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# Agglomeration in a linear city with heterogeneous households

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This paper develops a general equilibrium model with consumers heterogeneous in endowments and tastes. An agglomerative linear city is generated due to gains from trade rather than interaction externalities or market imperfections. An equal-treatment social optimum is shown to have a competitive price support. When preferences are location-independent, there is no population agglomeration. When a geographical factor, leisure, enters the utility function, the residential relocation process is agglomerative if the immobile and mobile goods are weakly Pareto complementary. Under general cross-preference structures, deglomeration may emerge, and the population density and the shadow price of land might not be positively correlated.

## 1. Introduction

Monocentric city models have been the most favored setup of regional scientists studying the behavior of spatially competitive consumers and the economic structure of a city [for example, see Alonso (1964), Mills (1967), Beckmann (1969) and Mirrlees (1972)]. The size of city can be either given exogenously [Alonso (1964)] or determined endogenously [Mills and de Ferranti (1971)]. The location of the central business district (CBD) is usually predetermined to consumers (see both port city and factory town models). Given the location of the CBD to which everyone must travel while paying positive transportation cost, the spatial structure of monocentric city in essence makes the landscape *heterogeneous*, thus resulting in agglomeration in which population per square foot is higher toward the center.[1] A

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[1]For further discussion on the concept of agglomeration, the reader is referred to Weber (1969) and Papageorgiou (1979). For a comprehensive survey of the existing literature in this field, see Fujita (1988).

natural question is whether an agglomerative city can emerge in an economy with purely homogeneous landscape, such as in a standard long-narrow city model or in $R^2$.

Along this line, Starrett (1978) produces a path-breaking result: a *spatial impossibility theorem.* Specifically, in a model with a closed, homogeneous landscape with perfect markets for all goods at all locations and without relocation cost, there is no competitive equilibrium with a positive aggregate transportation cost. Thus, population agglomeration cannot emerge competitively. In order to generate an agglomerative city with an exogenous center, one may assume market imperfections, increasing returns in production, or positive interaction externalities. In his seminal work, Mills (1967) used an imperfect competition model to drive agglomeration from spatial advantages and firms' increasing-returns-to-scale technologies. This framework is formalized by Starrett (1974) who, based upon increasing returns production technologies, generates a factory town in which the optimal degree of increasing returns can be determined [see also Papageorgiou and Thisse (1985) and Fujita (1986) for extensions]. Alternatively, Papageorgiou and Smith (1983) induced agglomeration by assuming positive interaction externalities among consumers. Analogous to the result in Starrett (1978), a uniform population distribution turns out to be a steady state, even in the presence of interaction externalities. Therefore, Papageorgiou and Smith (1983) concluded that an agglomerative process emerges only if this uniform steady state is unstable and perturbed.

In this paper, we draw on the structure of the linear city model developed by Solow and Vickery (1971) in which the landscape is homogeneous and the population is initially uniformly distributed. The main contribution of this paper is to generate an agglomerative city within a general equilibrium framework without assuming positive externalities, increasing returns, or market imperfections. The key to this result is to introduce *heterogeneities in endowments and preferences* so that the spatial impossibility theorem cannot be applied. With spatial heterogeneity, Wang (1990) shows that a competitively supportable, socially optimal marketplace can endogenously form. When tastes of different types of consumers for a location-dependent factor (say, leisure) are the same, the endogenous marketplace must be in the geographical center. In this paper, we extend Wang (1990) to incorporate immobile goods into the model under a standard continuum setup through which the residential relocation process is endogenously determined and characterized.

In contrast with standard agglomeration models, we generate residential relocation agglomeration without interaction externalities, increasing returns, or market imperfections. Instead, the classical idea of 'gains from trade' is used. Disutility of travel time or leisure–travel trade-off plays a key role in determining the residential location. This, however, allows location to enter the utility function directly, which may cause the failure of classical existence and welfare theorems [Berliant and ten Raa (1991) and Berliant et al. (1990)].[2] To circumvent this problem, we follow Negishi (1960) to focus mainly on the social planner's problem and use market equilibrium only to justify a proper social welfare weighting scheme. More specifically, the social optimum obtained is shown to have a competitive price support. Thus, we are able to use the socially optimal allocation to characterize the underlying competitive spatial equilibrium behavior.

