integration by parts yields

$$\int_\Omega D^\alpha u_m\phi\,dx = (-1)^{|\alpha|}\int_\Omega u_mD^\alpha\phi\,dx. \tag{11}$$

Since $u_m \to u$ and $D^\alpha u_m \to D^\alpha u$ in L^1_{loc}, letting $m\to\infty$ gives

$$\int_\Omega D^\alpha u\phi\,dx = (-1)^{|\alpha|}\int_\Omega uD^\alpha\phi\,dx, \tag{12}$$

so that $v = D^\alpha u$. ■

Note. By Young's inequality (Proposition 8.7 in Folland), the convolutions u_h satisfy $\|u_h\|_{L^p(\Omega)} \le \|u\|_{L^p(\Omega)}$ and, for any compact $K \subset \Omega$, $\|Du_h\|_{L^p(K)} \le \|Du\|_{L^p(K)}$ whenever $h < \mathrm{dist}(K, \partial\Omega)$. We will use this fact below, for the cases $p = 1$ and $p = \infty$.

3. Let $u, v \in L^1_{loc}(\Omega)$ be **weakly differentiable,** meaning that all weak derivatives of first order exist (and are locally integrable). Then, whenever all terms make sense, the **product rule** holds:

$$D(uv) = uDv + vDu, \tag{13}$$

where Du denotes the gradient vector $(\partial_{x_1} u, \ldots, \partial_{x_n} u)$. (You should state precise conditions, making them as weak as you can, under which (13) is true.)

Proof. For the weak gradient Du to be defined, as in (5), u and Du must both be in $L^1_{loc}(\Omega)$. The weakest conditions under which (13) could be true are therefore that u, Du, v, Dv, uv, and $uDv + vDu$ are all in $L^1_{loc}(\Omega)$. We will prove (13) under these conditions, which we will refer to as the minimal assumptions, in several steps.

Step 1. In addition to the minimal assumptions, assume that $u, v \in L^\infty(\Omega)$. Consider the regularizations u_h and v_h, fix $\phi \in C_c^\infty(\Omega)$, and let $K = \operatorname{supp}(\phi)$. Since all functions are infinitely differentiable, we may certainly integrate by parts and obtain

$$-\int_\Omega u_h v_h D\phi \, dx = \int (u_h Dv_h + v_h Du_h)\phi \, dx, \tag{14}$$

and the integrands converge pointwise as $h \to 0$ by (the proof of) Exercise 2. On the left hand side, $|u_h v_h| < \|uv\|_u$, by the note following Exercise 2, and since we are really integrating over the compact set K, the Dominated Convergence Theorem shows that

$$\int_\Omega u_h v_h D\phi \, dx \to \int_\Omega uvD\phi \, dx. \tag{15}$$

We will complete the proof by showing that the right hand side also converges. In fact, we will show that

$$\int (u_h Dv_h)\phi \, dx \to \int (uDv)\phi \, dx; \tag{16}$$

the full result then follows by interchanging u and v. But

$$\int (u_h Dv_h - uDv)\phi \, dx = \int u_h(Dv_h - Dv)\phi \, dx + \int (u_h - u)Dv\phi \, dx. \tag{17}$$

The first integral can be estimated (in absolute value) by $\|u_h\|_\infty \|Dv_h - Dv\|_{L^1(K)}$, which goes to 0 by Exercise 2 and its following remark. The second integrand goes to 0 pointwise, and is bounded by $2\|u\|_\infty \|Du\| \in L^1(K)$, so the Dominated Convergence Theorem shows that the second integral also goes to 0. Putting everything together, we have shown that

$$-\int_\Omega uvD\phi \, dx = \int (uDv + vDu)\phi \, dx \tag{18}$$

which is (13).

