Macroeconomic Dynamics, 1–35. Printed in the United States of America.
doi:10.1017/S1365100510000003

DECENTRALIZED EXCHANGE AND FACTOR PAYMENTS: A MULTIPLE-MATCHING APPROACH

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The emergence of fiat money is studied in a multiple-matching environment in which exchange is organized around trading posts and prices are determined with a dynamic monopolistically competitive framework. Each household consumes a bundle of commodities and has a preference for consumption variety. We determine the endogenous organization of exchange between firms and shoppers, the means of factor payment (remuneration), and the prices at which these trades occur. We verify that the endogenous linkage of factor payments with the medium of exchange can lead to a monetary equilibrium outcome where only fiat money trades for goods, an ex ante feature of cash-in-advance models. We also examine the long-run effects of money growth on equilibrium exchange patterns. A key finding, consistent with documented hyperinflationary episodes, is that a sufficiently rapid expansion of the money supply leads to the gradual emergence of barter, where sellers accept both goods and cash payments and workers receive part of their remuneration in goods.

Keywords: Product Variety, Factor Payments, Money vs. Barter

1. INTRODUCTION

The impact of inflation on both individual trading patterns and the exchange process is an integral concern in the evaluation of the economic effects of

We are grateful for valuable comments and suggestions from Dean Corbae, Peter Howitt, Nobu Kiyotaki, Yiting Li, Rob Reed, Shouyong Shi, Ross Starr, Neil Wallace, Randy Wright, Nicholas Yannelis, an editor and two anonymous referees, and seminar participants at Penn, Penn State, Purdue, Vanderbilt, the Econometric Society Meetings, the Midwest Economic Theory and International Trade Meetings, and the Midwest Macroeconomic Conference. Needless to say, the usual disclaimer applies. Address correspondence to: Victor Li, Department of Economics, Villanova University, Villanova, PA 19406, USA; e-mail: victor.li@villanova.edu.

monetary policy. Historical evidence of the disruptive nature of moderate to high inflation on normal patterns of exchange between buyers and sellers has been well documented by Lerner (1969), Tallman and Wang (1995), and Ericson and Ickes (2001), among others. For example, in the last, a study of the more recent Russian hyperinflationary experience, payments in kind were common as barter increased from 5% of sales in 1992 to 45% in 1997. Yet relatively little theoretical work has been done to formalize the mechanism behind this phenomenon. The two pioneering contributions by Kiyotaki and Wright (1989, 1993) have inspired a huge literature on reexamining substantive issues in monetary economics in a microfounded search–equilibrium framework.[1] Not only does such a framework offer penetrating insights into the foundations of monetary exchange, but it is also becoming increasingly clear that the underlying themes—informational frictions rooted in spatial separation and heterogeneous preferences—are also central to understanding the nature and organization of exchange itself. For example, Howitt (2005, p. 405) remarks that, in contrast to the random pairwise matching environment frequently used in the literature, "E]xchanges in actual market economies are organized by specialist traders, who mitigate search costs by providing facilities that are easy to locate."

Our objective is to advance this research program a step further along several dimensions by focusing on three related questions. First, how can we elucidate the endogenous transactions role of money in an environment characterized by organized rather than sequential bilateral search? Second, what role does the preference for consumption variety play in determining equilibrium exchange patterns? Finally and most important, what does such a framework imply about the effect of money growth on equilibrium trading patterns, and are these results consistent with historical observations pertaining to hyperinflation?[2]

To address these questions, we consider a monopolistically competitive environment in which production takes place in identifiable firms, using the labor supplied by households—compensated via suitable factor (wage) payments—and in which households purchase an assortment of goods from these firms. We advance a *multiple-matching approach* wherein buyers (*households*) and sellers (*firms*) meet to trade their goods and services at a common trading post. In this richer setting, our framework integrates the roles of fiat money as the principal means whereby households purchase goods *and* whereby firms make factor payments (in particular, the payment of a monetary wage to workers). For example, in modern economies, steelworkers are *typically* paid in cash rather than in steel bars, and they subsequently use their cash earnings to purchase other goods. The emphasis upon *typically* is important, for under certain circumstances workers may well be paid in both cash and kind, and attempt to subsequently barter the goods that their employers pay them.

Hence our market structure resembles the way transactions of goods and labor are organized in modern monetary economies. Our model possesses four distinctive features:

(i) Although individual preferences are specialized (in that not every household desires *every* good), households have a strict preference for consuming a *variety* of goods, which they accomplish by purchasing *baskets* of commodities.
(ii) The multiple-matching market structure is one in which each buyer sequentially meets a large number (i.e., a positive measure) of sellers every period, which overcomes distributional issues that are common in conventional money-search models.
(iii) Each worker's remuneration can be in the form of goods or money—that is, both the means of factor payments and the means of exchange are endogenously determined at equilibrium.
(iv) Sellers set both monetary and relative prices in a monopolistically competitive environment.

The merging of a monopolistically competitive pricing structure [via the Dixit–Stiglitz (1977) approach] and the preference for consumption variety into a matching model of money adds a strategic pricing mechanism that is not captured by the canonical Walrasian framework. We focus on steady-state symmetric equilibria satisfying subgame perfection.

We establish the existence of a *pure barter equilibrium* (PBE) in which money is not valued and workers are paid in kind. The potential absence of a monetary equilibrium is, of course, a desideratum in any model that seeks to provide an *equilibrium* role for money. We then study conditions that lead to the emergence of monetary equilibria. In particular, we show that, for a sufficiently low rate of nominal money growth, there is an equilibrium in which money is valued and used on one side of every transaction [the *pure monetary equilibrium* (PME)].[3] If, however, the rate of monetary expansion is sufficiently high—with a concomitantly rapid rate of inflation—the PME is nonsustainable and barter emerges. This leads to the *mixed-trading equilibrium* (MTE), which is characterized by the coexistence of monetary and barter exchange. Consequently, this endogenous link between the medium of exchange and the means of factor payments implies that an increase in the money growth rate can shift the entire pattern of equilibrium exchange as the PME unravels and the MTE emerges. Furthermore, within the MTE, the rate of inflation and the volume of barter transactions are positively related (indeed, in the limiting case, the MTE converges to the PBE). This finding is consistent with a commonly observed phenomenon during hyperinflationary episodes in which sellers accept both goods and cash and workers often receive part of their remuneration in the form of their employer's output.

We then examine equilibrium welfare levels within the context of pure barter and pure monetary equilibrium outcomes. Critically, in the PBE, the consumption basket that emerges is sparser (as measured by the *variety* of goods that are consumed) than in the PME (or the MTE). With money, households need only locate a good they want, whereas under barter the more stringent double coincidence of wants must be satisfied in order for trade to take place. Thus, our model points to the drawback of barter, relative to monetary exchange, as stemming from *atemporal* trade frictions that stymie consumption variety rather than the *temporal* frictions emphasized by the (random) search literature, in which the absence of a double coincidence of wants reduces the frequency of trade and consumption. Also, within the PME, the endogenously chosen means of factor payments consists only of cash, which gives rise to a cash-in-advance economy as an equilibrium outcome.

2. RELATED LITERATURE

It is useful to highlight those features of our model that are most fundamental to our results, and to compare our contributions to those approaches widely used in the literature.

First, fiat money is valued in our model because it expands trading opportunities by allowing agents to procure a wider *variety* of consumption goods. Hence, the *preference for consumption variety* is essential for the emergence of equilibrium monetary exchange. Although the root cause of the double-coincidence problem arises because of heterogeneous tastes, the preference for variety provides an additional motive for the use of money, not previously examined in the money-search literature.

Second, instead of pursuing a bilateral bargaining approach [cf. Trejos and Wright (1995)] one of the main innovations of this paper is incorporating a Dixit–Stiglitz (1977)-style monopolistically competitive pricing structure into a monetary model. Such a structure not only provides a natural pricing mechanism in the presence of product differentiation, but also has been proven to be an extremely versatile vehicle for macroeconomic analysis [e.g., Blanchard and Kiyotaki (1987)].

Third, by articulating the endogenous link between the means of factor payments and the medium of exchange. we can (i) characterize the cash-in-advance environment as an equilibrium outcome and (ii) link inflation to the equilibrium pattern of exchange.

Fourth, our multiple matching approach naturally leads to a degenerate distribution of money and inventory holdings by ironing out, as it were, the vicissitudes of the random trading environment. This feature offers an extremely tractable means of studying issues in monetary and macroeconomics that incorporate explicit trade frictions.[4] Hence this aspect of our framework is similar in spirit to the contributions of Shi (1997) and Lagos and Wright (2005), who construct environments that are amenable to studying monetary policy in search-theoretic settings.

Fifth, this paper complements Howitt (2005) and by extension to Starr and Stinchcombe (1999).[5] Howitt neatly integrates the informational and spatial frictions, emphasized by search theory, into a market exchange process organized around well-defined shops that trade only a limited set of goods and that are costly to run. Because of the double coincidence of wants problem, barter exchange fails to emerge in equilibrium. In essence, the flow of trade is too small to cover the costs of running the trading facility. Alternatively, these operating costs can be covered under monetary exchange, which requires only a single coincidence—under circumstances where barter would be infeasible. Similarly, we assume that each trading facility can trade only a limited set of commodities. As Howitt (2005, p. 409) has stressed, "Such a limitation is empirically plausible, given the casual observation that no retail outlet (even Walmart) in any economy of record trades more than a small fraction of tradeable objects." Moreover, whereas Howitt's model captures shopping at the canonical shoe store, our model captures shopping at a department or grocery store (in which consumers purchase *baskets* of differentiated goods). For simplicity, we do not model the endogenous formation of trading posts. This allows us to pursue our primary focus, which is elucidating the links between the means of factor payments and the exchange of final goods and services.

Finally, this paper lays the theoretical foundation for the monopolistically competitive multiple-matching trading environment used by Laing et al. (2007). There our focus is on the impact of monetary growth on market participation and production in the context of the PME outcome. Hence, the outcomes of that paper are indeed a subset of those considered by the generalized framework developed here. In particular, Laing et al. (2007) ruled out barter a priori, which, although simplifying the structure, precludes any attempt to study the issues regarding the organization of exchange and the equilibrium nature of factor payments that are central to this paper.

3. THE MODEL

Time is discrete and is indexed by $t \in \mathbf{N}$. The commodity space, $\Omega_0 = [0, N] \subseteq \mathbf{R}_+$, consists of a continuum of distinct varieties of goods, indexed by ω, which are arranged around a circle with circumference N. The economy is populated by a continuum of infinitely lived households, indexed by $h \in H_0 = [0, H]$, and a continuum of infinitely lived owners, indexed by $\hat{h} \in \hat{H}_0 = [0, N]$. (Throughout, we use the circumflex "^" to distinguish owners from households.) Although they discount the future at the common rate $\beta \in (0, 1)$, the two groups of agents differ in their endowments and preferences. Specifically, each household possesses an indivisible unit of labor that is supplied inelastically to at most one firm at a time, whereas each owner both owns and controls a firm that has unique access to the technology used to produce one type of the differentiated commodities $\omega \in \Omega_0$.[6] We assume that the set of firms in the economy is exogenously given. We denote measures by $\sigma[\cdot]$, and make the following normalizations: $\sigma[\hat{H}_0] = \sigma[H_0] = N = H = 1$.

3.1. Preferences

To capture the problem of the double coincidence of wants, we assume that agents possess idiosyncratic preferences. More specifically, a given household, h, derives utility only by consuming goods that belong to an idiosyncratic interval $\Omega(h)$. Each household draws its particular interval independently and at random, from

Ω_0 at the beginning of each period. Although all of such intervals, $\Omega(h)$ $(h \in H_0)$ are of equal length, we assume that their locations are uniformly distributed on the commodity circle. Similarly, we assume that owners derive utility by consuming goods and services that belong to idiosyncratic intervals $\hat{\Omega}(\hat{h})$ $(\hat{h} \in \hat{H}_0)$, which are drawn independently and at random from Ω_0 at the beginning of each period and have the same length as $\Omega(h)$.

Define the degree of "specialization" in tastes by $x = \sigma[\hat{\Omega}] = \sigma[\Omega] \in [0, 1]$. In a given meeting between two agents endowed with distinct goods ω and ω', the probabilities of a single coincidence of wants and the double coincidence of wants are x and x^2, respectively. Assumption 1 describes formally the preferences of households and owners.

Assumption 1 (Preferences).

(a) Household h's utility function is given by $U(D_t(h))$, where $U(\cdot)$ is strictly increasing and strictly concave, satisfying the boundary conditions $U(0) = 0$ and $\lim_{D\to\infty} U(D) = \bar{u} << \infty$, and where the consumption aggregator $D_t(h)$ takes the constant-elasticity-of-substitution form

$$D_t(h) = \left[\int_{\Omega(h)} c_t(\omega)^{\frac{\gamma-1}{\gamma}} d\omega\right]^{\frac{\gamma}{\gamma-1}}, \tag{1}$$

where $\gamma > 1$ and $c_t(\omega)$ is the date-t consumption of good ω.

(b) The utility function of owner $\hat{h}$, who produces good $\omega_t(\hat{h})$, is linear in the consumption aggregator,

$$\hat{D}_t(\hat{h}) = \widehat{C}(\omega_t(\hat{h})) + \int_{\hat{\Omega}(\hat{h})\setminus\{\omega(\hat{h})\}} \hat{c}_t(u, \omega(\hat{h}))\, du, \tag{2}$$

where (upper case) $\widehat{C}(\omega)$ is owner $\hat{h}$'s consumption of his own-produced good $\omega(\hat{h})$, and (lower case) $\hat{c}(u, \omega(\hat{h}))$ is his consumption of other goods $u \in \hat{\Omega}(\hat{h}) \setminus \{\omega(\hat{h})\}$ procured from exchange.