We prove the competitive price supportability of an equal-treatment social optimum; the associated social welfare function ensuring equilibrium supportability employs, under a regularity condition, equal weights. To form an agglomerative city, standard conditions used in the monocentric city literature, such as those imposed on 'Rmu' [Wheaton (1974)], are not sufficient in this more general framework (see the detailed discussion in section 5 below). Rather, we find that when the utility function is separable in leisure and goods consumption, the residential relocation process is agglomerative if the immobile and mobile goods are *weakly Pareto complementary.* Examples with well-behaved utility functions are established in which a deglomerative city may emerge even when the Rmu conditions are satisfied. Furthermore, in this model, the shadow price of land is found always to decrease in the distance measure of a location from the CBD even though the relocation process is not agglomerative, contrary to findings in previous work.

The organization of the remainder of this paper is as follows. In section 2 we present a linear city model with spatial heterogeneities in endowments and preferences. Section 3 defines a competitive spatial equilibrium, while section 4 establishes a social planner's maximization problem and constructs an equal-treatment social optimum that can be supported by equilibrium prices. The condition which results in an agglomerative linear city is studied in section 5. Section 6 is a summary of the main conclusions.

## 2. The model

In the present work, we consider a pure-exchange, two-region, linear city model. Different from the standard factory town literature, this paper uses heterogeneity in regional commodity endowments and preferences to create an incentive to trade between regions. In an economy without trust [Gale (1978)], intermediate transactions are not allowed; hence, there will exist a transactions concentrated central business district (CBD) to which every consumer travels. Under a simplifying symmetry assumption, one can set the CBD at the geographical center. Therefore, there will be a trade-off between

[2] In standard commuting-cost models, locational choice affects utility indirectly through consumption of (mobile and immobile) commodities. For models with a location-dependent preference structure, the reader is referred to Scotchmer (1985) and a small selection of papers cited by Berliant et al. (1990).

gains from trade and space congestion, which endogenously determines the residential relocation process of the initially uniformly distributed consumers.

The landscape is bounded, homogeneous, and linear; it is represented by the closed interval $Z=[-1,1]$. In contrast to the long-narrow city constructed by Solow and Vickery (1971), consumers residing at the east and west sides of the city are heterogeneous in preferences. Let the two types of a continuum of consumers (or households) be indexed by $h(h\in H=\{1,2\})$. Let consumers of type 1 reside in $Z^1=[-1,0]$ and let consumers of type 2 reside in $Z^2=[0,1]$. The population of type-$h$ consumers is $N^h(h\in H)$, and $N^1+N^2=N$. Notice that this linear landscape differs from that in de Palma and Papageorgiou (1988) in which the two zones represent the center and the suburb.

Each of the type-$h$ consumers has an endowment of a region-specific mobile commodity, $e_h\in R_+, 0<e_h<\infty$, and is endowed with a unit time endowment. The only market activity of labor in this pure-exchange economy is the travel effort to the CBD expended to transact. Consumers of the same type have identical preferences, represented by a utility function which is separable in (mobile and immobile) goods consumption and leisure. Assume that all consumers have identical preferences for leisure. This, together with the uniform time endowment, will imply that the unique endogenous marketplace (CBD) must be in the geographical center, $\{0\}$ [see the discussion in Wang (1990)].

For simplicity, we assume that there is no relocation cost nor physical transportation cost. That is, all consumers will carry goods to the CBD to trade. This results in a utility cost of travel time; however, they are free to choose location, and carrying goods across locations bears no pecuniary cost. Further assume that everyone has a region-specific 'knowledge' to collect the home good so that no one has an incentive to change habitation region.[3] Hence, the relocation process occurs only within each region. In other words, a type-$h$ consumer can be regarded as a good-$e_h$ collector or a region-$Z^h$ resident. To close the model, an *absentee* landlord is introduced. The landlord is endowed with the whole landscape of measure $2Q$, with density $Q$ at each location, and consumes mobile commodities only.

