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A General Two-Sector Model of Endogenous Growth with Human and Physical Capital: Balanced Growth and Transitional Dynamics*

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We examine a two-sector endogenous growth model with general constant-return-to-scale production technologies governing the evolution of human and physical capital. We prove the existence, uniqueness, and saddle-path stability of the balanced growth equilibrium. A dual approach drawing on techniques from international trade theory is used to provide complete characterization of the transitional dynamics of consumption, goods and education outputs, human and physical capital inputs, and the relative price of human capital investment. We investigate the long-run effects of changes in time preference and factor taxation, and show the emergence of instability or indeterminacy when factor taxes are too distortionary. *Journal of Economic Literature* Classification Number: D90.

1. Introduction

Recently, an increasing number of economists, led by Romer [18], Lucas [14], Rebelo [17], and Stokey [22], have presented models of economic growth in which the rate of economic growth is an endogenously

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determined variable. This new growth literature has analyzed the determinants of the rate of growth along a balanced growth path, in order to provide explanations of the diverse economic development experiences that have been observed. Due to the nonstationary nature of the problem, however, its transitional dynamics is generally unexplored. In this paper, we completely characterize the balanced growth path and the transitional dynamics of a two-sector model of endogenous growth with human and physical capital. We analyze a model in which endogenous growth arises from the fact that the two-sector technology exhibits constant returns to scale in the reproducible inputs. This model serves as a useful first step for understanding the transitional dynamics of two-sector models of endogenous growth and serves as a useful benchmark against which other models which include elements of increasing returns to scale can be compared. Moreover, our analysis enables us to investigate parameter perturbations as well as to evaluate factor tax and subsidy policies both in the long-run balanced growth equilibrium and along the short-run transition path.

We analyze a two-sector model of endogenous growth, consisting of a goods sector and an education sector, in which each sector produces the respective type of capital (i.e., physical or human) under conditions of constant returns to scale. The static version of the two-sector production model we consider has a long tradition in the theory of international trade and was introduced into the exogenous growth literature by Uzawa [24, 25]. The analysis of the static two-sector production model is substantially simplified by the fact that it has a block recursive structure (i.e., the factor returns can be determined as a function of output prices alone, independently of factor supplies, as in the *factor price equalization* property of trade models), under the assumptions of perfect competition in goods and factor markets and the equalization of factor rewards across sectors. Jones [11] illustrates how this structure leads to "magnification effects": an increase in the price of one sector's output leads to a more than proportional increase in the price of the factor used intensively in that sector (the *Stolper–Samuelson theorem*) and an increase in the quantity of a factor leads to a more than proportional expansion of the output of the good that uses that factor intensively (the *Rybczynski theorem*). We show that the two-sector endogenous growth model can be analyzed using these properties of static two-sector models, combined with an intertemporal no-arbitrage condition which requires an equality between the net rates of return on physical and human capital.[1] This approach allows us to obtain conditions under which there exists a unique balanced growth equilibrium with a positive common

[1] We are thus accounting for the "capital gain" on each reproducible capital stock as described in Shell et al. [20] within the exogenous growth framework.

growth rate, and also highlights the role played by the relative factor intensities of the two production sectors in determining the transitional dynamics of the system. This can be seen most clearly by taking a dual approach that focuses on the relative price of the education/learning output (in units of goods), which is the ratio of the respective costate variables in the dynamic optimization problem and will equal the value of an increment of human capital. The intertemporal no-arbitrage condition, which links the capital gain on human capital to the difference between the rentals on capital and the wage rate, thus becomes a differential equation determining the relative price of human capital investment and its dynamic adjustment in the neighborhood of the balanced growth equilibrium. The role of factor intensities in the transitional dynamics arises because of the *Stolper–Samuelson* effect. If the education sector is labor intensive, an increase in the price of human capital raises the wage rate and makes human capital investment more attractive, so the capital gain on human capital will fall to restore equality of rates of return to the two factors, implying a stable price adjustment process. If the education sector is capital intensive, however, the increase in the price of educational output will reduce the wage rate, requiring an increase in the capital gain on human capital.

We show that the model will exhibit saddle-path stability for either factor intensity ranking of the sectors. In the case where the education sector is capital intensive, we show that the price of education output must jump immediately to its steady-state value. The output adjustment process (at fixed prices) is stable in this case and results in convergence to the balanced growth path capital/labor ratio. In contrast, the output adjustment process at fixed prices is unstable for the case in which the education sector is labor intensive because of the *Rybczynski effect*: an increase in the relative supply of human capital results in an increase in the rate of human capital accumulation and a decrease in the output of the goods sector.[2] In this case, saddle-path stability results from adjustments in the price of education output along the transition path. For each factor intensity ranking, an unstable adjustment process in prices or quantities is offset by adjustment in the other variable and the system is saddle-path stable. The relative growth rates of consumption and factor supplies along the transition path will differ between the two cases, however. We also show that the stability of this adjustment process may be upset by factor taxation if the factor taxes are too distortionary.

[2] There is an interesting comparison between the instability of the quantity adjustment process described in this paper and that obtained in the two-sector growth model of Uzawa [24, 25] where only capital accumulation is endogenous. In the Uzawa model, instability may arise when the investment good is capital intensive relative to the consumption good. This is analogous to our case in which the education sector is labor intensive, since each factor is used intensively in its own production.

The intertemporal no-arbitrage condition is also useful for analyzing the effects of factor taxation, since it leads to a natural decomposition of the effects of factor taxation into static effects (the location of existing factor supplies across sectors) and dynamic effects (whether to invest in physical or human capital). We consider three types of factor taxes: taxes on physical capital, taxes on human capital in the goods sector, and subsidies to human capital in the education sector. We show that each type of factor tax will reduce the rate of growth while the subsidy will increase it, regardless of the sectoral factor intensities. The effects of factor taxation on the capital/labor ratio in the balanced growth path is, however, dependent on the factor intensity ranking.

Our approach differs from that of other recent work in this area, which has generally approached the problem by using simulation with specific functional forms. For example, King and Rebelo [12] investigate the two-sector neoclassical exogenous growth model with Cobb–Douglas production functions, Mulligan and Sala-i-Martin [15] calibrate a two-sector endogenous growth model with externalities using Cobb–Douglas production technologies, and Xie [26] and Benhabib and Perli [1] have used explicit functional forms to analyze the Lucas [14] framework.[3] Our results are thus more general than those using explicit functional forms to analyze endogenous growth without externalities, and serve as a benchmark against which to compare results from models with externalities. Note that the approach taken in this paper could also be extended to models with externalities.

Section 2 of the paper presents a general two-sector endogenous growth model and proves the existence and uniqueness of the balanced growth equilibrium. Section 3 shows that the system will be saddle-path stable and characterizes the transitional dynamics in the neighborhood of the balanced growth path. Section 4 provides applications of the model illustrating the effects of parameter and policy changes on the balanced growth equilibrium and the transition path. Section 5 concludes the paper.

2. A Two-Sector Model of Endogenous Growth

In this section we present a two-sector model of endogenous growth in which there are two reproducible factors of production, physical capital (K) and human capital (H). We then establish conditions under which

[3] In particular, Benhabib and Perli [1] show that in the Lucas [14] model with human capital externalities, one may obtain a saddle-path stable, balanced growth equilibrium if the subjective rate of time preference is greater than the maximal rate of human capital accumulation and the intertemporal elasticity of substitution is sufficiently high.

there exists a unique balanced growth path in which consumption and factor stocks grow at a common (constant) rate, and in which relative prices are constant.