In equation (1), $U(D)$ is the periodic utility a household derives by consuming the "basket" of goods $D_t(h)$. The concavity assumption is standard; the asymptotic upper bound $\bar{u}$—as explained later—ensures the convergence in the limit, as search frictions vanish, of welfare under barter and monetary exchange. Observe from (1) that the value obtained from any given basket of goods depends upon the *variety* of commodities contained therein [see Dixit and Stiglitz (1977)]. The parameter γ is the constant elasticity of substitution (CES) between goods. To ensure the existence of a well-defined monopolistically competitive pricing game we impose $\gamma > 1$, implying that goods are substitutes. Although firm owners also have specialized tastes, they have no desire for variety and goods within their consumption set are perfect substitutes. This is captured in (2), where we restrict the owner's periodic utility to be linear in $\hat{D}$.[7]

3.2. Technology

In the monopolistically competitive environment considered in this paper, each owner $\hat{h}$ owns a single firm, and each firm produces a unique product, $\omega \in$

Ω_0. Hence it is both possible to ease the notational burden by identifying each owner/firm, $\hat{h}$, with his unique product, ω. Although firms produce differentiated commodities, we assume they possess identical technologies in the sense that, for the same quantity of labor input, they produce the same quantity of output of their particular product variety. The force of this assumption is that firms are economically symmetric ex ante. Formally, denote $l_t(\hat{h}) \in \mathbf{R}_+$ as the firm's employment level, $y_t(\omega(\hat{h})) \in \mathbf{R}_+$ as the level of output of good $\omega(\hat{h})$, and the production technology as $F(\ell_t(\hat{h}), \hat{h})$. Then we have:

Assumption 2 (Technology).

(a) At each point in time t, each firm $\hat{h}$ has access to an identical technology, given by

$$y_t(\omega(\hat{h})) = F(\ell_t(\hat{h}), \hat{h}) = f(\ell_t), \quad \mathbf{(3)}$$

where $f(\ell)$ represents the quantity of each good produced with labor input ℓ and is strictly increasing and strictly concave, satisfying $f(0) = 0$ and $\lim_{\ell \to 0} f'(\ell) = \infty$.

(b) Households, which have no access to the production technology, are equally talented at producing any one of the differentiated commodities.

(c) After production occurs, firms and households are capable of storing any amount of their own produced good. Neither possess the technology required to store any other good. The only cost to storage is that inventory depreciates at the common rate $\delta \in [0, 1]$.

Consider a household h that is employed by a firm that produces good ω, at the beginning of period t. In what follows, we denote this household's initial inventory holding of good ω by $k_t(\omega, h)$. Because we have already identified the owner of the firm with its unique product, ω, the firm's initial inventory holdings are simply denoted by $\hat{k}_t(\omega)$.

3.3. Markets, Prices, and Contracts

There are two principal markets of interest: the labor market and the product market. We assume the labor market is competitive: firms can hire labor provided their contractual offer (see below) provides workers with a lifetime utility of at least V_0 (determined in a market for labor contracts). The competitive labor market is warranted by the assumed free mobility of labor, and the assumption that households are equally talented at producing any good ω.[8] Note that even though the labor market is frictionless, it is immaterial whether or not a worker accepts employment at a firm that produces a good in his consumption set. By virtue of the integral used to define the household's preferences for consumption variety [equation (1)], the contribution to utility from any such source is precisely zero. The twin assumptions in part (c) that agents can store their production good (in any amount) and only their production good are important. The former, by minimizing the significance of money as a store of value, enables us to focus on its role as a medium of exchange. The latter feature precludes the emergence of commodity monies, which would complicate the analysis considerably.[9]

In order to focus on the role of money as a medium of exchange, throughout we assume that neither firms nor workers have access to credit markets. This implies that firms must use beginning-of-period cash balances and/or inventory holdings to finance the firm's contractual obligations. Likewise, households can procure goods only using their current income and/or any savings they carried over from the previous period. Notably, the assumption that the firm cannot use *current* output to finance goods payments to workers is inconsequential. Finally, we assume that the product market is monopolistically competitive and is subject to trade frictions. In the remainder of this section we describe the labor contracts offered by firms; the prices they post; and the nature of the frictions that inhere in the product market.

Owners make all of the hiring, production, and pricing decisions relevant to the firm they control. Thus, in any given period, each firm, $\omega \in \Omega_0$, hires $\ell_t(\omega)$ workers by offering a *labor contract* $\nu_t(\omega) = (G_t(\omega), s_t(\omega))$, where $G_t(\omega) \geq 0$ is a monetary wage, and $s_t(\omega) \geq 0$, is a payment made in terms of the firm's output. As we shall see later, these labor contracts forge the link between equilibrium factor payments and the endogenous medium of exchange.

Each firm also posts prices $Q_t(\omega) = (P_t(\omega), \{r_t(\omega, \omega')\}_{\omega' \in \Omega_0 \setminus \{\omega\}})$, where (i) $P_t(\omega)$, is the (date-t) monetary price of the firm's product and (ii) $\{r_t(\omega, \omega')\}_{\omega' \in \Omega_0 \setminus \{\omega\}}$ are its (date-t) relative (goods-for-goods) prices. These relative prices determine its willingness to exchange its own good ω for goods ω' brought to it by other traders; the measurement units are units of ω' per unit ω. Intuitively, $r_t(\omega, \omega')$ equals the number of units of ω' that firm ω must receive in order to exchange a unit of ω. Under this convention it is then immediate that $1/r_t(\omega, \omega')$ is again the relative price posted *by* firm ω—this time measured in units of ω per unit of ω'. Notice that $r_t(\omega', \omega)$ is the relative price posted *by* firm ω' for good ω, measured in units of ω per unit ω'. In a monopolistically competitive environment, the relative goods-for-goods prices posted by two different sellers *for two identical goods may, and generally will, differ.* Heuristically, the apple producer might set a price of two bananas per apple, at the same time the banana producer sets a price of two apples per banana: i.e., there is no presumption that $1/r_t(\omega, \omega') = r_t(\omega', \omega)$. Later, we will see that a convenient feature of the symmetric properties of the model is that all relative prices take the simple form $r_t(\omega, \omega') \in \{0, r\}$, for each firm ω and for each good ω'. Consider

Assumption 3 (The Product Market).

(a) Matching takes place only between households and firms.

(b) During each period, each household is randomly matched with a subset of firms, $Z(h)_0 \subseteq \Omega_0$, with measure

$$\sigma(Z(h)_0) = \alpha \in (0, 1]. \tag{4}$$

(c) Anonymity.

Part (a) rules out direct household-to-household and firm-to-firm exchanges, which simplifies admissible steady-state exchange patterns. This pattern of exchange can be justified (at the cost of additional notation) as an endogenous outcome given more primitive assumptions on individual preferences and worker skills. (We show this formally in Appendix A.)

In part (b), each household matches with a continuum of firms of measure α. The parameter captures the extent of search frictions in the underlying environment (a frictionless economy is consequently one in which $\alpha = 1$). As an alternative to randomness in the shopping process, $\alpha < 1$ can also be interpreted as a measure of spatial friction; although the locations of desired goods are known, shoppers can only visit a subset of those shops in a given period. Whenever a household meets a firm, then (as an identity) a firm must also meet a household. Given our earlier population normalizations, α is also the fraction of households that each firm contacts during the period. Under suitable random matching assumptions, αx is the measure of contacts that satisfy the single coincidence of wants (from the perspective of both households and firms). This gives αx^2 as the measure of contacts that satisfy the more stringent double-coincidence of wants. Although agents may meet many times, the anonymity assumption in part (c) implies the lack of an appropriate record-keeping technology, which rules out the emergence of informal credit arrangements.[10]

The intuition we intend to capture via Assumption 3 is disarmingly simple. Think of a consumer who does his week's shopping at a local market or bazaar during a period of time of unit length. While it is at the market we view the household as, in essence, having time to match with the sellers of many products (but not every product in the economy), and for realism conceive of it as selectively purchasing a basket of commodities (but not every good offered for sale). The "large numbers" assumption is intended to capture the notion that, although the consumer may be uncertain about the specific group of goods offered for sale that week, he or she anticipates almost surely (a.s.) the nature of his or her end-of-period shopping experience (and the utility he or she will obtain as a result).[11] The force of this assumption is that almost every household perceives a fully deterministic planning environment during each period. In order to study both barter and monetary exchange, we assume that each market stall posts both monetary and goods-for-goods prices and allow households to finance its purchases using cash and/or goods.

3.4. Matching

Both households and firms desire only those goods that belong to their respective consumption sets $\Omega(h)$ and $\hat{\Omega}(\omega)$. Consequently, not every match described in Assumption 3 can result in beneficial exchange. In this section we describe those that do (and as a corollary, those that do not).

According to Assumption 3, at the beginning of each period, household $h \in H$ matches with a set of firms $Z(h)_0$,with measure $\sigma[Z(h)_0] = \alpha$. In what follows, we will have frequent recourse to consider the following subset of them:

$Z(h) = \{\omega \in Z(h)_0 : \omega \in \Omega(h)\} \subseteq Z(h)_0$. It consists of those matches that also belong to the given household's consumption set, $\Omega(h)$.

It is convenient to further partition the set Z into two subsets, Z_B and Z_M, which represent, respectively, those matches that satisfy the double coincidence of wants, and those that satisfy the household's (but not the owner's) single coincidence of wants. The significance of this distinction is that the household can finance its purchases of goods belonging to the set Z_B using a mixture of cash and goods. However, it is obliged to use money for matches that belong to the set Z_M, as they do not satisfy the double coincidence of wants. Finally, we denote the complementary set of matches that provide the household with no utility whatsoever by Z_N. It is easily checked that the respective measures of these sets are $\sigma[Z] = \alpha x$; $\sigma[Z_B] = \alpha x^2$; $\sigma[Z_M] = \alpha x(1 - x)$; and $\sigma[Z_N] = \alpha(1 - x)$.

Similar concepts can be defined from the perspective of each of the firms that populate the economy—with a slight twist. The owner of a given firm, ω, is not interested in the identities of the households it matches with per se; instead he or she is interested in the particular *goods* that they bring to market—in particular, those that belong to his or her own consumption set $\hat{\Omega}(\omega)$. Analogously to the case of households described above, define $\hat{Z}(\omega)_0 \subset H$ as the set of *households* that match with firm ω during the period, and define $\hat{Z}(\omega)$ to be the subset of them that have a product that the owner of firm ω desires. Just as was the case for households, the set $\hat{Z}(\omega)$ can be further partitioned into two subsets: $\hat{Z}(\omega)_B$ and $\hat{Z}(\omega)_M$. The former includes those matches that satisfy the double coincidence of wants; the latter are those matches that satisfy the *household's* (but not the firm's) single coincidence of wants.[12] Finally, $\hat{Z}(\omega)_N$ denotes the set of households that bring with them to market a product that firm ω does not value. Given a level of employment per firm normalized as one, according to Assumption 3, each firm matches with a set of employed consumers with measure $\sigma[\hat{Z}_0] = \alpha$.[13] The measures of the other sets are $\sigma[\hat{Z}] = \alpha x$, $\sigma[\hat{Z}_B] = \alpha x^2$, $\sigma[\hat{Z}_M] = \alpha x(1 - x)$, and $\sigma[\hat{Z}_N] = \alpha(1 - x)$.

3.5. Fiat Money

The aggregate stock of fiat money, at the beginning of time t, is M_t. Fiat money is not intrinsically valued by any agent; it cannot be privately produced (think of paper currency, for example); and it is perfectly divisible. We assume free disposal of cash balances, implying that

$$M_t \geq \int_{H_0} M_t(h)dh + \int_{\Omega_0} \hat{M}_t(\omega)d\omega, \tag{5}$$

where $M_t(h)$ and $\hat{M}_t(\omega)$ are, respectively, household h's and owner ω's nominal cash holdings. We assume that the money supply grows over time as a consequence of a lump-sum injection, T_t, from the monetary authority given to firms each period.[14] The stock of money evolves as follows:

$$M_{t+1} = M_t + T_t = (1 + \mu)M_t, \quad (6)$$

where $\mu \geq 0$ is the constant rate of monetary growth. Given the constant rate of monetary growth μ, we can transform all of the nominal variables by the common growth factor $(1 + \mu)^t$. Accordingly, let $m_t(h) = M_t(h)/(1 + \mu)^t$, $\hat{m}_t(\omega) = \hat{M}_t(\omega)/(1 + \mu)^t$, $g_t = G_t/(1 + \mu)^t$, and $p_t = P_t/(1 + \mu)^t$. We define $q_t(\omega) = (p_t(\omega), \{r_t(\omega, \omega')\}_{\omega' \in \Omega_0 \setminus \{\omega\}})$ as the vector of monetary and goods-for-goods prices posted by the firm. In what follows we shall consider only these transformed variables.

3.6. Time Sequence

The sequence of events, during any given period t, is described below. In stage I each household and firm begins the period with inventory holdings $k_t(\omega, h)$ and $\hat{k}_t(\omega)$ and money holdings $M_t(h)$ and $\hat{M}_t(\omega)$, respectively. The idiosyncratic preference shock is then realized, and both households and owners learn the respective intervals $\Omega(h)$ and $\hat{\Omega}(\omega)$ over which their preferences are defined for that period. In stage II the owner of each firm $\omega \in \Omega_0$ (i) offers $\ell_t(\omega)$ workers the contract $\nu_t(\omega) = \{g_t(\omega), s_t(\omega)\}$ and (ii) posts the prices $q_t(\omega)$. After firms make their hiring commitments for the period, production commences and the terms of the contract are executed (stage III). In stage IV, matching takes place and trading occurs. In stage V, firms receive the monetary transfer T_t from the government. Finally, in stage VI, each agent chooses a consumption and savings plan.

3.7. The Equilibrium Concept

We focus on stationary symmetric subgame-perfect equilibria, in which (given each household's optimal behavior) each firm's choices of employment, ℓ, the contract, ν, and its prices, q, are optimal given the perceived behavior of other firms.[15] Each firm is negligible in the continuum and treats as exogenous the worker reservation utility V_0 and the prices posted by other firms. Households optimally supply their labor on the basis of the contractual offers made by firms and take as given the prices set by firms. Each firm is fully cognizant of the fact that households have met many other sellers and that they will substitute toward other commodities if the price it sets is unfavorable.