This model structure can be regarded as a neoclassical version of Ricardian trade theory. It focuses on gains from trade which is implicitly motivated by comparative advantage between the two types of traders (consumers). In effect, one may imagine that such an advantage is due to production specialization (as in standard trade models) or heterogeneous immobile resources (as in traditional port-city models). Consider the eastern region as an apple orchard and the western region as a wheat field. When

[3]If consumers are allowed to move to another region, free riding may result in non-existence of a spatial equilibrium, even under the standard assumptions of the general equilibrium literature [see the discussion in Bewley (1981)].

apple and wheat are necessities to both types of consumers, agents from each region have to transport these mobile goods to a trade node (CBD). The delivered price of each good depends not only on the mill price but on the transportation cost, which is measured here by the disutility of travel. Suppose apples are more costly to transport than wheat. Then the gap between the delivered and the mill price for apples must be larger. This can be taken into account by giving the apple orchard residents a higher preference toward leisure. Thus, a simple modification of our model can indeed capture the spatial features of the traditional pecuniary transportation/commuting cost model. Furthermore, it is noteworthy that the framework developed here is a simplification of a general $N$-good model in which some mobile goods are *not* available in every location.

Let $(h,z)$ indicate a representative consumer of type $h$ who finally resides at $z$, $z \in Z^h$. Assume that travel effort is constant-return-to-scale in the travel distance with a travel time–distance coefficient $\frac{1}{2}$.[4] Thus an $(h,z)$-consumer needs to travel a round-trip distance of $2|z|$ and the corresponding travel time can be measured by $|z|$. This enables everyone to travel from their home to the CBD given a unity time endowment. Define an $(h,z)$-consumer's consumption of (mobile) good $i$ $(i=1,2)$, land at $z$, and leisure time as $c_i(z)$, $q(z)$, and $l(z)$, respectively. Furthermore, denote by $n(z)$ the final population density of all $(h,z)$-consumers (or, equivalently, population density at $z$).

An $(h,z)$-consumer's utility function can be specified as

$$U^h(c_1,c_2,q,l)=u^h(c_1,c_2,q)+v(l), \tag{1}$$

where the separable utility term for leisure $(v)$ is assumed common to both types of consumers.[5] In contrast to consumers, the absentee landlord's preference is represented by

$$U^{\mathrm{L}}(c_1^{\mathrm{L}},c_2^{\mathrm{L}})=u^{\mathrm{L}}(c_1^{\mathrm{L}},c_2^{\mathrm{L}}), \tag{2}$$

where the superscript L indicates the landlord. We now assume

*Assumption 1* (Well-behaved preferences). $u^h$, $v$, and $u^{\mathrm{L}}$ are strictly increasing, twice continuously differentiable, and strictly concave. There is a $\bar{c}_i>0$ $(i=1,2), \bar{c}_i \leqq N^i e_i/N$, such that, for all $\{c_1,c_2\}$ with $\prod_{i\in H} c_i=0, u^h(\bar{c}_1,\bar{c}_2,q)>u^h(c_1,c_2,q)$.

The second half of Assumption 1 implies that both mobile goods are

[4]Specifically, (travel time)$=(1/2)$ (round-trip travel distance). Therefore, travel time is nothing but the one-way travel distance.

[5]Notice that a social optimum might not exist if $v$ is not separable from $u$. A counter-example to the existence of social optima will be provided in section 4 below.

*necessary* to consumers, thus ruling out the potential autarkic allocation resulting from a no-trade equilibrium.[6]

The time constraint for an $(h,z)$-consumer can be written as

$$|z|+l(z)=1. \tag{3}$$

The location-by-location material balance condition for land, in the absence of idle land, is

$$n(z)q(z)=Q. \tag{4}$$

Material balance conditions for mobile good $i$ ($i=1,2$) are governed by

$$\int_Z n(z)c_i(z)\,\mathrm{d}z=(1-\theta_i)N^i e_i, \tag{5}$$

where $\theta_i=c_i^{\mathrm{L}}/(N^i e_i)$ represents the fraction of good $i$ consumed by the absentee landlord. Finally, the population identity is

$$\int_{Z^h} n(z)\,\mathrm{d}z=N^h. \tag{6}$$

Thus an *allocation* can be defined as a list of positive measurable functions, specifying quantities of mobile goods, land, and leisure consumed by consumers at each location; the associated population density; and the fraction of mobile goods consumed by the absentee landlord: $(\{\{c_i(z)\},q(z),l(z),n(z)\};\{\theta_i\})$.