2.1. *The Model*

The two sectors will be referred to as the goods sector, X, and the education sector, Y. The goods are produced according to a constant returns to scale technology $X = F(sK, uH) = uHf(k_x)$, where s and u are the shares of physical and human capital, respectively, allocated to the goods sector and $k_x \equiv (sK)/(uH)$ is the capital/labor ratio in the goods sector. Goods may be either consumed, C, or added to the capital stock. The evolution of the stock of physical capital is thus given by

$$\dot{K} = uHf(k_x) - \delta K - C, \tag{1}$$

where δ is the rate of depreciation of physical capital. In the education sector, Y, the production function is also assumed to exhibit constant returns to scale, with $Y = G((1-s)K, (1-u)H) = (1-u)Hg(k_y)$ and $k_y \equiv [(1-s)K]/[(1-u)H]$. Note that the educational process is assumed to require both physical capital (libraries, laboratories, etc.) and human capital (faculty and students). The output of the education sector raises the supply of effective labor units by adding to the stock of human capital,

$$\dot{H} = (1-u)Hg(k_y) - \eta H, \tag{2}$$

where η is the rate of depreciation of human capital. The output per unit labor functions, f and g, are assumed to be strictly increasing and strictly concave. Both products and factors are allowed to be disposed freely. This production structure can be thought of as a generalization of King and Rebelo [13] and Rebelo [17].

A representative agent's optimization problem is given as

$$\max_{C, s, u, K, H} \Omega(C) = \int_0^\infty \frac{C(t)^{1-\sigma}}{1-\sigma} e^{-\rho t}\, dt \tag{P}$$

subject to (1), (2), $H(0) = H_0 > 0$, and $K(0) = K_0 > 0$, where ρ is a constant subjective rate of time preference, and $\sigma \in (0, 1) \cup (1, \infty)$ is the inverse of the constant intertemporal elasticity of substitution. The utility functional form is chosen because it can yield a balanced growth path under some standard regularity conditions.[4]

[4] We exclude the case of log-linear utility to ensure homogeneity property of the utility function, which will simplify the analysis of the dynamics of the system below. For similar treatment, see Huffman [8].

A basic assumption that will be required is:

Condition M (*Maximal Growth*). The maximal attainable rate of consumption growth, $v_{\max}$, satisfies $\rho > (1-\sigma)\, v_{\max}$.

Condition M is similar to that in Brock and Gale [4], which imposes an upper bound on the maximal growth rate of the economy; it is more likely to be satisfied the greater is σ, which corresponds to a lower intertemporal elasticity of substitution. Under Condition M, Ω defined in (P) will converge and the results of Benveniste and Scheinkman [2] can be applied to yield necessary conditions for the optimal growth path. Define μ and λ as the costate variables associated with K and H, respectively, and apply the Pontriyagin Maximum Principle to obtain

$$C^{-\sigma} - \mu = 0 \tag{3a}$$

$$(\mu f' - \lambda g')\, K = 0 \tag{3b}$$

$$[\mu(f - k_x f') - \lambda(g - k_y g')]\, H = 0 \tag{3c}$$

$$\dot{\mu} = \mu(\rho + \delta) - \mu s f' - \lambda(1-s)\, g' \tag{3d}$$

$$\dot{\lambda} = \lambda(\rho + \eta) - \mu u(f - k_x f') - \lambda(1-u)(g - k_y g') \tag{3e}$$

$$\lim_{t \to \infty} e^{-\rho t} \mu(t)\, K(t) = 0 \tag{3f}$$

$$\lim_{t \to \infty} e^{-\rho t} \lambda(t)\, H(t) = 0. \tag{3g}$$

The necessary conditions (3), together with Eqs. (1) and (2), can be used to characterize the solution to (P).[5] Notably, the sufficient condition for a solution of (3) to solve (P) is that the optimal Hamiltonian be concave in (K, H), which is surely met given the specification of preference and technologies.

[5] In order to simplify the exposition, we have assumed that the constraints on the control variables ($0 \leqslant u \leqslant 1$, $0 \leqslant s \leqslant 1$, $C \leqslant F(uH, sK)$) are all satisfied with strict inequality on the optimal path. In the next section, we establish the existence of a unique balanced growth path in which the control variables are in the interior of the feasible set. It is possible, however, that for some initial values of K/L there will be regions of the path for which $\dot{K} = 0$ or $\dot{H} = 0$. This case can readily be handled by incorporating the constraints on control variables into (P1) and making the appropriate modifications of (3). Also note that in deriving the transversality conditions (3f) and (3g) for this problem, Benveniste and Scheinkman [2] impose an assumption that the objective function u is non-negative. For $\sigma > 1$ this condition will not be met by (P). However, the purpose of this assumption is to assure that the state vector and the costate variables are positive. Assumptions which accomplish this, and which are met by (P), can be used to show that (3f) and (3g) are necessary for this problem.

2.2. *Balanced Growth and the Reduced System*

We will prove the existence of a unique balanced growth path (BGP) satisfying (3) in which *C*, *K*, and *H* grow at a common rate (i.e., $\nu_C = \nu_H = \nu_K$, where ν_i is the growth rate of *i*), the costate variables grow at a common rate ($\nu_\lambda = \nu_\mu$), and *s* and *u* are constant. Since the costate variables represent the values of a unit of human capital and physical capital, respectively, we can define $p \equiv \lambda/\mu$ to be the relative price of human capital in units of goods, which will be constant along the BGP. In this section we present the reduced system of equations from (3) which yields solutions for the growth rates ν_C, ν_H, ν_K, and ν_p.

First, we note that Eqs. (3b) and (3c) require that real rates of return on physical or human capital be equalized across sectors at each point in time. Let $r \equiv f'(k_x)$ denote the market rental on capital and $w \equiv f(k_x) - k_x f'(k_x)$ denote the market wage expressed in units of goods. Assuming that both sectors are active and that there are no factor intensity reversals (i.e., the sign of $k_x - k_y$ is the same for all w/r), (3b)–(3c) can be used to solve for the factor intensities in each sector and factor prices as functions of the output price alone. Totally differentiating (3b) and (3c) and using the definition of factor prices yields

$$\begin{aligned} &\text{(a)} \quad k_x'(p) = \frac{rk_y + w}{(k_y - k_x)\, f''} \qquad &&\text{(b)} \quad k_y'(p) = \frac{rk_x + w}{(k_y - k_x)\, pg''} \\ &\text{(c)} \quad r'(p) = f'' k_x'(p) &&\text{(d)} \quad w'(p) = -k_x g'' k_x'(p). \end{aligned} \tag{4}$$

If there are factor intensity reversals, the $k_i(p)$ will be correspondences. The existence of a factor intensity reversal would not affect the existence and uniqueness result, but consideration of this case complicates the exposition. Therefore, discussion of the modifications necessary for this case will be relegated to footnotes.

The solutions to (4) can be combined with the full employment condition to derive the allocation of human capital between sectors and the scaled outputs of each sector. Define $k \equiv K/H$ as the aggregate factor proportions. Full employment requires that $uk_x(p) + (1-u)\, k_y(p) = k$, which yields

$$u(p, k) = [k - k_y(p)]/[k_x(p) - k_y(p)] \tag{5a}$$

$$x(p, k) \equiv X/H = u(p, k)\, f(k_x(p)) \tag{5b}$$

$$y(p, k) \equiv Y/H = [1 - u(p, k)]\, g(k_y(p)). \tag{5c}$$

Equations (4) and (5) determine the allocation of resources for given values of the costate variable p and the state variable *k*. We now utilize these results to derive expressions for the growth rates of the variables.