Our ultimate goal is to solve for the model's steady-state symmetric (subgame perfect) equilibria, which we do in three steps. We first characterize each household's demand functions for the differentiated products in their consumption baskets for a given price distribution. Next, we determine each firm's best response function around any given (stationary) symmetric price configuration. The third and final step uses these households' demand schedules and firms' best response functions. Generally, this third step would involve solving for the fixed point in the functional space of the price distribution. However, under our symmetry assumption, this step becomes trivial as it is nothing but a simple guess-and-verify exercise. That is, we guess a symmetric price configuration (a single point price distribution) and verify it as the equilibrium price posting by all firms.

4. HOUSEHOLD BEHAVIOR

We now examine the behavior of an arbitrary household $h \in H$ endowed with $k_t = k_t(\omega', h)$ of good ω', and with money holdings $m_t = m_t(h)$. We study the household's behavior within a stationary environment in which (i) the household is offered the stationary labor contract $\nu = (g,\ s)\ \forall t$, and (ii) each firm $\omega \in \Omega_0$ posts the stationary prices $q(\omega) = (p(\omega),\ r(\omega,\ \omega')_{\omega' \in \Omega_0 \setminus \{\omega\}})\ \forall t$.[16]

Recall from Section 3.4 that Z_0 denotes the set of goods that a given household h encounters during the matching process, and that $Z = Z_{\mathrm{M}} \cup Z_{\mathrm{B}}$ denotes the subset of them that provide it with positive utility. To solve the household's problem, it is helpful to decompose the procurement of each good in the barter set, $\omega \in Z_{\mathrm{B}}$, according to its *means of financing*. Thus define

$$c(\omega) = c(\omega)_b + c(\omega)_m, \quad \text{for all } \omega \in Z_{\mathrm{B}}, \tag{7}$$

where $c(\omega)_b$ is that part of $c(\omega)$ *financed* using goods' payments and $c(\omega)_m$ is that part *financed* with money. A household that is paid in kind with the particular good ω' solves

$$V(k, m) = \max_{c_b, c_m}[U(D) + \beta V(k^+, m^+)], \tag{8a}$$

$$\text{s.t.} \quad k^+ = (1-\delta)\left\{k + s - \int_{\omega \in Z_{\mathrm{B}}} r(\omega, \omega')c(\omega)_b\, d\omega\right\}, \tag{8b}$$

$$(1+\mu)m^+ = \left\{m + g - \int_{\omega \in Z_{\mathrm{M}}} p(\omega)c(\omega)\, d\omega - \int_{\omega \in Z_{\mathrm{B}}} p(\omega)[c(\omega) - c(\omega)_b]\, d\omega\right\}, \tag{8c}$$

$$\text{equation (7)},\ c \geq 0\ , c_b \geq 0, \text{ and } c - c_b \geq 0.$$

where V is the household's value function, $k = k_t(\omega', h)$, and D is the CES valuation of goods in the set Z [see equation (1)].[17] To simplify the notation, all current time period subscripts are suppressed, whereas variables in the next period are labeled with superscripts "+". When discussing a prototypical household, we also suppress the index h. Condition (8a) is the consumer's objective function and (8b) describes the evolution of the household's inventory of goods. The household augments its current inventory holdings, k, through its (in-kind) goods income s and depletes them through bartering with firms for goods that belong to the set Z_{B}. Analogously, equation (8c) is the law of motion for the household's accumulated money balances.

Lemma 1 describes the household's optimal inventory holdings of cash, m_t, and goods, k_t.

LEMMA 1 (Household Behavior). *Each consumer's optimal behavior is described by*

$$k = m = 0 \quad \forall t. \tag{9}$$

Proof. All proofs are presented in Appendix B. ■

The environment confronting each household is stationary and nonstochastic, implying the absence of a precautionary savings motive. With positive discounting, consumers optimally set their inventory, k, and cash, m, holdings to zero in the steady state [equation (9)].

The zero holding of inventory and cash across periods simplifies the analysis greatly—the dynamic optimization and intertemporal consumption demand become generically static. As shown in Appendix B, a household's consumption demand for good ω, procured via barter by trading good ω', can be specified as

$$c(\omega) = c(\omega)_b = \frac{r(\omega,\omega')^{1-\gamma}}{\int_{Z_{\mathrm{B}}} r(u,\omega')^{1-\gamma}\,du}\left[\frac{s}{r(\omega,\omega')}\right], \quad \omega \in Z_{\mathrm{B}}, \tag{10}$$

where the denominator is the monopolistically competitive price index. Similarly, a household's consumption demand for good ω purchased with cash is given by

$$c(\omega) = c(\omega)_m = \frac{p(\omega)^{1-\gamma}}{\int_{Z_{\mathrm{M}}} p(u)^{1-\gamma}\,du}\left[\frac{g}{p(\omega)}\right], \quad \omega \in Z_{\mathrm{M}}. \tag{11}$$

In each case, the constants of proportionality depend upon the consumer's contract $\nu = (g, s)$, and upon the integral of each pricing profile $r(\omega, \omega')$ and $p(\omega)$ [suitably defined over those matches whose goods provide the household with positive utility $Z(h)$].

However, in what follows, we focus on *symmetric equilibria* in which firms a.e. post identical monetary and relative goods-for-goods prices. As described below, this emphasis leads to very simple household demand functions. To see this, consider a generic firm ω that posts the prices $q = (p, r)$, where (i) $p = p(\omega)$ is its monetary price, and (ii) $r = r(\omega, \omega') \geq 0$ for $\omega' \in \hat{Z}(\omega)_{\mathrm{B}}$ (and $r = 0$ otherwise) are its relative goods-for-goods prices. Suppose further that a.e. the other monopolistically competitive firms, $u \in \Omega_0 \setminus \{\omega\}$, post the common prices $\widetilde{q} = q(u) = (\widetilde{p}, \widetilde{r})$, where (i) $\widetilde{p} = p(u)$ is their common monetary price, and (ii) $\widetilde{r} = r(u, \omega') \geq 0$ for $\omega' \in \hat{Z}(u)_b$ (and zero otherwise) are their common relative goods-for-goods prices. Consider

LEMMA 2 (Consumers' Demand Functions). *Consider some household h with current desirable matches $Z = Z_{\mathrm{M}} \cup Z_{\mathrm{B}}$, and a given firm ω that posts prices $q(\omega)$ when the other firms $u \in \Omega_0 \setminus \{\omega\}$ post a.e. the common prices $\widetilde{q}$. Then*

(a) *For all $u \notin Z$ the consumer's demand is $c(u) = c(u)_b = 0$.*

(b) *For all $u \in Z \setminus \{\omega\}$ the consumer's demand is*

(b1) *If $(g/\widetilde{p})/x(1-x) > (s/\widetilde{r})/x^2$ then*

$$c(\omega) = (1/\alpha x)[(g/\widetilde{p}) + (s/\widetilde{r})]. \qquad \textbf{(12)}$$

(b2) *If $(g/\widetilde{p})/x(1-x) < (s/\widetilde{r})/x^2$ then*

$$c(\omega) = c(\omega)_b = (1/\alpha x^2)(s/\widetilde{r}), \quad \forall \omega \in Z_{\mathrm{B}} \qquad \textbf{(13a)}$$

$$c(\omega) = c(\omega)_m = [1/\alpha x(1-x)](g/\widetilde{p}), \quad \forall \omega \in Z_{\mathrm{M}}. \qquad \textbf{(13b)}$$

(c) *Define $\hat{\rho} = \{[(1-x)/x)(\widetilde{r}/s)(g/\widetilde{p})]\}^{\gamma}$. If $\omega \in Z$ then optimizing consumer behavior is described by*

(c1) *If $(g/\widetilde{p})/x(1-x) \geq (s/\widetilde{r})/x^2$ then*

$$c(\omega, q; \widetilde{q}) = \frac{1}{\alpha x}\left[\left(\frac{g}{\widetilde{p}}\right) + \left(\frac{s}{\widetilde{r}}\right)\right]\left[\chi_{\mathrm{A}}\left(\frac{\widetilde{r}}{r}\right)^{\gamma} + (1-\chi_{\mathrm{A}})\left(\frac{\widetilde{p}}{p}\right)^{\gamma}\right], \qquad \textbf{(14)}$$

where $\chi_{\mathrm{A}} = 1$ if $\omega \in Z_{\mathrm{B}}$ and $r \leq \hat{\rho}(p/\widetilde{p})\widetilde{r}$; otherwise $\chi_{\mathrm{A}} = 0$.

(c2) *If $(g/\widetilde{p})/x(1-x) < (s/\widetilde{r})/x^2$ then*

$$c(\omega, q; \widetilde{q}) = \frac{1}{\alpha x(1-x)}\left[x(1-\chi_{\mathrm{B}})\frac{g}{\widetilde{p}} + (1-x)\chi_{\mathrm{B}}\frac{s}{\widetilde{r}}\right] \times \left[\chi_{\mathrm{B}}\left(\frac{\widetilde{r}}{r}\right)^{\gamma} + (1-\chi_{\mathrm{B}})\left(\frac{\widetilde{p}}{p}\right)^{\gamma}\right], \qquad \textbf{(15)}$$

where $\chi_{\mathrm{B}} = 1$ if $\omega \in Z_{\mathrm{B}}$ and $r \leq \hat{\rho}(p/\widetilde{p})\widetilde{r}$; otherwise $\chi_{\mathrm{B}} = 0$.

(c3) *(Financing). If $s > 0$ then*

$$c(\omega, q, \widetilde{q})_b = \begin{cases} c(\omega, q; \widetilde{q}) \\ (1/\alpha x^2)(s/\widetilde{r}) \\ 0 \end{cases} \quad \text{as } r \begin{cases} < \\ = \\ >, \end{cases} [\chi_{\mathrm{C}} + (1-\chi_{\mathrm{C}})\hat{\rho}](p/\widetilde{p})\widetilde{r} \qquad \textbf{(16)}$$

where $\chi_{\mathrm{C}} = 1$ if $(g/\widetilde{p})/x(1-x) \geq (s/\widetilde{r})/x^2$ and $\chi_{\mathrm{C}} = 0$ otherwise.

Part (a) is trivial: the household desires only those goods that belong to its consumption set $\Omega(h)$ and *can* purchase only from those firms that it matches with during the period: $Z(h)_0$. That is, it purchases neither goods it does not want nor those that it cannot.

Part (b) is explained as follows. Ideally, consumers seek uniform consumption levels of each of the differentiated products belonging to $Z \setminus \{\omega\}$, as each of them enters symmetrically into their strictly concave utility functions. However, because of trade frictions, this might not always be possible—an observation that is the key to the distinction between cases (b1) and (b2) in the lemma. For instance, in part (b2), the consumer has a relative abundance of goods he or she can trade if the discounted value of goods wage payments exceeds the discounted real value of cash wages: $(s/\widetilde{r})/x^2 > (g/\widetilde{p})/x(1-x)$. Under these circumstances, equations (13a) and (13b) imply that for any pair of goods $\omega_1 \in Z_{\mathrm{B}}$ and $\omega_2 \in Z_{\mathrm{M}}$,

$$c(\omega_1) = (1/\alpha x^2)(s/\widetilde{r}) > c(\omega_2) = [1/\alpha x(1-x)](g/\widetilde{p}).$$

This inequality illustrates how the problem of the double coincidence of wants distorts the household's consumption levels (relative to a world without trade frictions). More specifically, it impedes the household from using its real goods income, $s/\widetilde{r}$, to obtain uniform levels of consumption by affecting a simultaneous reduction in $c(\omega_1)$ and increase in $c(\omega_2)$.

In contrast, uniform consumption levels *are* feasible—and indeed chosen—in case (b1). Here, the household is abundant with cash as $(g/\widetilde{p})/x(1-x) > (s/\widetilde{r})/x^2$. The lemma shows that under these circumstances $c(\omega_1) = c(\omega_2) = \{(g/\widetilde{p})+(s/\widetilde{r})\}/(\alpha x)$ for $\omega_1 \in Z_{\mathrm{B}}$ and $\omega_2 \in Z_{\mathrm{M}}$, which indicates that the consumer simply uniformly spreads out his (periodic) real income $\{(g/\widetilde{p}) + (s/\widetilde{r})\}$ across all of the matches that provide him with positive utility.

The demand functions presented in part (c) of the lemma essentially take standard constant elasticity forms. Specific instances are readily recovered. For instance, in case (c1) the household has a relative surfeit of cash, as $(g/\widetilde{p})/x(1-x) \geq (s/\widetilde{r})/x^2$. Suppose that $\omega \in Z_{\mathrm{B}}$ and that the terms of goods-for-goods trading are "favorable"—in the sense that $r < (\widetilde{p}/p)\widetilde{r}$. Then $\chi_{\mathrm{A}} = 1$, implying

$$c(\omega) = c(\omega)_b = (1/\alpha x)\{(g/\widetilde{p}) + (s/\widetilde{r})\}(\widetilde{r}/r)^{\gamma}.$$

Hence, under these circumstances, the consumer finances its purchases of ω through barter alone. Notice that the pertinent price is the relative barter trading price $(\widetilde{r}/r)$. Alternatively, if $r > (\widetilde{p}/p)\widetilde{r}$, then according to the lemma, $\chi_{\mathrm{A}} = 0$, thus leading to

$$c(\omega) = c(\omega)_m = (1/\alpha x)\{(g/\widetilde{p}) + (s/\widetilde{r})\}(\widetilde{p}/p)^{\gamma}.$$

Hence, in this case, the household finances its purchases of ω using cash exclusively and the relative monetary price $(\widetilde{p}/p)$ is the relevant one.

A similar interpretation holds for case (c2), in which the household has an abundant supply of tradable goods relative to its cash holdings: $(s/\widetilde{r})/x^2 > (g/\widetilde{p})/x(1-x)$. Notice from its definition that under these circumstances $\hat{\rho} > 1$. It follows from equation (16) that the terms of monetary trade must be relatively attractive—i.e., $p/\widetilde{p}$ must be "low"—before a cash-poor household will finance its purchases of a good that belongs to the barter set Z_{B} exclusively using money.