## 3. Competitive spatial equilibrium

In this model, leisure that is location-dependent at an optimum or equilibrium enters into consumers' utility functions directly. Within the continuum framework, it is known that a competitive spatial equilibrium might not exist and a well-defined finite approximation might not be constructed [Berliant and ten Raa (1991)]. To circumvent this difficulty, we focus mainly on the social planner's problem and use market equilibrium only to justify a proper social welfare weighting scheme.

Normalize the region-1 mobile good as the numeraire. Let $p$ and $r(z)$ denote, respectively, the uniform price for mobile good 2 and land $q$ (both prices are in units of numeraire). Then, one can write consumers' budget constraints as

$$c_1^h+pc_2^h+r(z)q^h=E^h, \tag{7}$$

[6]This condition is weaker than the standard Inada condition used in the literature [for example, see de Palma and Papageorgiou (1979)].

where $E^h$ denotes a type-$h$ consumer's endowment income: $E^1 = e_1$ and $E^2 = pe_2$. So the representative consumer of each type will maximize his/her utility defined as in (1) subject to the corresponding budget constraint (7). Let $\mu^h$ be the Lagrange multiplier associated with a type-$h$ consumer's budget constraint. Then the first-order conditions can be derived as

$$u^h_{c_1} = \mu^h, \tag{8a}$$

$$u^h_{c_2} = \mu^h p, \tag{8b}$$

$$u^h_q = \mu^h r(z), \tag{8c}$$

where subscripts of $u$ denote partial derivatives in the usual manner.

To determine residential locations, one may require the Muth (1969) condition be satisfied:

$$\frac{\mathrm{d}r}{r\,\mathrm{d}|z|} = -\frac{v_l}{u_q q}, \tag{9}$$

from which one can immediately characterize the land rent profile: rent is decreasing in the distance from residence to the CBD. To obtain the land rent gradient, one needs a constant of integration, which involves lengthy discussion concerning the so-called closed city and open city models [see the survey by Wheaton (1979)]. In the present paper, consumers of the same type are completely identical (in both preferences and endowments). Thus, one may just adopt the Alonso (1964) condition:

$$U^h(c_1 . c_2, q, l) = U^h_0 \text{ (constant for all } z \in Z^h), \tag{10}$$

which can be referred to as a *locational equilibrium condition.*

Let the rent collection be $R = Q \int_Z r(z)\,\mathrm{d}z$. Then the landlord will maximize utility (2), subject to $c^L_1 + pc^L_2 = R$. Notice that this landlord maximization will recursively determine the landlord's shares of the mobile goods, $\{\theta_i\}$. Now, we can define a *competitive spatial equilibrium* as an allocation $(\{\{c^e_i(z)\}, q^e(z), l^e(z), n^e(z)\}; \{\theta^e_i\})$, with a list of positive prices $(p, \{r(z)\})$ for given space structure, endowments, and preferences, such that (a) consumers achieve maximal satisfaction, i.e. (8a)–(8c) hold; (b) the landlord achieves maximal satisfaction; and (c) the time constraint, (3), material balance conditions, (4)–(5), the population identity, (6), budget constraints, (7), and the locational equilibrium condition, (10), hold.

## 4. Social optimum

We are unable to establish standard general equilibrium properties (existence and efficiency) for competitive spatial equilibrium. Nonetheless, we can follow Negishi (1960) and Wang (1990) to study the social planner's problem and then find a social optimum that has a competitive price support.

Consider a simple utilitarian social welfare function :

$$SW=\sum_{h\in H}\int_{Z^h}\omega^h(z)n(z)[u^h(c_1(z),c_2(z),q(z))+v(l(z))]\,\mathrm{d}z+\psi u^{\mathrm{L}}(\{\theta_i N^i e_i\}), \tag{11}$$

where $\omega^h(z)$ is the social welfare weighting factor associated with the $(h,z)$-consumer and $\psi$ is the social welfare weighting factor associated with the landlord.