Equations (3d) and (3e) yield an expression for the growth rate of p, $v_p = (\dot{\lambda}/\lambda) - (\dot{\mu}/\mu)$, which by (4) is a function of p alone,

$$v_p = r(p) + w(p)/p + \eta - \delta. \tag{6}$$

This "intertemporal no-arbitrage condition" specifies the intertemporal relative-price adjustment necessary to equalize the net returns on human and physical capital. If $r - \delta > (w/p) - \eta$, the rental value of capital (net of depreciation) exceeds that on human capital, and there must be a capital gain earned on human capital investments to offset the difference in net rental values. We impose the following condition on the technology, which will ensure the existence of an equilibrium in which investment in both types of capital is profitable.

Condition FP (*Factor Price*). $\sup_p(r(p) - w(p)/p) > \delta - \eta > \inf_p(r(p) - w(p)/p)$.

This condition will be satisfied, for example, if both sectoral production functions satisfy the Inada conditions.[6] This condition can be used to give existence and uniqueness of the relative price of human capital consistent with a BGP.

LEMMA 1. *If FP holds, there exists a unique p^* at which $v_p = 0$.*

Proof. From (4), $Z(p) \equiv r(p) - w(p)/p = f'(k_x(p)) - [g(k_y(p)) - k_y(p)\, g'(k_y(p))]$ is a monotone function of p, which guarantees that (6) can have at most one solution with $v_p = 0$. Condition FP guarantees the existence of a solution.[7] Q.E.D.

[6] For example, suppose the production processes in both sectors have fixed coefficients, and let a_{ij} denote the amount of factor i ($i = K, L$) required to produce a unit of output in sector j ($j = X, Y$). It follows that $r \in [0, 1/a_{KX}]$ and $(w/p) \in [0, 1/a_{HY}]$, so $(r - w/p) \in [1/a_{KX}, -1/a_{HY}]$. On the other hand, if both sectors have Cobb–Douglas production functions, the factor return differentials are unbounded and Condition FP is met automatically.

[7] For the case of factor intensity reversals, a more general argument can be used to prove Lemma 1. Consider the values (r, w, p) that satisfy $\psi_X(w, r) = 1$ and $\psi_Y(w, r) = p$, where ψ_i is the unit cost function for sector i. Applying the implicit function theorem to the unit cost function for sector X, we obtain $w = j(r)$ with j continuous and decreasing $j'(r) < 0$ when both factors are used in sector X. Substitution of $j(r)$ into the sector Y cost function yields $p = \Gamma(r)$, where Γ is continuous and $\Gamma'(r) > 0$ iff $k_y > k_x$. It then follows that $w/p = \chi(r) \equiv j(r)/\Gamma(r)$ is a continuous function with $\chi'(r) < 0$. On the other hand, we assume that both factors are utilized in both sectors, but it should be noted that the Lucas [14] model (in which only labor is used in the Y sector and the X sector uses both factors with a Cobb–Douglas technology) also satisfies this condition. Letting B denote the output of Y per unit of labor, it follows that $w/p = B$. Since $\inf r(p) = 0$ and $\sup r = \infty$ with the Cobb–Douglas technology, Condition FP will be satisfied in the Lucas model if $B + \delta - \eta \in (0, \infty)$. This is guaranteed by the assumption that there is positive growth.

The growth rate of consumption is obtained by differentiating (3a) with respect to time and equating to (3d), which gives

$$\dot{C} = \nu_C(p)\, C, \tag{7}$$

where

$$\nu_C(p) = \sigma^{-1}[r(p) - (\rho + \delta)].$$

The following condition on the production technologies ensures that they are sufficiently productive to generate $\nu_C > 0$ at the price consistent with balanced growth.

Condition G (*Nondegenerate Growth*). $r^* - \delta > \rho$.

This condition is analogous to condition G in Jones and Manuelli [9].[8]

The growth rates of physical and human capital are derived from substituting (5) into (1) and (2),

$$\nu_K = \dot{K}/K = k^{-1}[x(p, k) - c] - \delta \tag{8a}$$

$$\nu_H = \dot{H}/H = y(p, k) - \eta, \tag{8b}$$

where $c \equiv C/H$ is consumption per effective labor unit. Notably, ν_i $(i = K, H)$ must be constant along a BGP and, from (6) and (7), it is clear that c and k must be constant if p is constant, implying that the balanced growth path is one in which C, H, and K all grow at the common rate ν^*. The following lemma establishing the constancy of p on the BGP can be proven by analyzing a transformed version of optimization problem (P).[9]

LEMMA 2. *In order for the BGP to have the common growth property, p must be constant.*

[8] In the two-sector model presented here, only a local condition is required. With constant returns to scale, it is possible to attain the balanced growth path from any initial factor endowment (K_0, H_0) by disposing freely a sufficient quantity of one of the factors until the aggregate factor proportion equals that associated with the balanced growth equilibrium.

[9] To prove this assertion, one needs to transform the optimization problem (P) to

$$H_0^{1-\sigma} V(k_0) = \max_{c,\, s,\, u,\, k} H_0^{1-\sigma} \int_0^{\infty} \frac{c(t)^{1-\sigma}}{1-\sigma} e^{-\Delta(t)}\, dt, \tag{P$'$}$$

where $\Delta(t) \equiv [\rho + (1-\sigma)\eta]\, t - (1-\sigma)[\int_0^t (1-u)\, g(k_y)\, ds]$ and the transition rule for k is $\dot{k} = [uf(k_x) - c - (1-u)\, g(k_y)\, k + (\eta - \delta)\, k]$. Applying the Uzawa transformation, we can derive $p = (1-\sigma)\, V(k)/V'(k) - k$, which ensures the assertion. The detailed proof is available upon request.

2.3. *Existence and Uniqueness*

We are now ready to establish the following result:

PROPOSITION 1 (Existence and Uniqueness of a Non-degenerate Balanced Growth Equilibrium). *Under Conditions M, FP, and G, there exists a unique balanced growth equilibrium.*

Proof. Lemmas 1 and 2 establish the existence of a unique price consistent with $\nu_p = 0$. From (7) we obtain a unique candidate growth rate $\nu^* = \nu(p^*)$, which is positive under Condition G. We next show that there exist unique solutions c^* and k^* satisfying (8) with $\nu_H = \nu_K = \nu^*$. Substituting (5) into (8b) yields the solution $k^* = k_x + (k_y - k_x)(\nu^* + \eta)/g^*$. Since full employment of factors with both sectors active requires $\min\{k_x, k_y\} < k^* < \max\{k_x, k_y\}$, this solution will be feasible if $0 < (\nu^* + \eta) < g^*$. Since $g^* > (w^*/p^*)$, $(w^*/p^*) - \eta > \nu^*$ is a sufficient condition for there to exist a unique solution k^* at which $\nu_H = \nu^*$. Using the fact that $\nu_p = 0$, which requires $r^* - \delta = (w^*/p^*) - \eta$, this inequality can be rewritten using (7) as $r^* - \delta = \rho + \sigma\nu^* > \nu^*$. This inequality is obviously satisfied for $\sigma > 1$. For $\sigma < 1$, Condition M can be used to rewrite the left-hand side of the inequality $r^* - \delta > (1 - \sigma)\,\nu_{\max} + \sigma\nu^*$, which must exceed ν^*. Therefore, there will exist a unique k^* consistent with balanced growth if Condition M is satisfied.