5. PURE BARTER EXCHANGE

In the PBE, all trade involves the exchange of goods for goods, and money is not valued. Each period, workers receive their remuneration in terms of their employers' output alone and, upon payment, search for trading partners. In order to establish the existence of a steady-state symmetric equilibrium, our analysis proceeds as follows. We first *assume* that money is valueless and derive each seller's best response given (i) the consumer demand functions in Lemma 2 and (ii) both the prices and labor contracts offered by other firms. We then solve for the symmetric steady-state full-employment PBE and finally check that no agent optimally accepts cash, which is trivial.

5.1. Firm's Behavior

We determine the best response behavior of an arbitrary firm, indexed ω, conditional on the demands presented in Lemma 2 given values of $\widetilde{q} = (\infty, \widetilde{r})$ and $\widetilde{\nu} = (0, s)$ offered and a level of employment per firm of one by other firms (assuming that money is valueless).

As noted in Section 3.4, firm ω matches with a set $\hat{Z}_0$ of employed customers, where $\sigma[\hat{Z}_0] = \alpha$. The owner of firm ω maximizes his or her lifetime utility $\hat{V}$,

$$\hat{V}(\hat{k}) = \max_{\{\hat{C}.\ell,s,r\}} [\hat{C} + x^2 rc(\omega, q; \widetilde{q})] + \beta \hat{V}(\hat{k}^+), \quad \textbf{(17a)}$$

$$\text{s.t.} \quad \hat{k}^+ = (1-\delta)[\hat{k} + f(\ell) - s\ell - \alpha x^2 c(\omega, q; \widetilde{q}) - \hat{C}], \quad \textbf{(17b)}$$

$$(s - \widetilde{s})\ell \geqslant 0, \quad \textbf{(17c)}$$

$$\hat{k} \geqslant s\ell, \quad \textbf{(17d)}$$

where $\hat{k} = \hat{k}_t(\omega)$. As a consequence of symmetry, the firm's relative price is $r = r(\omega, \omega'')$ for $\omega'' \in \hat{Z}_{\mathrm{B}}$ and $r = 0$ otherwise.

In (17a) the owner of firm ω derives utility by consuming his or her own product ($\hat{C}$) in conjunction with goods in $\hat{Z}_{\mathrm{B}}$ acquired after bartering with households.[18] Equation (17b) describes the evolution of the owner's inventory holdings; any output not used to pay workers is either consumed by the owner, sold to households, or stored for the future. Condition (17c) is the workers' participation constraint. The firm must offer a goods' payment of at least $\widetilde{s}$ to be accepted by workers. The inequality (17d) reflects the absence of capital markets: all payments to workers are financed from beginning-of-period inventory holdings. The first-order conditions (with respect to $\{\hat{C}, \ell, s, r\}$) and the Benveniste–Scheinkman condition (with respect to $\hat{k}$) are

$$\hat{c}[1 - \beta(1-\delta)\hat{V}_k] = 0 \text{ with } 1 - \beta(1-\delta)\hat{V}_k \leq 0, \ \hat{c} \geqslant 0, \quad \textbf{(18a)}$$

$$\beta(1-\delta) f' = \widetilde{s} (= s), \quad \textbf{(18b)}$$

$$-\hat{V}_k + \varphi = 0, \quad \textbf{(18c)}$$

$$\alpha x^2 c(\omega, q; \widetilde{q})[\gamma\beta(1-\delta)\hat{V}/r - (\gamma - 1)] = 0, \quad \textbf{(18d)}$$

$$\hat{V}_k{}^+ = \beta(1-\delta)\hat{V}_k + \mu_{\mathrm{B}}, \quad \textbf{(18e)}$$

where $\hat{V}_k = d\hat{V}(k)/dk$, $f' = df(\ell)/d\ell$, and φ and μ_{B} are the Lagrange multipliers on the constraints (17c) and (17d), respectively. The complementary slackness condition (18a) reflects the possibility that the firm might, after paying workers, optimally exchange all of its residual output with consumers and set $\hat{C}(\omega) = 0$. Condition (18b) says the firm hires workers up to the point at which the marginal benefit of labor equals its marginal cost (all measured in terms of real output). The other conditions, (18c), (18d), and (18e), possess similar routine interpretations.

5.2. Steady-State Equilibrium

In a symmetric steady-state equilibrium with full employment, the numbers of workers per firm are equalized ($\ell = 1$), each firm sets a common price ($r = \widetilde{r} = r^*$), and all firms offer the same payment to workers $s = \widetilde{s} = s^*$. Also, in the PBE, cash is valueless ($p^* = \infty$), and money wages are not paid to workers $g^* = 0$.

In order to avoid the tedious duplication of results in the boundary case $\hat{C} = 0$, in which the owner trades away all of his or her residual output, consider

Condition U. $\beta \leq \gamma/\{\gamma + (1-\delta)(1-\gamma)\}$.

Condition U ensures that owners discount the future sufficiently rapidly so that it is optimal, at the margin, for them to consume unsold output beyond that required to pay for the next period's labor. We assume throughout that Condition U is satisfied.

THEOREM 1 (Pure Barter Equilibrium: PBE). *Under Condition U a unique symmetric steady-state PBE exists. It is described by*

$$\ell^* = 1, \quad \textbf{(19a)}$$

$$\nu^* = \{g^*, s^*\}, \quad g^* = 0, \textit{ and } s^* = \beta(1-\delta)f'(1), \quad \textbf{(19b)}$$

$$p^* = \infty \textit{ and } r^* = \gamma/(\gamma - 1), \quad \textbf{(19c)}$$

$$\hat{C}^* = f(1) - \beta(1-\delta)f'(1)\{1 + [r^*/(1-\delta)]\}/r^* > 0, \quad \textbf{(19d)}$$

$$c^* = (s^*/\alpha x^2 r^*) \quad \forall \omega \in Z_{\text{B}} \textit{ and } c(\omega)^* = 0 \textit{ otherwise}, \quad \textbf{(19e)}$$

$$\hat{k}^* = s^* \textit{ and } \hat{m}^* \leqslant M_0. \quad \textbf{(19f)}$$

Equation (19b) says that workers are hired up to the point at which the value of their goods payment, s^*, equals the net value of their marginal product [adjusted by $(1-\delta)\beta$, reflecting discounting and the depreciation of inventory]. Equation (19c) determines equilibrium pricing. The condition $r^* = \gamma/(\gamma-1)$ is standard in models of monopolistic competition. It equals each consumer's common marginal rate of substitution between all goods in his or her consumption set. With $p^* = \infty$, it is optimal neither for workers to exchange their labor for money nor for firms to trade their goods for money. Given symmetric pricing, each household uniformly allocates its periodic real income s^*/r^* among all commodities that satisfy the double coincidence of wants $\omega \in Z_{\text{B}}$. From (19b) and (19c), real income is

$$(s^*/r^*) = [(\gamma - 1)/\gamma](1-\delta)\beta f'(1). \quad \textbf{(20)}$$

In (20) the term $1/r^* = (\gamma - 1)/\gamma < 1$ is the wedge between workers' real incomes and their (suitably) discounted marginal product that arises by virtue of each firm's monopoly power. As $\gamma \rightarrow \infty$, consumers regard all goods as close substitutes. In this case each firm's monopoly is minimal and both real incomes, s^*/r^*, and the relative prices, r^*, converge to their "competitive" value equations (19c) and (20).

6. MONETARY EXCHANGE UNDER STEADY-STATE INFLATION

Although pure barter exchange is always an equilibrium, our model also admits monetary equilibria. Two cases may be distinguished. First, in the PME, cash is used on one side of every transaction (goods and labor). Second, in the MTE, monetary exchange and barter coexist. Which one of these two exchange regimes pertains depends crucially upon the storability of goods relative to money, measured by $\Delta = (1-\delta)(1+\mu)$. This parameter captures the comparative advantage of barter relative to monetary exchange: barter is more attractive the lower is the rate of depreciation of goods, δ, and the higher is the rate of monetary growth, μ.

The basic strategy used to prove the existence of a steady-state equilibrium and to characterize its properties is essentially identical to that used for the PBE in Section 4. The main difference is ruling out the possibility that in the PME, a firm will defect from the proposed equilibrium and offer its employees a contract that includes both goods and cash payments, which workers optimally accept. Unlike fiat money, goods *are* intrinsically valuable.

6.1. Firm's Behavior

We determine the best response of an arbitrary firm, indexed ω, conditional upon the consumer demand functions presented in Lemma 2 given values of $\widetilde{\nu}= (\widetilde{g},\widetilde{s})$ and $\widetilde{q}= (\widetilde{p},\widetilde{r})$ offered and a level of employment per firm of one, a.e., by other firms.

With the lump sum cash transfer from the authorities, if the firm employs ℓ workers at a wage G, its cash balances evolve as

$$\hat{M}_{t+1} = \left[\hat{M}_t + \mu M_0(1+\mu)^t + \int_{u\in\widehat{Z}} P_t c_{mt}\, du - G_t \ell_t\right], \qquad \textbf{(21)}$$

where $\mu M_0(1+\mu)^t$ is the nominal value of the periodic cash transfer and c_{mt} is the (money financed) demand for the firm's product, ω, by household, $u \in \hat{Z}$. [It is determined using the condition $c(\omega)_m = c(\omega) - c(\omega)_b$—see (7)—and Lemma 2.] The firm augments its money holdings through cash sales to consumers and depletes them through money wage payments to workers ($G\ell$). By using the transformations of nominal variables in conjunction with the measure $\sigma[\hat{Z}] = \alpha x$, (21) becomes

$$(1+\mu)\hat{m}_{t+1} = \hat{m}_t + \mu M_0 + \alpha x p_t c_{mt} - g_t \ell_t. \qquad \textbf{(22)}$$

Given the evolution constraint, (22), and the measures $\sigma[\hat{Z}_{\mathrm{B}}]$ and $\sigma[\hat{Z}_{\mathrm{M}}]$, the owner of firm ω solves

$$\hat{V}(\hat{k},\hat{m}) = \max_{\{\hat{C},\ell,g,p\}} [\hat{c} + \alpha x^2 r c(\omega_2, q; \widetilde{q}) + \beta \hat{V}(\hat{k}^+, \hat{m}^+)], \tag{23a}$$

$$\text{s.t.} \quad \hat{m}^+ = [\hat{m} + \mu M_0 + \alpha x p[(1-x)c(\omega_1, q; \widetilde{q}) + x c(\omega_2, q; \widetilde{q}) - g\ell](1+\mu)^{-1}, \tag{23b}$$

$$\hat{k}^+ = (1-\delta)[\hat{k} + f(\ell) - s\ell - \alpha x[(1-x)c(\omega_1, q; \widetilde{q}) + x c(\omega_2, q, \widetilde{q})] - \hat{c}], \tag{23c}$$

$$U[D] \geqslant (1-\beta)V_0, \tag{23d}$$

$$\hat{k} \geqslant s\ell, \tag{23e}$$

$$\hat{m} \geqslant g\ell, \tag{23f}$$

$$(s - \widetilde{s})\ell \geqslant 0, \tag{23g}$$

$$(g - \widetilde{g})\ell \geqslant 0, \tag{23h}$$

where $\omega_1 \in \hat{Z}_{\mathrm{M}}$ and $\omega_2 \in \hat{Z}_{\mathrm{B}}$. The possibility of barter implies that owners can derive utility by consuming their own product, and from those goods they acquire after trading with households (23a). Equation (23b) restates the law of motion describing the evolution of the firm's money holdings. Notice that in (23b) households in $\hat{Z}_{\mathrm{B}}$ and $\hat{Z}_{\mathrm{M}}$ may well finance their purchases differently: members of the former set may use cash and goods, whereas members of the latter must use cash. In (23c)—the participation constraint—$U[D]$ is the periodic utility derived by the firm's employees from the contract ν, given that other firms set, a.e., prices $\widetilde{q} = (\widetilde{p}, \widetilde{r})$.[19]

It is important to emphasize that inequality (23f) is an ex post finance constraint, which arises due to the absence of capital markets. Correctly interpreted, it is not an ex post cash-in-advance constraint (restricting both the means of payment and exchange). The reason is that firms have the option of paying workers in terms of their own output (which workers can use to barter for goods with other firms). The object of the present exercise is to circumscribe the conditions under which this latter possibility either is or is not optimally exercised.

6.2. Steady-State Equilibrium

In a symmetric steady-state full-employment equilibrium: employment per firm is equalized ($\ell = \ell^* = 1$); each firm sets a common price $p = \widetilde{p} = p^*$; and all firms offer the same contract $\nu = \widetilde{\nu} = (g^*, s^*)$. In the PME workers are not paid in goods, $s = \widetilde{s} = s^* = 0$, whereas in the MTE barter and monetary exchange coexist ($g^* > 0$ and $s^* > 0$).[20]

Theorem 2 establishes the existence of monetary equilibria.

THEOREM 2 (Monetary Equilibria). *Given condition U, there is a stationary symmetric monetary equilibrium a.e.*

(A) *If $\mu < \delta/(1-\delta)$, it is a PME characterized by, $g^* > 0$ and $s^* = 0$.*
(B) *If $\mu > \delta/(1-\delta)$, it is a MTE characterized by, $g^* > 0$ and $s^* > 0$.*
(C) *If $\mu = \delta/(1-\delta)$, there is a unique PME.*

Once again Condition U ensures that $\hat{C} > 0$ in either regime. In part (A) the condition that $\mu < \delta/(1-\delta)$ $(\Delta < 1)$ implies there is a comparative advantage of monetary exchange relative to barter. Here, the rate of inflation is not too high and firms optimally offer their employees only cash payments. However, this is not so in (B), and as a consequence, $s^* > 0$—workers are paid in both cash and in kind. In the knife-edge case $\mu = \delta/(1-\delta)$, neither monetary exchange nor barter has a comparative advantage. Accordingly, firms and workers are indifferent to any contract $\nu = (g, s)$ offering workers (equilibrium) utility V^*, provided that $g \geqslant p^*[(1-x)/x](s/r^*)$. The reason is that, under these circumstances, (i) households secure uniform consumption levels of all goods in their consumption set $\Omega^*(h)$ (Lemma 1) and (ii) at the margin, money wage payments, w, and payments in kind, s, are equally costly to the firm.