Step 2. In addition to the minimal assumptions, assume that $uDv \in L^1_{loc}(\Omega)$ and $vDu \in L^1_{loc}(\Omega)$. Consider the truncated function u_n defined by

$$u_n(x) = \begin{cases} n, & u(x) > n \\ u(x), & |u(x)| \le n \\ -n, & u(x) < -n, \end{cases} \tag{19}$$

and similarly for v_n. (Note that the subscript here doe not indicate regularization.) Since $u_n = \min\{n, \max\{u, -n\}\}$, we can apply Exercise 5 below, and deduce that u_n is weakly differentiable, with

$$Du_n(x) = \begin{cases} 0, & u(x) > n \\ Du(x), & |u(x)| \le n \\ 0, & u(x) < -n, \end{cases} \tag{20}$$

Since u_n and v_n satisfy the conditions of Step 1, we deduce that

$$-\int_\Omega u_n v_n D\phi \, dx = \int (u_n Dv_n + v_n Du_n)\phi \, dx. \tag{21}$$

The Dominated Convergence Theorem now allows us to let $n \to \infty$, so (13) holds in this case.

Step 3. In addition to the minimal assumptions, assume that $u \ge 1$ and $v \ge 1$. (The idea for this part of the proof–to truncate u where the product uv is large—is taken from a Mathematics Stack Exchange post entitled "Product rule of weak derivatives", by Slm2004.) We define $u_n = \min\{u, n/v\}$. Note that this implies that $0 \le u_n v \le n$. Since we are assuming $v \ge 1$, $n/v \in L^1_{loc}(\Omega)$, and so $u_n \in L^1_{loc}(\Omega)$. Moreover, by using the Chain Rule (Exercise 4 below, with $f(t) = 1/t$) and Exercise 5, we see that

$$Du_n = \begin{cases} Du, & uv < n \\ -n\frac{Dv}{v^2}, & uv \ge n. \end{cases} \tag{22}$$

Recalling that $u \ge 1$ and $v \ge 1$, it is easy to show that

$$|vDu_n| \le \begin{cases} n|Du|, & uv < n \\ n|Dv|, & uv \ge n, \end{cases} \tag{23}$$

so that $vDu_n \in L^1_{loc}(\Omega)$. Finally,

$$vDu_n + u_n Dv = \begin{cases} vDu + uDv, & uv < n \\ 0, & uv \ge n, \end{cases} \tag{24}$$

so $Du_n + u_n Dv \in L^1_{loc}(\Omega)$. (It follows that $u_n Dv \in L^1_{loc}(\Omega)$, which may also be seen directly.) The product rule therefore holds for $u_n v$, by Step 2, and Dominated Convergence allows us to take the limit as $n \to \infty$.

Step 4. We now remove the assumption that $u \ge 1$ and $v \ge 1$. First assume that $u \ge 0$ and $v \ge 0$, and apply Step 3 to the functions $u+1$ and $v+1$. The result is

$$-\int (u+1)(v+1)D\phi \, dx = \int ((u+1)Dv + (v+1)Du)\phi \, dx, \tag{25}$$

from which (13) follows.

If u and v are only assumed to be real, we may use the identity

$$uv = (u^+ - u^-)(v^+ - v^-) = u^+v^+ + u^-v^- - u^+v^- - u^-v^+ \tag{26}$$

and apply (13) to each of these four products of positive functions.

Finally, if u and v are complex-valued, we may similarly write them in terms of their real and imaginary parts. The product rule (13) is therefore seen to be true in all cases, under only the minimal assumptions. ∎

4. **(Chain Rule)** Let $f \in C^1(\mathbb{R})$, $f' \in L^\infty(\mathbb{R})$, and let u be weakly differentiable in Ω. Then the composite function $f \circ u$ is weakly differentiable in Ω, and $D(f \circ u) = f'(u)Du$.

   Proof. Choose a sequence $\{u_n\}_{n=1}^\infty \subset C^\infty(\Omega)$ such that $u_n \to u$ and $Du_n \to Du$ in $L^1_{loc}(\Omega)$. By passing to a subsequence if necessary, we may assume $u_n \to u$ almost everywhere in Ω. Fix $\phi \in C_c^\infty(\Omega)$, and let $K = \mathrm{supp}(\phi)$. Then

$$\begin{aligned}
\left| \int (f(u_n) - f(u)) D\phi \, dx \right| &\leq \sup |f'| \int |u_n - u| \, |D\phi| \, dx \to 0 \text{ as } m \to \infty, & (27)\\
\left| \int (f'(u_n)Du_n - f'(u)Du)\phi \, dx \right| &\leq \int |f'(u_n)Du_n - f'(u_n)Du| \, |\phi| \, dx & (28)\\
&\quad + \int |f'(u_n) - f'(u)| \, |Du| \, |\phi| \, dx \\
&\leq \sup |f'| \int |Du_n - Du| \, |\phi| \, dx & (29)\\
&\quad + \int |f'(u_n) - f'(u)| \, |Du| \, |\phi| \, dx.
\end{aligned}$$