The solution for c consistent with $\nu_K = \nu^*$ is obtained from (8a) to be $c^* = (r^* - \delta - \nu^*)\,k^* + [w^* - p^*(\eta + \nu^*)]$, where we have used the fact that normalized national income can be written as $x + py = w + rk$ and $y^* = \eta^* + \nu^*$ from (8b). But $r^* - \delta - \nu^* > 0$ and $w^* - p^*(\eta + \nu^*) > 0$ by the arguments above, so $c^* > 0$. These results have established the existence of a unique BGP that solves conditions (3a)–(3e) of (P). It remains to show that this solution also satisfies the transversality conditions (3f) and (3g). From (3d), μ will grow at a rate $\rho + \delta - r^*$ on the balanced growth path. Since K is growing at rate ν^*, the transversality condition (3f) will be satisfied if $r^* - \delta > \nu^*$, which has been shown above to be guaranteed by Condition M. A similar argument establishes that $r^* - \delta > \nu^*$ ensures that (3g) will hold. Q.E.D.

It is noteworthy that our uniqueness result generalizes, in part, that of Xie [26] and Benhabib and Perli [1], in which a locally unique balanced growth equilibrium obtains in a Lucas [14] economy without human capital externalities.[10]

[10] In the presence of sufficiently large spillover externalities, it is possible to have a continuum of balanced growth equilibria with positive growth rates, as found in Boldrin and Rustichini [3], Xie [26], and Benhabib and Perli [1].

3. Transitional Dynamics

In this section we analyze the transitional dynamics of the system in the neighborhood of the balanced growth path by examining the linearized dynamic system around the BGP. We first show that the dynamical system exhibits saddle-path stability and then show how the transitional dynamics of the system are affected by the relative factor intensity ranking of the sectors. In presenting the results, it will be assumed that we have an interior solution in which both factors are being accumulated along the transition path. This amounts to assuming that $k \in (k_x(p), k_y(p))$ along the optimal path, since capital stocks in this range are consistent with production in both sectors from (5a). The analysis can readily be extended to the case in which the aggregate endowment lies outside this range, which requires one of the sectors to be shut down until sufficient factor accumulation has occurred that the endowment ratio is consistent with production of both goods.[11]

3.1. *Saddle-Path Stability*

The dynamics of the system can be described by the behavior of the endogenous variables p, c, and k. The evolution of p is given by (6). Utilizing (7) and (8), we obtain the dynamics of consumption and factor supplies,

$$\frac{\dot{c}}{c} = \frac{\dot{C}}{C} - \frac{\dot{H}}{H} = v_C(p) - y(p, k) + \eta \tag{9}$$

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{H}}{H} = k^{-1}[x(p, k) - c] - y(p, k) + \eta - \delta. \tag{10}$$

The linearized dynamic system around the BGP is given by

$$\begin{bmatrix} \dot{p}/p \\ \dot{c}/c \\ \dot{k}/k \end{bmatrix} = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} p - p^* \\ c - c^* \\ k - k^* \end{bmatrix}, \tag{11}$$

where $a_{11} = r' - (w' - w/p)/p$, $a_{21} = -\partial y/\partial p + v'_C$, $a_{23} = -\partial y/\partial k$, $a_{31} = (\partial x/\partial p)/k - \partial y/\partial p$, $a_{32} = -1/k$, and $a_{33} = -(x - c)/k^2 + (\partial x/\partial k)/k - \partial y/\partial k$.

[11] Define $p_{\min}(k)$ as the price solving $k_y(p) = k$, which is the lowest value of p at which operation of the Y sector is profitable. If $\lambda/\mu < p_{\min}(k)$, then the Y sector will shut down. Factor prices will be determined by marginal productivities in the X sector with $k_x = k$, and physical capital will accumulate until λ/μ has risen sufficiently that production of Y is profitable. For the X sector, the form of the utility function ($U'(C) \to \infty$ as $C \to 0$) ensures that production of X will not shut down. However, the constraint $C \leqslant F(uH, sK)$ may bind so that $U'(C) > \mu$. In this case there will be no accumulation of physical capital until λ/μ falls sufficiently that accumulation of physical capital is profitable.

Two points should be noted regarding the properties of the dynamical system (11) which result from the two-factor, two-good structure of the production model. The first is that the model has a block recursive structure, since the dynamics of the relative price is independent of c and k. This is a result of the *factor price equalization* property of 2×2 models, under which the prices of the factor inputs are determined by prices alone (i.e., they are independent of the relative factor supplies). This substantially simplifies the analysis of the transitional dynamics. The second is that the signs of $r'(p)$ and $w'(p)$ are dependent on which sector is more capital intensive (the *Stolper–Samuelson Theorem*), as are the signs of $\partial x/\partial k$ and $\partial y/\partial k$ (the *Rybczynski Theorem*). These results are well known from the literature on international trade theory (e.g., Jones [11]), and are derived from (4) and (5). These results can be used to derive the sign pattern of the Jacobian matrix in (11). We collect these results for future reference in the following lemma:

LEMMA 3. (i) *If* $k_x < k_y$, *then* $a_{11} > 0$, $a_{23} < 0$, $a_{33} < 0$.

(ii) *If* $k_x > k_y$, *then* $a_{11} < 0$, $a_{23} > 0$, $a_{33} > 0$.

(iii) *In all cases*, $a_{31} < 0$, $a_{32} < 0$.

We now establish the stability property of the dynamical system:

PROPOSITION 2 (Stability). *The balanced growth equilibrium is saddle-path stable regardless of the factor intensity ranking.*

Proof. To examine stability of the above 3×3 system, we evaluate its Jacobian matrix at the balanced growth equilibrium, denoted as J_3^*. The eigenvalues of J_3^* are the solutions of its characteristic equation

$$-\gamma^3 + \mathrm{Tr}(J_3^*)\,\gamma^2 - R(J_3^*)\,\gamma + \mathrm{Det}(J_3^*) = 0,$$

where the trace of J_3^* is $\mathrm{Tr}(J_3^*) \equiv a_{11} + a_{33}$, the determinant is $\mathrm{Det}(J_3^*) \equiv -a_{11}a_{23}a_{32}$, and $R(J_3^*) \equiv a_{11}a_{33} - a_{23}a_{32}$. Utilizing Lemma 3, it can be seen that $\mathrm{Det}(J_3^*)$ is negative for either factor intensity ranking. However, the signs of $\mathrm{Tr}(J_3^*)$ and $-R(J_3^*) + \mathrm{Tr}(J_3^*)/\mathrm{Det}(J_3^*)$ are both ambiguous. Straightforward application of the Routh Theorem implies that the characteristic equation specified above will have either one or three roots with negative real parts.