7. CHARACTERIZATION OF THE PURE MONETARY EQUILIBRIUM AND THE MIXED-TRADING EQUILIBRIUM

In this section we characterize formally the properties of the PME and MTE described in Theorem 2 and discuss the implications of our results.

THEOREM 3 (The PME and the MTE).
(A) *In any symmetric steady-state monetary equilibrium,*

$$\ell^* = 1, \tag{24a}$$

$$\nu^* = \{g^*, s^*\}, \quad \textit{where } M_0 = \hat{m}^* = g^*\ell^*, \tag{24b}$$

$$r^* \geq \gamma/(\gamma - 1), \quad \textit{with equality whenever barter trades occur.} \tag{24c}$$

(B) *If $\mu \leqslant \delta/(1-\delta)$, then in the PME*

$$s^* = \hat{k}^* = 0, \tag{25a}$$

$$p^* = M_0(1+\mu)r^*/[\beta f'(1)], \tag{25b}$$

$$\hat{C}^* = f(1) - \{\beta(1-\delta)f'(1)\}/(r^*\Delta) > 0, \tag{25c}$$

$$c^* = (g^*/p^*)(1/\alpha x), \quad \forall \omega \in Z. \tag{25d}$$

(C) *If* $\mu > \delta/(1-\delta)$*, then in the MTE*

$$s^* = (1-\delta)\beta f'(1)\left\{\frac{x}{x+(1-x)\Delta^{1-\gamma}}\right\} > 0, \tag{26a}$$

$$p^* = \left\{\frac{M_0}{\beta f'(1)}\right\}\left\{\frac{x\Delta^{\gamma-1}+(1-x)}{(1-x)}\right\}, \tag{26b}$$

$$\hat{C}^* = f(1) - \left\{\frac{[xr^*/(1-\delta)]+[x+(1-x)\Delta^{-\gamma}]}{r^*[x+(1-x)\Delta^{1-\gamma}]}\right\}\beta(1-\delta)f'(1) > 0, \tag{26c}$$

$$c(\omega_1) = c_b^* = \frac{1}{\alpha x^2}\frac{s^*}{r^*} > c(\omega_2) = c_m^* = \frac{1}{\alpha x(1-x)}\frac{g^*}{p^*},\ \omega_1 \in Z_{\text{B}} \textit{ and } \omega_2 \in Z_{\text{M}}, \tag{26d}$$

$$\hat{k}^* = s^* > 0. \tag{26e}$$

The competitive labor market assumption, in conjunction with full wage and price flexibility, implies that all workers are employed in any putative symmetric equilibrium (24a). Moreover, in a monetary equilibrium, the money stock is optimally held across each of the periods. Indeed, with $m(h) = 0\ \forall h \in H_0$, firms hold all of the money balances at the end of each period and in an amount just sufficient to cover next period's wage bill. Notice that the barter trading price is $r^* = \gamma/(\gamma-1)$, as was the case for the PBE (24c).

Inspection of (25b) and (26b) indicates that the price level is simply proportional to the (initial) stock of money M_0. Further examination of the system of equations (25) and (26) reveals that money is neutral, as the real variables in the model are independent of M_0.

From (24b), (24c), and (25b) it follows that each household's real income in the PME is

$$(g^*/p^*) = [(\gamma-1)/\gamma](1-\delta)\beta f'(1)/\Delta. \tag{27}$$

As in equation (20), the term $(\gamma-1)/\gamma < 1$ stems from the monopolistically competitive structure. Notably, (27) differs from the real income obtained in the PBE (20) only in the inclusion of the factor $1/\Delta$, reflecting the (possible) depreciation of goods necessarily stored under barter and the deleterious effects of anticipated inflation. A comparison of (25b) and (27) indicates that money is not superneutral. An increase in the monetary growth rate, μ, redistributes wealth from households to the owners of firms. From the household's perspective, the Friedman Rule, which contracts the money growth rate at the rate of time preference, $\mu = \beta - 1$, would be optimal. However, because of this redistributive effect between households and firms, the equilibrium allocations resulting from different inflation rates are Pareto noncomparable. A similar finding is obtained by Casella and Feinstein (1990), in which the monetary infusion is applied to one of two separate sectors. The underlying nominal variables are easily recovered. For instance, the nominal price level is $P_t^* = p^*(1+\mu)^t$, which indicates a constant steady-state rate of inflation that equals the monetary growth rate μ. Notice also that the lack of a savings motive, which leads households to optimally spend all of their cash balances each period, implies a unitary velocity of money that is invarient to the inflation rate. As we shall see below, this counterfactual implication will no longer be present in the MTE.

In the PME workers are not paid in goods, $s^* = 0$, and thus cannot subsequently engage in barter. This implies that in equilibrium the value of the relative price r^* is inconsequential for the payoff accruing to any given firm and, indeed, that witnessing a worker with goods for sale is an "out-of-equilibrium" event. As a consequence, there are multiple equilibria, all yielding the same payoffs. As a refinement, one may consider a perturbed game in which an exogenous fraction $\varepsilon' > 0$ of workers are endowed at the beginning of each period with goods alone. One can then establish that, under these circumstances, as $\varepsilon' \to 0$, the optimal barter goods-for-goods price is $r^* = \gamma/(\gamma - 1)$. Given this particular refinement, the conditions of the theorem imply that there is no barter (as $s^* = 0$). Moreover, this is the optimal choice of r^* if a small amount of barter were to take place.

As might be expected, the MTE possesses many features in common with both the PBE described earlier and the PME described above. For the purposes of the present discussion, the key feature of the equilibrium is that $s^* > 0$ and $g^* > 0$, implying that monetary exchange and barter coexist. (Moreover, because $s^* > 0$, in contrast to the case of the PME, barter *is* an equilibrium event and the price r^* is unique and is perfectly well defined.) Given that $\gamma > 1$ and $\Delta = (1+\mu)(1-\delta) > 1$, it is easily seen from (26a) that $ds^*/d\mu > 0$. Thus, further increases in the rate of expansion of the money supply (and hence the rate of inflation) raise the steady-state volume of barter transactions. Hence, in the MTE, the velocity of money will be strictly increasing in the inflation rate. These finding are consistent with the commonly observed patterns of exchange under hyperinflation in which barter emerges as sellers accept goods and cash payments and in which workers receive part of their remuneration in terms of their employer's output.[21]

The mechanism of our result differs from those obtained by Casella and Feinstein (1990) and by Shi (1997). In Casella and Feinstein, an increase in the monetary growth rate affects the relative bargaining power of buyers and sellers under a given exchange protocol. Absent lump sum redistributive taxation, this tends to improve the steady-state welfare of sellers relative to buyers. Shi considers endogenous exchange patterns and uncovers an interesting trading opportunity effect. This arises because each household fails to recognize the trading externality arising from its choice of the fraction of money holders in the family. A higher money growth rate encourages households to trade money away by increasing this fraction (which promotes economic activity).[22] In our model, the nonsuperneutrality result stems from the fact that we endogenize both the medium of exchange and the means of factor payments. At higher rates of inflation, each firm optimally adjusts the terms of its contractual offer to workers by substituting away from cash payments toward (less costly) payments in kind.

As we have seen earlier (Lemma 1), consumers seek to spread their periodic real incomes uniformly across all goods they contact and desire. In view of this, the result reported in equation (26d), which indicates that $c_b^* > c_m^*$, reflects the distorting effects of (hyper)inflation on steady-state consumption patterns. For sufficiently rapid rates of monetary growth (in which $\Delta > 1$), consumers substitute away from those goods they can procure through cash payments alone toward those that they can obtain through barter. Manipulation of (26b) in conjunction with the other first-order conditions gives $(c_m^*/c_b^*) = \Delta^{-\gamma} < 1$. In the limit $\gamma \to \infty$ all goods are close (perfect) substitutes, and households gain little from consuming a wide variety of goods. Hence, provided that $\Delta > 1$, they can drive their consumption of c_m^* close to zero with little utility loss (i.e., $\lim_{\gamma\to\infty}(c_m^*/c_b^*) = \lim_{\gamma\to\infty} \Delta^{-\gamma} = 0$). As in the PME described above, money is neutral: a once and for all anticipated increase in M_0 simply raises all prices in direct proportion without real effects. Equation (26b) implies that $\partial^2 p^*/\partial M_0 \partial\mu \propto [(1-x) + \gamma x \Delta^{\gamma-1}] > 0$. This says that increases in the initial money stock, M_0, have proportionately greater effects on the price level, p^*, the greater the rate of inflation, μ. This conclusion that anticipated inflation crowds out real balances is often imposed as a key assumption of ad hoc money demand functions in the hyperinflation literature. It arises here because, as the volume of monetary transactions declines, a given monetary infusion (M_0) is used to procure ever fewer goods. The term $x\gamma\Delta^{\gamma-1}$ reflects the rate at which households are willing to abandon cash-financed consumption and switch to barter. In the case of perfect substitutes, $\gamma \to \infty$ and hence $\lim_{\gamma\to\infty} \gamma\Delta^{\gamma-1} \to \infty$. Here, even small differences in μ have a dramatic effect on the volume of barter transactions and hence upon the sensitivity of the price level, p^*, to the money stock M_0.

Casella and Feinstein (1990) obtain a similar result but for quite different reasons. Their model is characterized by *predetermined* (monetary) exchange patterns, overlapping generations of different search vintages (corresponding to a buyer's duration in the market), with a maximal vintage (at which point a buyer's money holdings have atrophied to a point of obsolescence). Increases in the monetary growth rate decrease this maximal vintage and the steady-state population of buyers in the market, reducing the average time buyers hold cash and increasing the velocity of circulation. Consequently, any new injection of cash has a proportionately greater effect on prices with a higher rate of money creation. In contrast, our finding is a direct consequence of endogenous adjustments of monetary and barter transactions undertaken in equilibrium. This latter mechanism is precluded in Casella and Feinstein, because an exogenous exchange role for money is prescribed a priori.

Turning now to workers' real incomes, they are

$$(g^*/p^*) + (s^*/r^*) = [(\gamma-1)/\gamma](1-\delta)\beta f'(1)\left[\frac{x+(1-x)\Delta^{-\gamma}}{x+(1-x)\Delta^{1-\gamma}}\right]. \quad \textbf{(28)}$$

It is instructive to consider this value in the limit, $\lim_{\mu\to\infty}$. Consider

THEOREM 4. *As the rate of monetary growth becomes arbitrarily large (i.e.,* $\lim_{\mu\to\infty}$ *), the MTE converges to the PBE described in Theorem 1.*

In particular, from (28), we have $\lim_{\mu\to\infty}(g^*/p^*) = 0$ and hence

$$(s^*/r^*)_{\text{PBE}} = \lim_{\mu\to\infty}\{(g^*/p^*) + (s^*/r^*)\} = [(\gamma-1)/\gamma]\beta(1-\delta)f'(1),$$

indicating, from equation (20), that each worker's real income converges to that of the PBE $(s^*/r^*)_{\text{PBE}}$. However, for any finite rate of inflation, the monetary component of the real wage is strictly positive, $g^*/p^* > 0$ (provided of course cash is still valued). The continued circulation of money is a consequence of each household's preference for consumption variety. Even if μ is extremely large, small holdings of real money balances allow workers to secure an additional $\alpha x(1-x)$ goods relative to the basket they could obtain using barter alone (i.e., if $g^* = 0$). By virtue of their relative scarcity of these goods in the household's consumption basket, they possess extremely high marginal utilities of consumption and command a commensurately high "willingness to pay."

8. WELFARE ANALYSIS

We now compare the welfare properties of the PBE and PME. In order to ensure that the conditions of Theorem 2 are satisfied, assume throughout that $\Delta \leqslant 1$. First, what elements of our model are essential for monetary exchange to improve welfare relative to barter? Second, what are the welfare implications in the limiting case where trade frictions vanish? Theorems 1 and 3 may be used to compute each agent's steady-state lifetime discounted utility in the PBE (B) and the PME (M),

$$V^{*\text{B}} = U(1-\beta)^{-1}\left[(\alpha x^2)^{\frac{1}{\gamma-1}}(s^*/r^*)\right], \tag{30a}$$

$$\hat{V}^{*\text{B}} = (1-\beta)^{-1}\left[f(\ell^*) - (s^*/r^*)\ell^*\left\{1 + \frac{\delta r^*}{(1-\delta)}\right\}\right], \tag{30b}$$

$$V^{*\text{M}} = U(1-\beta)^{-1}\left[(\alpha x)^{\frac{1}{\gamma-1}}(g^*/p^*)\right], \tag{30c}$$

$$\hat{V}^{*\text{M}} = (1-\beta)^{-1}[f(\ell^*) - (g^*/p^*)\ell^*]. \tag{30d}$$

Using equations (20) and (27), it is readily verified that periodic real incomes in the PBE and in the PME may be written as $(s^*/r^*) = (1-\delta)\beta f'(\ell^*)$ and $(g^*/p^*) = (s^*/r^*)/\Delta$, respectively. In order to better understand the role played by trade frictions, α, and by the problem of the double coincidence of wants, $x \leqslant x^2 \leqslant 1$, it is instructive to first examine the benchmark case in which goods are perfectly storable ($\delta = 0$) and in which there is no monetary growth ($\mu = 0$). In this case $\Delta = 1$ and, as a result, $(g^*/p^*) = (s^*/r^*)$. With these, we have

THEOREM 5. *Welfare Properties of the PBE and the PME with $\mu = \delta = 0$).*

(A) $\hat{V}^{*\text{M}} = \hat{V}^{*\text{B}}$

(B) *For finite α if (a) $x < 1$, then $V^{*\text{B}} < V^{*\text{M}}$ and (b) $x = 1$, then $V^{*\text{M}} = V^{*\text{B}}$.*

Given that $\mu = \delta = 0$, owners are equally well off in either the PBE or the PME. This is natural: they have *no* preference for consumption variety and under the conditions of the theorem, there is an intrinsic disadvantage neither of barter (depreciation of inventory) nor of monetary exchange (inflation). However, Theorem 5 shows that even with $(g^*/p^*) = (s^*/r^*)$, workers' welfare levels are strictly lower in the PBE than in the PME whenever $\alpha < \infty$ and $x < 1$. The drawback of barter exchange is that the problem of the double coincidence of wants stymies the *variety* of the resultant consumption basket [which may be seen by comparing $x > x^2$ in equations (30a) and (30c)]. However, if $x = 1$, agents are "generalists" in consumption. Accordingly, all trades are beneficial, and hence are consummated in equilibrium.[23]

The welfare properties of the general model in which $\mu > 0$ and $\delta > 0$ then follow in a straightforward manner. An increase in the depreciation rate of goods, δ, lowers the steady-state welfare of both households and firms in the PBE, leaving welfare levels in the PME unchanged. Similarly, an increase in the monetary growth rate, μ, is deleterious (to households) in the PME, but irrelevant in the PBE, because money is not valued.