   The penultimate integral is bounded by $\sup |f'| \int |Du_n - Du| \, |\phi| \, dx$, and so converges to 0 as $m \to \infty$. The last integrand is bounded by $2 \sup f' |Du||\phi| \in L^1(\Omega)$, and converges pointwise to 0, so the integral converges to 0 by Dominated Convergence. Conclusion: we may pass to the limit as $m \to \infty$ in

$$-\int f(u_n) D\phi \, dx = \int f'(u_n)(Du_n)\phi \, dx \tag{30}$$

   and deduce that

$$-\int f(u) D\phi \, dx = \int f'(u)(Du)\phi \, dx, \tag{31}$$

   which is the required result. ■

5. Let u be weakly differentiable in Ω. Then so are u^+, u^-, and $|u|$, and

$$Du^+ = \begin{cases} Du & \text{if } u > 0 \\ 0 & \text{if } u \leq 0 \end{cases}, \tag{32}$$

   with similar formulas for Du^- and $D|u|$, which you should state explicitly. (Hint. Apply the chain rule, with the function f replaced by f_ϵ, $\epsilon > 0$, where

$$f_\epsilon(t) = \begin{cases} (u^2 + \epsilon^2)^{1/2} - \epsilon & \text{if } u > 0 \\ 0 & \text{if } u \leq 0.) \end{cases} \tag{33}$$

   Proof. Observing that f_ϵ satisfies the conditions of Exercise 4, we have

$$\int f_\epsilon(u) D\phi \, dx = -\int_{[u>0]} \frac{uDu}{(u^2 + \epsilon^2)^{1/2}} \phi \, dx, \tag{34}$$

   and letting $\epsilon \to 0$ shows that

$$\int u^+ D\phi \, dx = -\int_{[u>0]} \phi Du \, dx, \tag{35}$$

which proves the result for Du^+. The results for Du^- and $|u|$ follow from writing $u^- = (-u)^+$ and $|u| = u^+ + u^-$; the appropriate formulas are

$$Du^- = \begin{cases} 0 & \text{if } u \geq 0 \\ -Du & \text{if } u < 0 \end{cases} \tag{36}$$

and

$$D|u| = \begin{cases} Du & \text{if } u > 0 \\ 0 & \text{if } u = 0 \\ -Du & \text{if } u < 0 \ . \end{cases} \tag{37}$$

■

6. Let u be weakly differentiable in Ω. Then $Du = 0$ almost everywhere on any set where u is constant.

   Proof. That $Du = 0$ (almost everywhere) on the set where $u = 0$ follows from the last exercise, since $Du = Du^+ - Du^-$. That $Du = 0$ (a.e.) on the set where $u = c$, for any constant c, now follows from considering the function $u - c$. ■

7. Let $n = 1$, and let $\Omega = (a, b) \subseteq \mathbb{R}$. Suppose that $u \in L^1_{loc}((a, b))$ and $u' \equiv Du = 0$ (a.e.) on (a, b). Show that u is constant on (a, b). (More precisely, that u is equal a.e. to a constant function.) This is Exercise 2e on the "Weak Derivatives in One Dimension" class exercise set from the start of the semester; you might want to refresh your memory of the proofs of the other parts of that exercise and, if necessary, include them in your proof.

   Proof. Let Ω' be any connected, bounded, open set contained in Ω, such that $\text{dist}(\Omega', \partial\Omega) > 0$. Whenever $h < \text{dist}(\Omega', \partial\Omega)$, we have $Du_h = (Du)_h = 0$ on Ω', and so u_h is constant on Ω'. Since $u_h \to u$ in $L^1_{loc}(\Omega)$, it follows that u is constant (almost everywhere) on Ω'. Since Ω is a countable union of sets of the form of Ω', we conclude that u is constant (a.e.) on Ω.
   ■

   Note. The above proof is equally valid in higher dimensions, provided that f' is replaced by Df: all that is required is that Ω is a **domain:** a connected, open set in $\mathbb{R}$.