Next, we utilize the block-recursive nature of the 3×3 dynamical system: its characteristic equation specified above contains a real root, $\gamma_1 = a_{11}$ (which is positive for $k_x < k_y$ and negative for $k_x > k_y$), with the remaining two roots satisfying the characteristic equation of a 2×2 subsystem (which

Jacobian matrix evaluated at the balanced growth equilibrium is denoted as J_2^*)

$$\gamma^2 - \mathrm{Tr}(J_2^*)\,\gamma + \mathrm{Det}(J_2^*) = 0,$$

where $\mathrm{Tr}(J_2^*) \equiv a_{33}$ and $\mathrm{Det}(J_2^*) \equiv -a_{23}a_{32}$. Applying Lemma 3, both $\mathrm{Tr}(J_2^*)$ and $\mathrm{Det}(J_2^*)$ are negative and the discriminant $(a_3^2 + 4a_{23}a_{32})$ is positive for the case $k_x < k_y$, so the remaining roots are real and of opposite signs. In this case, we must have one negative real root and two positive real roots. If $k_x > k_y$, both $\mathrm{Tr}(J_2^*)$ and $\mathrm{Det}(J_2^*)$ are positive, so the remaining roots will have positive real parts. In either case, there is one negative real root and two roots with positive real parts, implying saddle-path stability. Q.E.D.

It is intriguing that we obtain saddle-path stability for either factor intensity ranking. This is due to the fact that for each factor intensity ranking, an unstable adjustment process in prices or quantities is offset by a stable adjustment process in the other variables. This "exactly polar property" in price and quantity dynamics is obtained by Bruno [5] in a two-sector (consumption/investment sectors) exogenous growth model in which there is a single fixed proportions production process in each sector. In Section 4 below, we will show that this stability property may be upset by the introduction of factor taxation when such taxes are too distortionary.

3.2. *Characterization of the Dynamics*

As indicated by the results in Lemma 3, both the price and quantity adjustment processes depend on the factor intensity ranking, so we examine each case separately.

Case I. $k_x < k_y$. The relative price adjustment process, which depends only on p, is unstable in this case since $a_{11} > 0$. Therefore, p must jump immediately to the balanced growth value. The remaining two roots corresponding to the (c, k) dynamics are real and of opposite sign, so we can focus our analysis on the 2×2 subsystem containing the (c, k) dynamics illustrated in Fig. 1. The $\dot{c} = 0$ locus will be vertical, since from (9) there is a unique k (given p) at which C and H grow at the same rate. In this case an increase in the capital/labor ratio reduces the relative output of good x ($a_{33} < 0$), which is a stabilizing force because it reduces the accumulation of capital relative to labor (holding c constant). The $\dot{k} = 0$ must then be downward sloping, since $dc/dk = -(a_{33}/a_{32}) < 0$ by Lemma 3. An increase in k results in a decrease in the relative supply of X sector output when the Y sector is capital intensive, so c must fall to keep the two factors accumulating at the same rate. Examination of Fig. 1 establishes that the saddle path must be upward sloping. For example, starting from an initial factor endowment ratio, $k_0 < k^*$, we must have $\nu_K > \nu_H$ and $\nu_C > \nu_H$ during

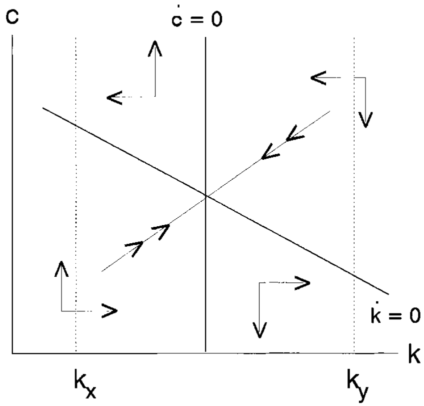

FIG. 1. Saddle-path stability: $k_x < k_y$ (projection on c–k space).

the transition to balanced growth. Since the price remains constant at p^* throughout the transition, the growth rate of consumption will be constant at the steady state level ν^* from (7).

The only remaining question is to rank the growth rate of capital relative to that of consumption during the transition to the BGP. From (8a), the slope of the locus of values in Fig. 1 along which ν_K is constant will be downward sloping for this case, since $(x - c)/k$ is decreasing in k and c. Thus, ν_K must be falling as k rises along the saddle path. Since $\nu_K = \nu^*$ on the BGP, $\nu_K > \nu_C = \nu^*$ along the transition to the balanced growth path.

Case II. $k_x > k_y$. In this case, the price adjustment process is stable. An immediate jump of p to the balanced growth value is not possible in this case, as it was in Case I, because the adjustment process of k and c at fixed prices is not stable. From (5), an increase in k raises the output of x and

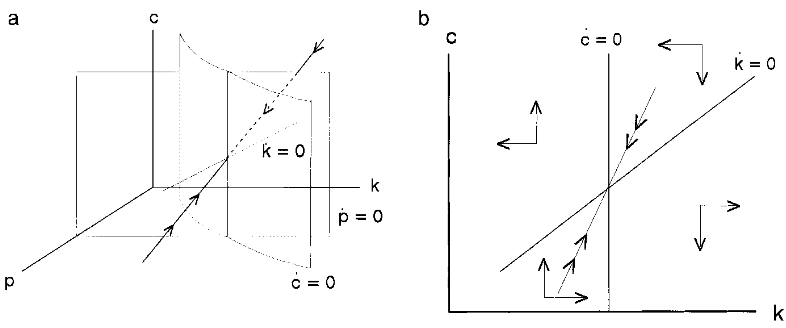

FIG. 2. Saddle-path stability: $k_x > k_y$. Projection on c–k space: $k_x > k_y$.

reduces the output of y, leading to a further increase in k. Similar arguments can be applied to the adjustment process for c. Thus, continual adjustments of the price along the optimal path will be required. However, it is still possible to analyze the projected dynamics in (c, k) space by making use of the fact that the value function is homogeneous of degree $1-\sigma$ (which is ensured as $\sigma \neq 1$). Since the costate variables are the derivatives of the value function with respect to the respective state variables, the costate variables are homogeneous of degree $-\sigma$. Therefore, $p \equiv \lambda/\mu$ is a function of k alone. Also, $p(k)$ will be non-decreasing in k from the concavity of the value function. The dynamics in the three-dimensional representation may be illustrated as in Fig. 2a.

Substituting $dp = p'(k)\,dk$ into (11) gives the projected dynamical system onto (c, k) space:

$$\begin{pmatrix} \dot{c}/c \\ \dot{k}/k \end{pmatrix} = \begin{pmatrix} 0 & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} c - c^* \\ k - k^* \end{pmatrix}, \tag{11'}$$

where

$$b_{12} = a_{23} + a_{21}\, p'(k) = -\partial y/\partial k + [v'_C(p) - (\partial y/\partial p)]\, p'(k)$$

$$b_{21} = a_{32} = -1/k < 0$$

$$b_{22} = a_{33} + a_{31}\, p'(k) = [-(x-c)/k^2 + (1/k)(\partial x/\partial k) - (\partial y/\partial k)]$$
$$+ [(1/k)(\partial x/\partial p) - (\partial y/\partial p)]\, p'(k).$$

Since $b_{11} = 0$, the projected $\dot{c} = 0$ locus must be vertical. The term b_{12} contains two conflicting forces of k on the growth rate of c. First, an increase in k reduces the output of y (at constant prices) since the Y sector is labor intensive in this case, which tends to raise the growth rate of c. Second, an increase in k causes a rise in p, which lowers the growth rate of consumption (i.e., $v'_C < 0$ for $k_x > k_y$) and raises the output of y, both of which tend to reduce the growth rate of c. However, it can be shown that the latter effect dominates so that $b_{12} < 0$. Proposition 2 shows that the system has a saddle path. The projection of this saddle path onto (c, k) space implies that the characteristic roots of this projected 2×2 system must have real parts of opposite sign. Thus, the determinant of the associated 2×2 Jacobian matrix evaluated at the balanced growth equilibrium, $\mathrm{Det}(J_p^*) \equiv -b_{12}b_{21}$, is negative. Since $b_{21} < 0$, it follows that $b_{12} < 0$.