9. CONCLUDING REMARKS

In this paper, we develop a monopolistically competitive model where decentralized exchange occurs through the multiple matching of buyers and sellers. The resultant structure, which highlights the necessary role of trade frictions in explaining the use of money, resembles how market exchange for goods and labor services are organized in modern economies. As such, it has proven to be highly tractable and we have used it to examine the endogenous patterns of exchange and pricing, as well as how inflationary monetary policies affects these equilibrium trading outcomes.

We believe that the framework admits a number of interesting extensions. One avenue is to use our model to study quantitatively the welfare cost of inflation when both trading and payment patterns are endogenously determined. Moreover, future work can incorporate a variety of assets (including share holdings and dividend payments) as well as a credit market. This exercise expands the scope of instruments at the government's disposal and permits a much richer analysis of the effects of monetary policy. The lack of a precautionary savings motive and the model's complete symmetry lead to a simple degenerate distribution of cash balances ex post, with firms holding all of the money in the economy at the end of each period. If, instead, we assume that households are subject either to idiosyncratic taste shocks or to shocks to their endowment of human capital, a nondegenerate cash distribution would emerge in equilibrium. It would be of interest to explore the effects of monetary policy on such distributions. Finally, the explicit inclusion of firms is significant. This feature provides a natural forum for admitting endogenous capital accumulation and for thus exploring the links between inflation and growth. These are enduring and important issues in monetary theory, but until recently they have proven to be difficult subjects of study when viewed under the conceptual lens of extant search theory.

NOTES

1. See Rupert et al. (2000) and Li (2001) for a detailed review of the origins of this literature as well as earlier extensions of the prototype model to include price-setting mechanisms.

2. This final question can hence be seen to bridge the gap between Casella and Feinstein's (1990) analysis of hyperinflation on decentralized exchange patterns in a model which money has value a priori and the search-theoretic literature originating with Kiyotaki and Wright.

3. As in Howitt (2005), this feature of our model is an equilibrium outcome; it does not call for special assumptions being made concerning the joint distribution of tastes and endowments—the Wicksellian triangle.

4. The random nature of sequential search implies that direct extensions of KW generally lead to an endogenous distribution of cash and inventory holdings. The resulting distributions are analytically complex, limiting the applicability of these models [e.g., see Camera and Corbae (1999) and Molico (2006)].

5. Starr and Stinchombe develop a structure organized around an endogenous trading post network, in which each shop at a particular location optimally chooses to trade a specialized good for a common commodity money. Much of this recent literature is rooted in the celebrated contribution of Shubik (1973).

6. As in Diamond and Yellin (1990), this structure allows us to avoid explicitly modelling an equity market or the Arrow–Debreu redistribution of firms' profits. Incorporating this feature into a barter environment is problematic, because dividend payments are in the form of goods. The current ownership structure avoids this problem; puts barter and monetary exchange on the same footing; and allows a precise characterization of the difficulties of the former relative to the latter grounded in tastes (the problem of the double coincidence) and trade frictions.

7. This restriction eliminates wealth effects on each owner's price-setting behavior. It is innocuous given our focus on ex post symmetric equilibrium.

8. As is standard in (optimal) contracting environments, only the distribution of utility between workers and firms depends upon the competitive-labor-market assumption and not the (essential) properties of the contract. Thus, if V_0 is determined in either a monopsonistic labor market—or even one characterized by search frictions—then firms must simply offer contracts, ν, that provide at least this reservation utility.

9. This can be endogenized by simply positing a small but positive transactions cost. Given the symmetry of all goods, agents would *not* accept commodity monies when they could always barter their own production good (and avoid the cost). This rules out the possibility for a household to resell goods paid by the employer to another agent. Shevchenko (2004) relaxes this assumption by considering middlemen who optimally store a variety of goods in inventory.

10. The importance and the role of this assumption is explored in Kocherlakota and Wallace (1989).

11. As is common in the monopolistic competition literature, we assume that although prices are fully flexible across periods, they are constant, within them. In the present context, this means that the prices the consumer faces at each stall are independent of the order in which he or she executes his or her shopping plan.

12. The trade structure is one in which households purchase goods from firms—it is a property of the symmetric equilibrium considered in this paper that no firm can gain by offering to purchase goods *from* consumers for cash.

13. Precisely, we postulate that at equilibrium everyone is employed and then verify this conjecture.

14. Modeling cash injections to firms that use them to finance wage payments is fairly standard in the monetary business cycle literature [e.g., Fuerst (1992)]. As we shall see below, this assumption is without loss of generality given that households face no finance constraints and will choose not to carry cash across periods.

15. Each period defines a proper subgame because the environment is stationary and nonstochastic. See, for example, Aliprantis et al. (2007).

16. The stationary environment is one with the nominal wage $\tilde{w}$ and the price level $\tilde{p}$ growing at the common rate μ.

17. Note that $c(\omega)_b = 0$ for all $\omega \in Z_{\mathrm{M}}$, as households must use cash for meetings that do not satisfy the double coincidence of wants.

18. Goods ω and ω'' are exchanged only if the double coincidence of wants is satisfied (i.e., only if $\omega'' \in \hat{Z}_{\mathrm{B}}$). In equation (17a), the value of goods acquired by the owner from trading with households (in utility terms) is $rc\alpha x^2$. It is derived as $\int_{\hat{Z}_{\mathrm{B}}} c(\omega'')d\omega'' = \int_{\hat{Z}_{\mathrm{B}}} r(\omega, \omega'')c(\omega)d\omega'' = rc\int_{\hat{Z}_{\mathrm{B}}} d\omega'' = rc\alpha x^2$. The first equality follows from the identity that income equals expenditure $[c(\omega'') = r(\omega, \omega'')c(\omega)]$. The second follows from symmetry ($r = r(\omega, \omega'')$ for all $\omega \in \hat{Z}_{\mathrm{B}}$) and the third from the law of large numbers ($\sigma[\hat{Z}_{\mathrm{B}}] = \alpha x^2$).

19. The term D is the consumer's valuation of the basket of goods acquired during the period. Formally, $D = \{\alpha x^2 c(\omega_1)^{1-1/\gamma} + \alpha x(1-x)c(\omega_2)^{1-1/\gamma}\}^{\gamma/(\gamma-q)}$, where $\omega_1 \in \hat{Z}_{\mathrm{B}}$ *and* $\omega_2 \in \hat{Z}_{\mathrm{M}}$.

20. The MTE considered here is quite distinct from the "mixed-monetary equilibrium" (MME) analyzed by Kiyotaki and Wright (1993). Indeed, the MME corresponds to a mixed-strategy equilibrium, in which each agent is indifferent between accepting and rejecting money provided that among the population of agents it is accepted with a specific critical probability. As we explain below, the MTE is a *pure strategy* equilibrium and it emerges only in specific regions of the parameter space.

21. In Lerner's (1969) study ofhyperinflation during the Civil War he notes that "As early as 1862 some Southern firms stopped selling their products for currency alone, and customers were forced to offer commodities as well as notes to buy things."

22. This is similar to the positive effect of inflation on trading effort and the consequences of search externalities first identified in the search-theoretic model of money by Li (1995).

23. The PBE and the PME converge in welfare terms as trade frictions vanish and there are no more limitation on consumption varieties (i.e., $\alpha \to \infty$, with $N \to \infty$ and $Z_0 \to \infty$).

REFERENCES

Aliprantis, Charalambos D., Gabriele Camera and Daniela Puzzello (2007) A random matching theory. *Games and Economic Behavior* 59, 1–16.

Blanchard, Olivier and Nobuhiro Kiyotaki (1987) Monopolistic competition and the effects of aggregate demand. *American Economic Review* 77, 647–666.

Camera, Gabriele and Dean Corbae (1999) Money and price dispersion. *International Economic Review* 40, 985–1008.

Casella, Allesandra and Jonathan Feinstein (1990) Economic exchange during hyperinflation. *Journal of Political Economy* 98, 1–27.

Diamond, Peter A. and Janet Yellin (1990) Inventories and money holdings in a search economy. *Econometrica* 58, 929–950.

Dixit, Avinash and Joseph Stiglitz (1977) Monopolistic competition and optimum product diversity. *American Economic Review* 67, 297–308.

Ericson, Richard and Barry Ickes (2001) A model of Russia's virtual economy. *Review of Economic Design* 6, 185–214.

Fuerst, Timothy S. (1992) Liquidity, loanable funds, and real activity. *Journal of Monetary Economics* 33, 3–24.

Howitt, Peter (2005) Beyond search: Fiat money in organized exchange. *International Economic Review* 46, 405–429.

Kiyotaki, Nobuhiro and Randall Wright (1989) On money as a medium of exchange. *Journal of Political Economy* 97, 927–954.

Kiyotaki, Nobuhiro and Randall Wright (1993) A search theoretic approach to monetary economics. *American Economic Review* 83, 63–77.

Kocherlakota, Narayana and Neil Wallace (1989) Incomplete record-keeping and optimal payment arrangements. *Journal of Economic Theory* 81, 272–289.

Lagos, Ricardo and Randall Wright (2005) A unified framework for monetary theory and policy analysis. *Journal of Political Economy* 113, 463–484.

Laing, Derek, Victor E. Li and Ping Wang (2007) Inflation and productive activity in a multiple-matching model of money. *Journal of Monetary Economics* 54, 1949–1961.

Lerner, Eugene M. (1969) Inflation in the Confederacy, 1861–1865. In Milton Friedman (ed.), *The Optimum Quantity of Money and Other Essays.* Chicago: Aldine.

Li, Victor E. (1995) The optimal taxation of fiat money in search equilibrium. *International Economic Review* 36, 927–942.

Li, Victor E. (2001) Is why we use money important? *Federal Reserve Bank of Atlanta Economic Review* (First Quarter), 17–30.

Molico, Miguel (2006) The distribution of money and prices in search equilibrium. *International Economic Review* 47, 701–722.

Rupert, Peter, Anderi Shevchenko, Martin Schindler and Randall Wright (2000) The search-theoretic approach to monetary economics: A primer. *Federal Reserve Bank of Cleveland Economic Review* 36, 10–28.

Shevchenko, Anderi (2004) Middlemen. *International Economic Review* 45, 1–24.

Shi, Shouyong (1997) A divisible search model of fiat money. *Econometrica* 65, 467–496.

Shubik, Martin (1973) Commodity money, oligopoly, credit and bankruptcy in a general equilibrium model. *Western Economic Journal* 11, 24–38.

Starr, Ross M. and Maxwell B. Stinchcombe (1999) Exchange in a network of trading posts. In *Markets, Information and Uncertainty: Essays in Economic Theory in Honor of Kenneth J. Arrow*, pp. 217–234. Cambridge, UK: Cambridge University Press.

Tallman, Ellis and Ping Wang (1995) Money demand and the relative price of capital goods in hyperinflations. *Journal of Monetary Economics* 36, 375–404.

Trejos, Alberto and Randall Wright (1995) Search, bargaining, money and prices. *Journal of Political Economy* 103, 118–141.

APPENDIX A: JUSTIFICATION OF ASSUMPTION (3a)

We demonstrate that a suitable choice of a Wicksell preference/production structure ensures that only household–firm trades arise in equilibrium, thus endogenizing the main features of Assumption (3a). For this purpose, assume that there are $J \geqslant 3$ separate classes of goods and types of households indexed $j = 1, 2, \ldots, J$. Within each class, normalize the measure of firms and households to unity. Assume that households in class j (i) consume only goods that belong to class $j+1$ (all modulo J) and (ii) possess the skills necessary for working in any firm in class j. Assume that the owners of firms in class j derive utility from either their own good or goods in class $j-1$. Under this schema, household–household trades do not arise in equilibrium. A household that owns inventory in class j desires goods in class $j+1$. However, households that work for firms in class $j+1$ desire goods in class $j+2$ and so on. Interfirm trades do not arise for similar reasons. However, household–firm trades may arise. A household in class j desires goods in $j+1$ and the owner of a firm that produces goods in class $j+1$ desires goods in $(j+1)-1 = j$.

APPENDIX B: PROOFS OF LEMMAS 1–3 AND THEOREMS 1–5

Consider the recurrence relation $V = U + \beta V_{+1}^{+}$ given in (8a), corresponding to a generic household $h \in H$ that is employed by some firm ω' (recall the shorthand, $V = V(k_t, m_t)$ and $V_{+1}^{+} = V(k_{t+1}, m_{t+1})$). To ease the notational burden, we eschew writing $Z(h)$ and so on, preferring the shorter Z.

Assume that $s > 0$ and that $g > 0$ (if $sg = 0$ the problem is trivial). According to equation (7), $c(\omega) = c(\omega)_b + c(\omega)_m$. Consequently, we can write the following constraints for each good in the double-coincidence set Z_{B}:

$$c(\omega) - c(\omega)_b \geqslant 0 \quad \text{and} \quad c(\omega)_b \geqslant 0, \quad \text{for all } \omega \in Z_{\text{B}}, \qquad [\mu(\omega)_{\text{B}}], \tag{B.1}$$

where $\mu(\omega)_{\text{B}}$ is the (nonnegative) Lagrange multiplier associated with $c(\omega) - c(\omega)_b \geqslant 0$ [i.e., $c(\omega)_m \geq 0$] for each $\omega \in Z_{\text{B}}$ [$c(\omega)_b = 0$ for $\omega \in Z_{\text{M}}$, as barter is infeasible]. The first-order conditions with respect to $\{c(\omega)|_{\omega \in Z_{\text{M}}}, c(\omega)|_{\omega \in Z_{\text{B}}}, c(\omega)_b\}$ and the Benveniste–Scheinkman conditions with respect to $\{k, m\}$ are

$$\Lambda c^{-1/\gamma} = p(\omega)(1+\mu)^{-1}\beta V_m, \quad \omega \in Z_{\text{M}}, \tag{B.2a}$$

$$\Lambda c^{-1/\gamma} = p(\omega)(1+\mu)^{-1}\beta V_m - \mu_{\text{B}}, \quad \omega \in Z_{\text{B}}, \tag{B.2b}$$

$$\left.\begin{array}{l} -\beta(1-\delta)V_k r(\omega,\omega') + p(\omega)(1+\mu)^{-1}\beta V_{\text{M}} - \mu_{\text{B}} \leqslant 0 \\ c_b \geqslant 0 \end{array}\right\} \text{Comp.}, \omega \in Z_{\text{B}} \tag{B.2c}$$

$$k \cdot V_k[1 - \beta(1-\delta)] = 0, \tag{B.2d}$$

$$m \cdot V_m[1 - (1+\mu)^{-1}] = 0, \tag{B.2e}$$

where $\Lambda = [\partial U/\partial D]D^{1/(\gamma-1)}$ and "Comp." refers to a complementary slackness condition.

LEMMA 1

By assumption $\beta < 1$, $(1-\delta) \leqslant 1$, and $(1+\mu) \geqslant 1$. It is then immediate from the complementary slackness conditions (B.2d) and (B.2e) $k = m = 0$. This follows as $V_k > 0$ and $V_m > 0$, from (B.2a)–(B.2c), establishing Lemma 1.

LEMMA 2

(a) This part is trivial: consumers desire and can purchase only those goods that belong to the set Z.

The demand functions for each of the goods belonging to Z are determined from the first-order conditions presented above, together (in view of Lemma 1) with the stationary inventory and cash evolution equations (3.2b) and (3.2c). A number of cases must be considered, in view of the complementary slackness condition (B.2c).

For example, suppose that $\mu(\omega)_{\text{B}} > 0$ for all $\omega \in Z_{\text{B}}$. From complementary slackness, $c(\omega) = c(\omega)_b > 0$ for all such ω. Yet it is then immediate that (B.2c) must hold with equality for all $\omega \in Z_{\text{B}}$ and that (B.2b) can be written

$$\Lambda c^{-1/\gamma} = r(\omega,\omega')(1-\delta)\beta V_k, \quad \omega \in Z_{\text{B}}. \tag{B.2b$'$}$$

Consider some good $\omega \in Z_{\mathrm{M}}$ and another generic good $u \in Z_{\mathrm{M}}$. According to equation (B.2a), we have

$$\left[\frac{c(\omega)}{c(u)}\right]^{-1/\gamma} = \frac{p(\omega)}{p(u)}.$$

Rearranging gives

$$c(\omega)p(u)^{1-\gamma} = p(\omega)^{-\gamma}[p(u)c(u)].$$

Integrating both sides over Z_{M} (with respect to u) and using the cash-evolution equation (3.2b) that $g = \int_{Z_{\mathrm{M}}} p(u)c(u)du$ gives (11). Similar manipulations, applied to equation (B.2b′), imply (10).

Great simplicity is afforded by considering the consumer's demand for a generic product $\omega \in Z$ at the prices $p = p(\omega)$ and $r = r(\omega, \omega')$ when a.e. other producers set prices $\widetilde{p} = \widetilde{p}(u)$ and $\widetilde{r}=\widetilde{r}(u, \omega')$ for $u \in \hat{Z}_{\mathrm{B}}$ (and zero otherwise).

(b) To begin, we note that $\sigma[Z_{\mathrm{B}}] = \alpha x^2$ and $\sigma[Z_{\mathrm{M}}] = \alpha x(1-x)$. There are two cases to consider.

(b1) Let $(g/\widetilde{p}) \geqslant (s/\widetilde{r})[(1-x)/x]$. If $\mu_{\mathrm{B}} > 0$, then $c(\omega) = c(\omega)_b)$ from complementary slackness. Also, (B.2a) and (B.2b) give $c(\omega_1) < c(\omega_2)$, where $\omega_1 \in Z_{\mathrm{M}}$ and $\omega_2 \in Z_{\mathrm{B}}$. Constraints (8b) and (8c) imply $c_b = (s/\widetilde{r})/(\alpha x^2) > c_m = (g/\widetilde{p})/[\alpha x(1-x)]$. This contradicts (a). It follows that $\mu_{\mathrm{B}} = 0$ and that $c = c(\omega),\ \forall \omega \in Z_{\mathrm{M}} \cup Z_{\mathrm{B}}$. Notice that in this case household income equals the integral over the associated consumption basket and yields a measure of $\sigma[Z_{\mathrm{M}} \cup Z_{\mathrm{B}}] = \alpha x^2 + \alpha x(1-x) = \alpha x$. Constraints (8b) and (8c) give $c_b = (s/\widetilde{r})/\alpha x^2$ and $g/\widetilde{p} = \alpha x c - \alpha x^2 c_b$. In turn this yields $c = \{(g/\widetilde{p}) + (s/\widetilde{r})\}/(\alpha x) \geqslant c_b$.

(b2) Let, $(g/\widetilde{p}) < (s/\widetilde{r})[(1-x)/x]$. If $\mu_{\mathrm{B}} = 0$, then the previous argument gives $\{(g/\widetilde{p}) + (s/\widetilde{r})\}/(\alpha x) \geqslant c_b = (s/\widetilde{r})/(\alpha x^2)$ a contradiction. So $\mu_{\mathrm{B}} > 0$, implying that $c(\omega) = c_b,\ \forall \omega \in Z_{\mathrm{B}}$. The constraints (8b) and (8c) then yield $c(\omega_1) = (g/\widetilde{p})/(\alpha x(1-x)) < c(\omega_2) = (s/\widetilde{r})/(\alpha x^2),\ \forall \omega_1 \in Z_{\mathrm{M}}$ and $\omega_2 \in Z_{\mathrm{B}}$.

(c) For this part, consider the good $\omega \in Z$ (with prices $q = (p, r)$), and any other generic good $u \in Z \setminus \{\omega\}$ (with prices $\widetilde{q}= (\widetilde{p},\widetilde{r})$).

From (B.2a) and from (B.2b),

$$[c(\omega)/c(u)]^{-(1/\gamma)} = (\widetilde{p}/p), \text{ where } \omega \in Z_{\mathrm{M}}, \text{ and either } u \in Z_{\mathrm{M}} \text{ or } u \in Z_{\mathrm{B}} \text{ and } c(u)_b = 0 \qquad \textbf{(B.3a)}$$

$$[c(\omega)/c(u)]^{-(1/\gamma)} = (\widetilde{r}/r), \text{ where } \omega \in Z_{\mathrm{B}}, \text{ and either } u \in Z_{\mathrm{B}} \text{ or } u \in Z_{\mathrm{M}} \text{ and } c(u)_b > 0. \qquad \textbf{(B.3b)}$$

The demand functions reported in part (c2) of the lemma are derived as follows. First consider $g/\widetilde{p} < [(1-x)/x](s/\widetilde{r})$. The argument used to prove part (b) implies that $\mu_{\mathrm{B}} > 0$ and $c = c_b = (s/\widetilde{r})/(\alpha x^2)\ (\forall u \in Z_{\mathrm{B}}) > c_m = (g/\widetilde{p})/[\alpha x(1-x)]\ (\forall u \in Z_{\mathrm{M}})$. Equations (B.3a) and (B.3b) give

$$c(\omega)_b = c(\omega) = c_b(\widetilde{r}/r)^{\gamma}, \qquad \text{if } c_b > 0, \qquad \textbf{(B.4a)}$$

$$c(\omega)_m = c(\omega) = c_m(\widetilde{p}/p)^{\gamma}, \qquad \text{otherwise.} \qquad \textbf{(B.4b)}$$

Also, because $\mu(\omega)_{\mathrm{B}} \geqslant 0$, $c(\omega)_b \geqslant c(\omega)_m$. The R.H.S of (B.4a) is decreasing in r. Recall that $\hat{\rho} = \{[(1-x)/x](s/\widetilde{r})(\widetilde{p}/g)\}^{\gamma}$. Thus, using (B.4a), (B.4b) in conjunction with Lemma 3 gives

$$c(\omega) = (s/\alpha\widetilde{r}x^2)(\widetilde{r}/r)^{\gamma}, \quad \text{if } r \leqslant \hat{\rho}(\widetilde{r}/\widetilde{p}) \text{ and } \omega \in Z_{\mathrm{B}} \qquad \textbf{(B.5a)}$$

$$c(\omega) = [g/\alpha\widetilde{p}x(1-x)](\widetilde{p}/p)^{\gamma}, \quad \text{if } \omega \in Z_{\text{M}} \text{ or if } r > \hat{\rho}(\widetilde{r}/\widetilde{p}) \text{ and } \omega \in Z_{\text{B}}. \tag{B.5b}$$

Equations (B.5a) and (B.5b) are compactly written by defining the indicator function χ_{B} as is done in the lemma. Case (c1) follows analogously. Finally, part (c3) follows from the complementary slackness condition reported in (B.2c). If $s = 0$, the consumer's demand functions are derived directly from (8b), (8c), and (B.2a) with $c_b = 0$. Likewise, if $g = 0$, then (8b), (8c), and (B.2b) are used.

THEOREM 1 (The PBE)

Given the stationary values $(\widetilde{s},\widetilde{r})$, equations (18) in the text uniquely define the representative firm's best response behavior. Because $f(\ell)$ is strictly concave, there is a unique value s^* at which point $(1-\delta)\beta f'(\ell^*) = s^*$ and $\ell^* = 1$. If $\hat{C} > 0$, then complementary slackness gives, $r^* = \gamma/(\gamma-1)$ as the unique best response for r. However, under Condition U, $\hat{C} = 0$ is impossible. This follows as $\{\gamma + (1-\delta)(\gamma-1)\}/\gamma \leqslant 1$, and the strict concavity of $f(\cdot)$ implies that

$$\hat{C}/\ell^* = f(\ell^*) - f'(\ell^*)\{\gamma + (1-\delta)(\gamma-1)\}/\gamma \geqslant f(\ell^*)/\ell^* - f'(\ell^*) > 0,$$

establishing the uniqueness of s^*, r^*, ℓ^*. Equation (18e) implies that $\mu_{\text{B}} > 0$. Hence, from complementary slackness and (17d), $k^* = s^*\ell^*$. Finally, Lemma 1 gives consumers' equilibrium demands as $c(\omega)^* = (s^*/r^*)/(\alpha x^2)$ for all $\omega \in Z_{\text{B}}$.

THEOREMS 2 AND 3 (Monetary Exchange)

Let $\Delta = (1-\delta)(1+\mu) < 1$. We first establish that with $\Delta < 1$ there is a stationary PME and characterize its properties [parts (A) and (B) of Theorem 3]. Given that $s^* = 0$, the basic proof of the uniqueness of the symmetric steady-state equilibrium is virtually the same as that used in Theorem 1, once obvious adjustments are made to the first-order conditions, analogous to (10) reported in the text. The only caveat is that we must prove that it is not optimal for a firm to defect from the proposed equilibrium and to offer workers $s > 0$, contrary to the theorem. Let $(s^*, r^*, c^*, g^*, p^*, \ell^*)$ be the values reported in parts (A) and (B) of Theorem 3. In the proposed equilibrium the firm is assured a periodic utility,

$$\hat{C}^* = f(\ell^*) - (g^*/p^*)\ell^* > 0. \tag{B.6}$$

Consider an arbitrary firm, ω, that sets $q = (p, r)$ and offers $s > 0$ (if $s = 0$, there is nothing to prove). Let φ, μ_B, and μ_M be the Lagrangian multipliers associated with (23d)–(23f), respectively. The first-order conditions for the firm's problem with respect to $\{\hat{C}, \ell, g, s, p\}$, evaluated in the steady state, are easily derived with the aid of the recurrence relation $\hat{V} = c + \beta\hat{V}^+_{+1}$ and the results that $\mu_B = \hat{V}_k$ and $\mu_M = \hat{V}_m$,

$$\hat{V}_k = 1/\beta(1-\delta),$$

$$f'(\ell) = s\hat{V}_k + g\hat{V}_m,$$

$$-\hat{V}_m + \varphi/p^* = 0,$$

$$\hat{V}_k + \varphi/r^* = 0 \ (s > 0),$$
$$\gamma/p^* = (\gamma - 1)\beta(1-\delta)\hat{V}_m/\Delta,$$

where $\hat{V}_j = dV/dj,\ j = k, m$. Simple manipulation of these conditions, denoting $r^* = \gamma/(\gamma - 1)$ and $\varphi > 0$, gives

$$p/p^* = \Delta < 1,$$
$$\beta(1-\delta)f'(\ell)/r^* = g^*/p^*,$$
$$g/p^* + s/r^* = (g^*/p^*).$$

However, because $\beta(1-\delta)f'(\ell^*)/(r^*\Delta) = g^*/p^*$, it implies that $\ell^* \geq \ell$, as $\Delta < 1$ and $f(\cdot)$ is strictly concave. Under the proposed defection, the firm's steady-state periodic payoff is derived from $k = s\ell = (1-\delta)\{f(\ell) - \hat{C} - \alpha x\ell^* c_m\}$ and $c_m = c^*(p^*/\widetilde{p})^\gamma$ as

$$\begin{aligned}\hat{C} &= f(\ell) - [s\ell/(1-\delta)] - \alpha x\ell^* c^*(p^*/\widetilde{p})^\gamma \\ &= f(\ell) - [s\ell/(1-\delta)] - \ell^*(g^*/p^*)\Delta^{-\gamma},\end{aligned} \tag{B.7}$$

where (B.7) follows, as $p^*/\widetilde{p} = \Delta$ and $c^* = (1/\alpha x)(g^*/p^*)$. Finally, comparing (B.7) and the steady-state payoff (B.6) gives

$$\hat{C}^* - \hat{C} = \{f(\ell^*) - f(\ell)\} + \ell^*(g^*/p^*)\{\Delta^{-\gamma} - 1\} + s/(1-\delta) > 0. \tag{B.8}$$

The inequality in (B.8) follows because $\ell^* \geq \ell$, $(1/\Delta)^\gamma \geq 1$, and $s > 0$. This establishes that it is strictly suboptimal for the firm to defect from the proposed equilibrium and to offer $s > 0$, given that $r^* = \gamma/(\gamma - 1)$.