The term b_{22} also involves conflicting forces. An increase in k will increase the relative output of good x when $k_x > k_y$ (i.e., $a_{33} > 0$). However, the rising p will reduce the relative output of good x (i.e., $a_{31} < 0$). The fact that the system has a saddle path is not sufficient to determine which of these effects dominates. If the former effect dominates, then $b_{22} > 0$ and the projected $\dot{k}=0$ locus will be upward sloping since $dc/dk=-b_{22}/b_{21}>0$. An increase in k raises relative supply of x, which requires an increase in c to restore $\dot{k}=0$. The phase diagram for this case is illustrated in Fig. 2b, which shows that the projected saddle path must be upward sloping as well. If, on the contrary, the stabilizing effect of p on relative outputs dominates, then $b_{22}<0$ and the projected $\dot{k}=0$ locus will now be downward sloping. The analysis for this case is similar to that illustrated in Fig. 1. Therefore, in either case we have that if $k_0<k^*$, $v_K>v_H$ and $v_C>v_H$ along the transition to the BGP. These two results are similar to those obtained for the case $k_x<k_y$.

The difference from the previous case where $k_x<k_y$ is that the rising k must cause p to rise. Recall that the growth rate of consumption is a decreasing function of p when $k_x>k_y$. The growth rate of consumption will be falling along the transition path, so $v_C>v^*$ during the transition to balanced growth. There are, however, conflicting forces operating on the growth rates of both types of capital during the transition. From (8), the increase in k tends to raise the growth rate of physical capital and reduce the growth rate of human capital since $\partial x/\partial k>0$ and $\partial y/\partial k<0$ when $k_x>k_y$. The consequent increase in the relative price of human capital has the opposite effect. Therefore, it will not in general be possible to rank v_H and v_K relative to v^* for the transition. Similarly, it will not in general be possible to rank v_K relative to v_C for this case. To be specific, the slope of the locus along which $v_K=v_C$ is $dc/dk=-(b_{22}-b_{12})/b_{21}$. For the case $b_{22}>0$ in Fig. 2b, this locus must be steeper than the $\dot{k}=0$ locus, and its slope cannot be ranked in general relative to the saddle path. Similarly, for the case $b_{22}<0$ (as in Fig. 1), the locus may be upward sloping so its slope cannot be ranked relative to the saddle path. In the case $k_x>k_y$, we obtain a wider range of possible rankings of growth rates because of the conflicting role of price effects and factor accumulation effects on the outputs of the two goods.[12]

The above results regarding dynamic adjustment can now be summarized:

PROPOSITION 3 (Characterization of Dynamic Adjustment). *Starting from $k_0<k^*$, then in the neighborhood of the balanced growth path we have physical capital and consumption both growing more rapidly than human capital. If $k_x<k_y$, then $v_K>v_C=v^*>v_H$; if $k_x>k_y$, then $v_C>v^*$ and* $\min(v_K, v_C)>v_H$.

[12] As in the previous case, this analysis holds assuming that the non-negativity constraints on factor accumulation are not binding. This is equivalent to assuming that $k\in[k_y(p), k_x(p)]$ along the optimal path. This restriction cannot be simply illustrated in (c, k) space, since p (and thus the bounds of the feasible region) are changing along the path.

4. Applications

In this section, we illustrate how the model can be used to analyze the long-run balanced growth and short-run dynamic effects of changes in the rate of time preference and public policy parameters.

4.1. *Changes in Time Preference*

We begin by studying the balanced growth effects of a decrease in the time preference rate.

Proposition 4 (Effects of Changes in Time Preference). *A decrease in the time preference rate, ρ, raises the balanced growth rate, but leaves the BGP values of w, r, and p unaffected. The level of consumption per effective unit (c) is higher, and the capital/labor ratio (k) rises iff $k_x > k_y$. In transition, k adjusts monotonically to the new BGP value. c initially overshoots its BGP value if $k_x > k_y$, and c falls initially if $k_x < k_y$.*

Proof. From (4), it is clear that the balanced growth value of p^* (and hence r and w) is unaffected by ρ. Differentiating (7) yields $dv^*/d\rho = -1/\sigma < 0$. From (8), $dk/dv^* = (\partial y/\partial k)^{-1} = -1/a_{23} < (>)\ 0$ as $k_x > (<)\ k_y$ and $dc/dv^* = [(\partial x/\partial k) - k(\partial y/\partial k) - (x-c)/k]/(\partial y/\partial k) = -ka_{33}/a_{23} < 0$, where the inequalities follow from application of Lemma 3.

From (6), (9), and (10), it can be seen that an increase in ρ will leave the $\dot{k}=0$ and $\dot{p}=0$ loci unaffected, but will shift the $\dot{c}=0$ locus. Referring to Fig. 1 for the case of $k_x < k_y$, it can be seen that the $\dot{c}=0$ locus will shift to the left, leading to lower k and higher c on the BGP. Since the saddle path is upward sloping, c must jump to a value above its new BGP value, and then decline to the BGP level. For the case of $k_x > k_y$ (Fig. 2b), the $\dot{c}=0$ locus shifts rightward, leading to higher k and c on the new BGP. Since the saddle path is upward sloping in this case as well, c will fall initially, and then rise to the new balanced growth level. Q.E.D.

4.2. *Changes in Public Policy*

The growth effects of factor taxation have been an important issue in the current research of the endogenous growth theory.[13] Our analysis here simply highlights the application of our methodology to examine the short- and long-run effects of public policies. In particular, we focus on three representative public policy programs: capital and labor taxation and an education subsidy. Capital taxation is assumed to consist of a tax at rate τ_K on the earnings from all physical capital. Labor taxation is modeled as a tax at rate τ_H on earnings from human capital in the goods sector. The

[13] See Stokey and Rebelo [23] for a comprehensive survey of the literature in this area.

asymmetry between capital and labor taxation is that it is assumed that labor in the education sector is untaxed; i.e., the foregone earnings of labor are not taxed during the process of human capital accumulation. The education subsidy, at rate z_H, is one applied only to the labor input in the education sector.

Letting T denote the net tax revenues of the government, the goods market equilibrium condition can be written as

$$\dot{K} = uHf(k_x) - T - \delta K - C. \tag{1$'$}$$

Government tax revenues consist of the net factor tax revenues plus the proceeds of a lump sum tax, T_0. Tax revenues can be expressed as $T = r\tau_K K + w\tau_H uH + T_0 - wz_H(1-u)H$, where w and r are the pre-tax returns to labor and capital, respectively. In order to abstract from issues related to the choice of the level of public spending and to facilitate comparison with results of the previous section, we will assume that the factor taxes are redistributed to households in lump sum fashion so that $T = 0$ in all time periods.