Next, given employment per firm of one, prices $\widetilde{q} = (\widetilde{p}, \widetilde{r})$, and labor contracts $\widetilde{\nu} = (\widetilde{g}, \widetilde{s})$, the owner of firm ω solves the program

$$\text{(P)} \qquad \hat{V}(\hat{k}, \hat{m}) = \max[\hat{C} + x^2 r c_b + \beta\hat{V}(\hat{k}^+, \hat{m}^+)],$$
$$\text{s.t. } m'(1+\mu) = [\hat{m} + \mu M_0 + \alpha x p[(1-x)c(\omega_1)_m + xc(\omega_2)_m - g\ell],$$
$$\forall \omega_1 \in \hat{Z}_M,\ \omega_2 \in \hat{Z}_B,$$
$$\hat{k}' = (1-\delta)[\hat{k} + f(\ell) - s\ell - \alpha x[(1-x)c(\omega_1) + xc(\omega_2)] - \hat{C}],$$
$$\forall \omega_1 \in \hat{Z}_M,\ \omega_2 \in \hat{Z}_B,$$
$$U[D] \geq (1-\beta)\hat{V},$$
$$\hat{k} \geq s\ell,$$
$$\hat{m} \geq g\ell.$$

The basic strategy of proof is virtually identical to that used above. The only caveat is that—as indicated by Lemma 2(C)—there is a discontinuity in the means used by consumers to finance their purchases from firm ω. This must be dealt with before the firm's best-response function is derived. For this purpose we introduce a convexification that avoids the discontinuity and ensures that households and firms accrue payoffs at least as great as without it. Call this extended program (P*). We show that the solution of the extended program (P*) is also a solution of (P). Consider firm ω and recall that

$\hat{Z}_B$ is the set of the firm's customers that satisfy the double coincidence of wants. The convexification takes the following form. We assume that the firm assigns to each member $\omega'' \in Z_B$ the indicator $I(\omega, \omega'') \in \{0, 1\}$. If $I = 0$ the household must finance all their purchases using goods, whereas if $I = 1$ the customer must use money. The firm chooses $\theta(\omega) = \Pr[I(\omega, \omega'') = 0 | \omega'' \in Z_B]$. Under this scheme, the firm's receipts are continuous in its prices q. The firm's demand functions are given by the following conditions:

(a) If $(g/\widetilde{p})[(1-x)/x](s/\widetilde{r})$, then

$$c(\omega)_b = (1/\alpha x)\{(g/\widetilde{p}) + (s/\widetilde{r})\}(\widetilde{r}/r)^{\gamma} \quad \text{if } \omega \in Z_B \text{ and } I(\omega, \omega'') = 0, \tag{B.9a}$$

$$c(\omega)_m = (1/\alpha x)\{(g/\widetilde{p}) + (s/\widetilde{r})\}(\widetilde{p}/p)^{\gamma} \quad \text{otherwise.} \tag{B.9b}$$

(b) If $(g/\widetilde{p}) < [(1-x)/x](s/\widetilde{r})$, then

$$c(\omega)_b = (1/\alpha x^2)\{s/\widetilde{r}\}(\widetilde{r}/r)^{\gamma}, \quad \text{if } \omega \in Z_B \text{ and } I(\omega, \omega'') = 0, \tag{B.9c}$$

$$c(\omega)_m = [1/\alpha x(1-x)]\{(g/\widetilde{p})\}(\widetilde{p}/p)^{\gamma}, \quad \text{otherwise.} \tag{B.9d}$$

Under the convexification, the firm solves

$$(\mathrm{P}^*) \qquad \hat{V}(\hat{k}, \hat{m}) = \max_{\{\theta, \hat{C}, \nu, q\}} [\hat{C} + \alpha x^2 r \theta c_b + \hat{V}(\hat{k}^+, \hat{m}^+)],$$

$$\text{s.t. } \hat{m}^+(1+\mu) = [\hat{m} + \mu M_0 + \alpha x p[(1 - x\theta) c_m - g\ell], \tag{B.10a}$$

$$\hat{k}^+ = (1-\delta)[\hat{k} + f(\ell) - s\ell - \alpha x[(1-\theta x)c_m + \theta x c_b] - \hat{C}], \tag{B.10b}$$

$$U[D] \geq (1-\beta)^{-1} V_0, \tag{B.10c}$$

$$\hat{k} \geq s\ell, \tag{B.10d}$$

$$\hat{m} \geq g\ell, \tag{B.10e}$$

$$1 \geq \theta \geq 0, \tag{B.10f}$$

where c_m and c_b are given by equations (B.9), φ, μ_B, μ_M are (nonnegative) Lagrange multipliers on the constraints (B.10c)–(B.10e), and μ_θ is a Lagrange multiplier on the constraint $1 \geq \theta$, ensuring that the mixing probability cannot exceed unity. To show that P* implements P, let $(g/\widetilde{p})[(1-x)/x](s/\widetilde{r})$. In Program (P), we consider three cases.

(a) $r < \widetilde{r}(p/\widetilde{p})$: then $c(\omega) = (1/\alpha x)\{(g/\widetilde{p}) + (s/\widetilde{r})\}(\widetilde{p}/p)^{\gamma}$ *for* $\omega \in Z_M$ and $c(\omega) = c(\omega)_b = \{(g/\widetilde{p}) + (s/\widetilde{r})\}(\widetilde{r}/r)^{\gamma}$ for $\omega \in Z_B$. Program (P) gives

$$\hat{V}(\hat{k}, \hat{m}) = [\hat{C} + \alpha x^2 r c + \hat{V}(\hat{k}^+, \hat{m}^+)], \tag{B.11a}$$

$$\hat{m}^+(1+\mu) = [\hat{m} + \mu M_0 + \alpha x p[(1-x)c_m - g\ell], \tag{B.11b}$$

$$\hat{k}^+ = (1-\delta)[\hat{k} + f(\ell) - s\ell - \alpha x[(1-x)c(\omega_1) + x c(\omega_2)] - \hat{C}], \omega \in \hat{Z}_M, \omega_2 \in \hat{Z}_B. \tag{B.11c}$$

In Program (P*), let $\theta = 1$ and the same set of conditions can be obtained. This shows that $(P^*) \Longrightarrow (P)$.

(b) $r > \widetilde{r}(p/\widetilde{p})$: then in (P) $c(\omega) = c_m = (1/\alpha x)\{(g/\widetilde{p}) + (s/\widetilde{r})\}(\widetilde{p}/p)^{\gamma} > c_b = 0$, $\forall\omega \in Z_B \cup Z_M$. In this case,

$$\hat{V}(\hat{k}, \hat{m}) = [\hat{C} + \beta\hat{V}(\hat{k}^+, \hat{m}^+)],$$

$$\hat{m}^+(1+\mu) = [\hat{m} + \mu M_0 + xpc_m - g\ell],$$

$$\hat{k}^+ = (1-\delta)[\hat{k} + f(\ell) - s\ell - \alpha x c_m] - \hat{C}].$$

In (P*), setting $\theta = 0$ produces the same set of conditions.

(c) $r = \widetilde{r}(p/\widetilde{p})$: then in (P) $c(\omega) = (1/\alpha x)\{(g/\widetilde{p}) + (s/\widetilde{r})\}(\widetilde{p}/p)^{\gamma} \geq c_b$. In (P*) setting $\theta = \{(1/x)(s/\widetilde{r})\}/\{(g/\widetilde{p}) + (s/\widetilde{r})\} \leq 1$ implements (P). Similar arguments for $(g/\widetilde{p}) < [(1-x)/x](s/\widetilde{r})$ establish that (P*) $\Longrightarrow$ (P). The first-order conditions for (P*) with respect to $\{\hat{C}, \ell, r, p, \theta, g, s\}$ and the Benveniste–Scheinkman conditions with respect to $\{\hat{m}, \hat{k}\}$ are readily derived:

$$1 - \hat{V}_k\beta(1-\delta) \leq 0, \quad \hat{C} \geq 0, \quad [1 - \hat{V}\beta(1-\delta)]\hat{C} = 0, \tag{B.12a}$$

$$(1-\delta)\beta\hat{V} f(\ell)' = s\hat{V}_k + g\hat{V}_m, \tag{B.12b}$$

$$\alpha x^2\theta c_b\left\{\frac{\gamma(1-\delta)\beta\hat{V}_k}{r} - (\gamma-1)\right\} = 0, \tag{B.12c}$$

$$\alpha x(1-\theta x)\beta V_M(1+\mu)^{-1}\{\Delta[(\hat{V}_k/\hat{V}_M)(\gamma/p)] - (\gamma-1)\} = 0, \tag{B.12d}$$

$$\left.\begin{array}{l}\alpha x^2[c_b\{r - \beta(1-\delta)\hat{V}_k\} + \beta(1-\delta)c_m\{\hat{V}_k - \hat{V}_m p/\Delta\} - \mu_\theta] \leq 0\\ \theta \geq 0\end{array}\right\}\ \text{Comp.,} \tag{B.12e}$$

$$-\hat{V}_m g + \varphi\Lambda D_g \leq 0, \quad \text{and} \quad g \geq 0, \tag{B.12f}$$

$$-\hat{V}_k s + \varphi\Lambda D_s = 0, \quad \text{and} \quad s \geq 0, \tag{B.12g}$$

$$\hat{m} = g\ell = \alpha x(1-x)pc_m, \tag{B.12h}$$

$$\hat{k} = s\ell = (1-\delta)\{\hat{k} + f(\ell) - s\ell - \alpha x[(1-x)c_m + xc_b] - \hat{C}\}, \tag{B.12i}$$

where $\Lambda = [\partial U/\partial D](D)^{1/(\gamma-1)}$, $D_g = \partial D/\partial g$, and $D_s = \partial D/\partial s$. There are three subcases to consider.

(i) Let $\Delta < 1$. Condition U implies that $\hat{C} > 0$, in which case $\hat{V}_k = 1/[\beta(1-\delta)]$. Assume that $s^* > 0$ and that $(g^*/p^*) > [(1-x)/x](s^*/r^*)$ [if $s^* = 0$, there is nothing to prove]. Equations (B.12d) and (B.12f)–(B.12g) yield, respectively, $\Delta\hat{V}_k/\hat{V}_m = (p^*/r^*) = \hat{V}_k/\hat{V}_m$, which is a contradiction. Now suppose that $0 < (g^*/p^*) < [(1-x)/x](s^*/r^*)$. In this case (B.12f)–(B.12g) give $\hat{V}_k/\hat{V}_m = (D_g/D_s) = \{[(1-x)p^*s^*]/(r^*g^*x)\}(r^*/p^*)$. Equations (B.12c) and (B.12d) imply that $(r^*/p^*)\Delta = V_M/V_k$. It follows that $1 > \Delta = \{[(1-x)p^*s^*]/(r^*g^*x)\}^{1/\gamma} > 1$, which is a contradiction. Consequently, whenever $\Delta < 1$, $s^* = 0$ is the only candidate equilibrium. Part (a) of the proof establishes the existence of the PME under these circumstances. Thus (P^*) implements (P) with $s = \widetilde{s} = s^* = 0$ and $\theta'' = 0$.

(ii) Let $\Delta > 1$. In any putative symmetric steady-state equilibrium $\ell = \ell^* = 1, r = \widetilde{r} = r^*$, etc. Suppose that $\Delta > 1$, that Condition U is satisfied, and that contrary to claim $\theta'' < 1$ and $\hat{C}^* > 0$. Then $\mu_\theta = 0$ from complementary slackness. Also, using (B.12c) and (B.12d) in (B.12e) gives

$$(c_b^* - c_m^*)(r^* - 1) \leq 0,$$

which implies that $c_b^* \leq c_m^*$ as $r^* > 1$. Thus,

$$[(g^* r^*)/(s^* p^*)]\{1 - x\}/(x).$$

Also, (B.12f) and (B.12g) give

$$(g^*/s^*)\beta(1 - \delta)\hat{V}_m = (1 - x)/(x).$$

Hence,

$$[(g^* r^*)/(s^* p^*)]\Delta = \{1 - x\}/(x) \leq [(g^* r^*)/(s^* p^*)],$$

which is a contradiction, as $\Delta > 1$. Thus, $\theta'' < 1$ and $\hat{C}^* > 0$ is not optimal. Tedious manipulation of the first-order conditions shows that under Condition U, $\hat{C}^* > 0$. Thus, in any putative equilibrium, $\theta'' = 1$ and $\hat{C}^* > 0$. It is straightforward to verify that the expressions reported in Theorem 3 are the unique solutions to the optimality conditions (B.12) for (P*). This is also a solution to Program (P) at $r = (\widetilde{r}/\widetilde{p})p$ and $s/\widetilde{r} \geq (g/\widetilde{p})[x/(1 - x)]$.

(iii) Let $\Delta = 1$. In this case, the arguments used in part (a) of the proof show that $s^* = 0$ uniquely defines a stationary symmetric PME.

THEOREMS 4 AND 5 (Convergence and Welfare)

Theorem 4 is a direct consequence of Theorems 1 and 3, noting that $\lim_{\mu \to \infty} \Delta^{\gamma - 1} = 0$, because $\Delta > 0$ and $\gamma - 1 > 0$. Theorem 5 follows from Theorem 1 and 3 and equation (18).