The presence of factor taxes requires only slight modification of the necessary conditions (3). Equalization of after-tax returns to each factor across sectors implies that

$$r \equiv f'(k_x) = pg'(k_y) \tag{12a}$$

$$w \equiv [f(k_x) - k_x f'(k_x)] = p[g(k_y) - k_y g'(k_y)](1+z_H)/(1-\tau_H). \tag{12b}$$

Equations (12) yield solutions $k_i(p, \tau_H, z_H)$ (for $i = x, y$), $r(p, \tau_H, z_H)$, and $w(p, \tau_H, z_H)$. Utilizing these solutions, the intertemporal no-arbitrage condition and the consumption growth rate can be expressed as

$$v_p = r(p, \tau_H, z_H)(1-\tau_K) - \frac{w(p, \tau_H, z_H)(1-\tau_H)}{p} + \eta - \delta \tag{13a}$$

$$v_C = \sigma^{-1}[r(p, \tau_H, z_H)(1-\tau_K) - (\rho + \delta)]. \tag{13b}$$

Since (13a) is independent of c and k, it can be solved for the value p^* that is consistent with balanced growth as in Section 2.2. Utilizing p^* in (13b) yields the balanced growth rate, v^*. Existence of a unique p^* and corresponding growth rate $v^* > 0$ can be ensured if Conditions FP and G are modified to require that the net of tax factor returns satisfy the respective conditions. Note that these factor taxes affect the BGP through both the intersectoral resource allocation (reflected in (12)) and intertemporal resource allocation (13).

Substitution of the sectoral factor intensities into the full employment conditions and production functions in (5) yields $u(p, \tau_H, z_H, k)$,

$x(p, \tau_H, z_H, k)$, and $y(p, \tau_H, z_H, k)$. These output supply equations can be utilized in (8) to obtain

$$\frac{\dot{c}}{c} = v_C(p, \tau_H, z_H, \tau_K, k) - y(p, \tau_H, z_H, k) + \eta \tag{14a}$$

$$\frac{\dot{k}}{k} = k^{-1}[x(p, \tau_H, z_H, k) - c] - y(p, \tau_H, z_H, k) + \eta - \delta, \tag{14b}$$

which yield solutions for the BGP values c^* and k^* given the value of p that solves (13a). Proof of existence of these solutions proceeds along the same lines as in Proposition 1. The transitional dynamics of the system can then be analyzed using the system formed by (13a) and (14), as in the analysis of (11) in the previous section. This yields the following result on saddle-path stability:

PROPOSITION 5 (Stability with Distortionary Taxes). *The BGP will exhibit saddle-path stability iff sign*$(k_x - k_y) =$ *sign*$(k_x(1-\tau_H) - k_y(1+z_H))$. *If* $k_x - k_y > 0 > k_x(1-\tau_H) - k_y(1+z_H)$, *the system is unstable. If* $k_x - k_y < 0 < k_x(1-\tau_H) - k_y(1+z_H)$, *indeterminacy emerges in which the system has a continuum of equilibria.*

Proof. Stability analysis can be performed by linearizing the dynamic system consisting of equations (13a) and (14) around the BGP. It is readily seen that the Jacobian matrix of this system has the same pattern of 0 entries as does (11), with non-zero entries given by $a_{11} = (\partial r/\partial p)(1-\tau_K) - (1-\tau_H)((\partial w/\partial p) - (w/p))/p$, $a_{21} = \partial v_C/\partial p - \partial y/\partial p$, $a_{23} = -\partial y/\partial k$, $a_{31} = (\partial x/\partial p)/k - \partial y/\partial p$, $a_{32} = -1/k < 0$, and $a_{33} = -(x-c)/k^2 + (\partial x/\partial k)/k - \partial y/\partial k$. Differentiation of (12) and (13a) can be used to establish that sign $a_{11} = -\text{sign}(k_x(1-\tau_H) - k_y(1+z_H))$, while differentiation of (5) shows that sign $a_{23} =$ sign $a_{33} = \text{sign}(k_x - k_y)$ as in Lemma 3. If signs of $(k_x(1-\tau_H) - k_y(1+z_H))$ and $(k_x - k_y)$ are the same, then the determinant of the Jacobian, $\text{Det}(J_3) = -a_{11}a_{32}a_{23}$, is negative and saddle-path stability is guaranteed following the argument of Proposition 2. In the case where $(k_x - k_y) > 0 > (k_x(1-\tau_H) - k_y(1+z_H))$, it can be shown that all three roots will be positive and the system will be dynamically unstable. If $k_x(1-\tau_H) - k_y(1+z_H) > 0 > (k_x - k_y)$, which would require a tax on labor in the education sector and/or a subsidy to labor in the goods sector, one root will have a positive real part and the other two will have negative real parts. Q.E.D.

In the presence of distortionary taxation, the properties of the price adjustment process, which are determined by sign sign a_{11}, depend on the sign of $k_x(1-\tau_H) - k_y(1+z_H)$. This is referred to by Jones [10] as the factor intensity ranking in the value sense, since this expression is positive iff capital costs are a greater share of total costs in sector X than in sector Y. In contrast, the quantity adjustment process depends on the factor intensity ranking in the physical sense, $k_x - k_y$. Proposition 5 establishes that saddle-path stability arises iff the same sector is capital intensive on both of these rankings.

When factor taxes are too distortionary, the two intensity rankings need not be consistent, which leads to the possibility of either an unstable node or indeterminacy. Note first that when factor and value intensity rankings differ, the share of labor devoted to sector X will be decreasing in the relative price of X output (i.e., $\partial u / \partial p > 0$). The unstable case emerges if the education sector is value capital intensive, which requires p to jump immediately its balanced growth value. However, the quantity adjustment process at fixed prices is unstable in this case as well, since the goods sector is capital intensive in the physical sense. Indeterminacy arises if the education sector is labor intensive in the value sense but capital intensive in the physical sense. In this case, there will be a continuum of paths converging to the balanced growth path that satisfy (12)–(14), and expectations about the future path of factor prices will be self-fulfilling. Interestingly, factor market distortions may give rise to a continuum of equilibria, a situation which is usually associated with increasing returns due to externalities in endogenous growth models, as in Benhabib and Perli [1], Boldrin and Rustichini [3], and Xie [26].[14]

Assuming that saddle-path stability emerges, we can analyze the effects of factor taxation on the BGP by total differentiation of the system consisting of Eqs. (12)–(14). Results for the effects of factor taxes on k_x and k_y and p can be derived from differentiation of (12) and (13a) due to the block-recursive structure of the system, and are reported in Table I. Note that each of the three factor market interventions are distinct in their impact on these variables. The capital tax does not alter allocation of capital across sectors, but has the effect of raising the price of the capital-intensive good and reducing the return to investment in physical capital.

[14] It has been recognized in the international trade theory literature (e.g., Jones [10]) that paradoxes arise when factor market distortions are sufficiently large that the signs of $k_x - k_y$ and $k_x(1-\tau_H) - k_y(1+z_H)$ differ. Neary [16] shows that in a static two-sector model, this case is associated with instability in the inter-sectoral dynamic adjustment process when the speed of adjustment is proportional to the factor price differential across sectors. The instability that arises in the present case differs in that it refers to instability of dynamic factor accumulation process. Furthermore, the possibility of a continuum of equilibria does not arise in the static model. In the three-sector exogenous growth model of Bruno [5] with discrete technologies, a stable node may arise for either price adjustment or quantity adjustment. However, while the adjustment process in prices and quantities may both be unstable, they cannot both be stable under his framework.

TABLE I

Balanced Growth Effects of Changes in Public Policy

	Changes in						
				p^*		k^*	
Increase in	k_x^*	k_y^*	v^*	$k_x < k_y$	$k_x > k_y$	$k_x < k_y$	$k_x > k_y$
τ_K	−	−	−	+	−	−	−[a]
τ_H	+	−	−	−	−	?	?
z_H	−	−	+	−	−	−[a]	−

Note. All results assume that the $\text{sign}(k_x - k_y) = \text{sign}(k_x(1-\tau_H) - k_y(1+z_H))$, so that the system exhibits saddle-path stability by Proposition 5.

[a] The result holds as long as the intertemporal elasticity of substitution is sufficiently small.

This increase in the price of the capital-intensive good raises the pre-tax rental on capital, causing substitution toward the use of human capital in each sector. The tax on human capital in X and the subsidy to human capital in Y have similar effects on the intersectoral allocation of labor: both make the use of human capital in X less attractive and lower the relative price of Y output. However, they differ in their impact on the returns to investment in human capital (i.e., $w(1-\tau_H)/p$ from (13a)), since the tax on X labor reduces the after tax real wage while the subsidy to Y labor raises it.

These results can now be used to derive the effects of factor taxation on the growth rate on the BGP.

PROPOSITION 6. *Each of the factor taxes has the effect of reducing the balanced growth rate, while the subsidy raises it. The growth rate effects are larger the greater is* $1/\sigma$, *the intertemporal elasticity of substitution.*

Proof. The comparative statics results are obtained by differentiation of (12) and (13). The effect of an increase in τ_H can be used to illustrate which factors are most important in determining the magnitude of the growth rate effects,

$$\frac{dv_C}{d\tau_H} = \frac{1}{\sigma}\left(\frac{-(1-\tau_K)\,g'(k_y)\,k_y w}{(1-\tau_k)\,pg(k_y) + (1-\tau_H)\,k_y f(k_x)/p}\right) < 0.$$

The effect of labor taxation in the goods sector will be larger the greater the intertemporal elasticity of substitution and the greater the share of capital costs in the education sector, $g'(k_y)\,k_y/g(k_y)$. Similar results are obtained for the other factor market distortions. Q.E.D.

The effects of taxes on the balanced growth value of k can be obtained from the condition that human capital grow at the balanced growth rate ν^*, which requires $\nu^* = y(p^*, \tau_H, z_H, k) - \eta$. Using (5a) and (14a) and letting $k_i^*(\tau_H, z_H, \tau_K) = k_i(p^*(\tau_H, z_H, \tau_K), \tau_H, z_H)$ denote the balanced growth values of the sectoral factor intensities and $\nu^*(\tau_H, z_H, \tau_K)$ the growth rate on the BGP, the value of k on the BGP is given by

$$k^* = \left(1 - \frac{\eta + \nu^*(\tau_H, \tau_K, z_H)}{g(k_y^*(\tau_H, \tau_K, z_H))}\right) k_x^*(\tau_H, \tau_K, z_H) + \left(\frac{\eta + \nu^*(\tau_H, \tau_K, z_H)}{g(k_y^*(\tau_H, \tau_K, z_H))}\right) k_y^*(\tau_H, \tau_K, z_H). \qquad (15)$$

The aggregate capital/labor ratio is a weighted average of the sectoral capital/labor ratios, with the weight determined by the output of Y required to keep human capital growing at rate ν^*. The effect of a change in a factor tax parameter ϕ (for $\phi = \tau_H, \tau_K, z_H$) on the capital/labor ratio can be decomposed into its effect on the sectoral factor intensities and its effect on the growth rate,

$$\frac{dk^*}{d\phi} = \left[1 - \left(\frac{\eta + \nu^*}{g(k_y)}\right)\right] \frac{\partial k_x^*}{\partial \phi} + \left(\frac{\eta + \nu^*}{g(k_y)}\right) \times \left[\frac{(g - g'(k_y)\, k_y) + g'(k_y)\, k_x}{g(k_y)}\right] \frac{\partial k_y^*}{\partial \phi} + \left(\frac{k_y - k_x}{g(k_y)}\right) \frac{\partial \nu^*}{\partial \phi}, \qquad (16)$$

where both of the bracketed terms are positive. Note that increases in the growth rate on the BGP result in an increase in the fraction of the capital stock located in the Y sector in order to raise the growth rate of human capital. This effect increases k if the Y sector is capital intensive.

An increase in τ_K reduces the sectoral capital/labor ratios and also reduces the growth rate from Table I. If $k_x < k_y$, these effects all work in the same direction and we obtain the intuitive result that an increase in the tax rate on physical capital will reduce the BGP capital/labor ratio. If $k_x > k_y$, the reallocation of capital to the more capital-intensive sector that results when the growth rate falls could actually result in a rise in k. The outcome in this case will be affected by the magnitude of the change in the growth rate relative to the change in the sectoral factor intensities. The entry in Table I for this case is based on the assumption that the growth rate effect is relatively small, which by Proposition 6 is expected if the intertemporal elasticity of substitution is relatively low. An increase in z_H will also reduce both sectoral capital intensities, but causes the growth rate to rise. For this case, these effects work in the same direction when $k_x > k_y$, and the aggregate capital/labor ratio must fall. If $k_x < k_y$, then k could rise if the growth rate effect is sufficiently large. Finally, an increase in τ_H will cause the sectoral factor intensities to move in opposite directions and will reduce the growth rate. The impact on k for this case is ambiguous.

The normalized consumption level on the BGP, c^*, is determined from (14b) to be

$$c^* = \left(1 - \frac{\eta + v^*(\tau_H, \tau_K, z_H)}{g(k_y^*(\tau_H, \tau_K, z_H))}\right) f(k_x^*(\tau_H, \tau_K, z_H)) - (v^*(\tau_H, \tau_K, z_H) + \delta)\, k^*(\tau_H, \tau_K, z_H), \tag{17}$$

where use has been made of the fact that $u = (1 - (\eta + v)/g)$ on the BGP from (5c) and (14a). The consumption level will be increasing in k_x^* and k_y^* and decreasing in v^* and k^*. Utilizing the results in Table I, it can be seen that the effects of factor taxes on c^* will be ambiguous for all cases. For example, consider the effect of an increase in τ_K. The resulting decrease in sectoral factor intensities will tend to reduce c^*, but the coincident fall in v^* and k^* will work in the opposite direction. Whether consumption rises or falls cannot be determined without further assumptions regarding the production and preference parameters, such as the elasticities of substitution between factors in each sector and the intertemporal elasticity of substitution.

5. Concluding Remarks

The main objective of this paper has been to develop a general two-sector model of endogenous growth with human capital and physical capital accumulation, where labor and capital are used in both production sectors. We have shown how the use of the intertemporal no-arbitrage condition yields a simplified analysis of the balanced growth path and have shown how this technique can be applied to analyze effects of parameter and public policy changes.

We conclude with some brief comments on how the techniques in this paper can be applied to other extensions of the model. The first issue concerns the welfare analysis of taxation. We have briefly analyzed the effect of factor taxes on the rate of growth, but a more complete analysis would compare the effects of the two taxes on welfare given a government revenue objective. In addition, the model can be used to analyze other forms of taxes, such as sectoral output taxes. A second extension is to incorporate leisure time into the analysis. If leisure time enters the utility function, households must also choose the amount of human capital to be allocated to leisure activities. Such an extension does not affect the intertemporal no-arbitrage condition, but requires adjustments in the output equations to distinguish between the stock of human capital and the stock that is employed in production. Finally, the model could also be extended, at the expense of analytical generality, to allow for externalities from the stock of capital of the type analyzed by Romer [18] and Lucas [14].

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