A *social optimum* is an allocation $(\{\{c_i^*(z)\},q^*(z),l^*(z),n^*(z)\};\{\theta_i^*\})$ which maximizes the social welfare defined as in (11), subject to constraints (3)–(6). If one ignores the problem with sets of zero measure associated with the continuum setup, one may apply standard concave programming techniques to verify the existence of a social optimum, following Mirrlees (1972) and Wang (1990). To ensure this existence property, it is, meanwhile, necessary to have the location-dependent variable, leisure, separated from goods consumption. When the utility function is not separable in goods and leisure, one can easily establish a counter-example to the existence of social optima: $U^1=c_1^{1/3}(c_2+1-|z|)^{1/3}q^{1/3}$ and $U^2=(c_1+1-|z|)^{1/3}c_2^{1/3}q^{1/3}$. To be specific, let $\gamma(z)$, $\lambda_i$, and $\eta$ be, respectively, the Lagrange multipliers associated with (4), (5), and (6), where $\lambda$ and $\eta$ are constant across locations. For convenience, consider the equally weighted social welfare maximization problem as in Mirrlees (1972) [i.e. $\omega^h(z)=1$]. Suppose such a social optimum exists. Straightforward application of concave programming techniques leads to $U^h(\{c_i(z)\},q(z),1-|z|)-\sum_{i\in H}U^h_{c_i}c_i(z)-U^h_q q(z)=\eta^h$. However, by manipulation, one obtains $(1-|z|)=\eta^1/\lambda_2=\eta^2/\lambda_1$, which cannot hold up for all $z$.

In order to link the social optimum with a spatial equilibrium in which no one has the incentive to change his/her habitation, we restrict our attention to an *equal-treatment social optimum* such that the locational equilibrium condition, (10), holds. It has been pointed out by Mirrlees (1972) that, in an economy with identical consumers, an equally weighted social optimum may not be associated with a uniform utility level across all locations (within $Z^h$). Such an 'unequal treatment of equals' paradox has received increasing attention [see the survey by Wheaton (1979)].[7]

The present paper endogenously determines a set of social welfare weights such that an equal-treatment social optimum can be supported by equili-

[7]Recently, Wildasin (1986) has noted that this paradox disappears if the income elasticity of land demand is zero.

brium prices. Substituting (3) into (11) and applying the Kuhn–Tucker theorem, an interior social optimum satisfies the following conditions:

$$\omega^h(z)u^h_{c_i}(\{c_i^*(z)\}, q^*(z)) = \lambda_i^*, \tag{12a}$$

$$\omega^h(z)u^h_q(\{c_i^*(z)\}, q^*(z)) = \gamma^*(z), \tag{12b}$$

$$\omega^h(z)[u^h(\{c_i^*(z)\}, q^*(z)) + v(1-|z|)] - \sum_{i\in H} \lambda_i^* c_i^*(z) - \gamma^*(z)q^*(z) = \eta^{h*}, \tag{12c}$$

$$\psi u^{\mathrm{L}}_{c_i^{\mathrm{L}}}(\{N^i e_i \theta_i^*\}) = \lambda_i^* N^i e_i. \tag{12d}$$

Eqs. (12a) and (12b) are the no-arbitrage conditions for mobile and immobile goods, respectively. Eq. (12c) specifies the leisure–travel tradeoff, while (12d) determines the landlord's share of mobile goods consumption.

Applying the technique developed by Wang (1990), one can link an equal-treatment social optimum characterized by (3)–(6), (10), and (12a)–(12d) with a competitive spatial equilibrium characterized by (3)–(7), (10), (8a)–(8c) and the landlord optimization condition.[8] Since landlord maximization is entirely recursive to both social optima and spatial equilibria, one can always obtain a weight, $\psi$, to generate a set of landlord's shares of mobile goods with $\{\theta_i^e = \theta_i^*\}$, depending recursively on the social welfare weight, $\psi$, and consumer's marginal rates of substitution between land and mobile commodities. Landlord consumption is, therefore, irrelevant to the consumer's optimization and will be omitted from the rest of the paper.

Thus, we restrict our attention to the comparison between the first-order conditions for the consumers, (8a)–(8c) versus (12a)–(12c), given (7). From (8a)–(8c) and (12a) and (12b), one obtains

$$\lambda_1 = \omega^h(z)\mu^h, \tag{13a}$$

$$\lambda_2 = \omega^h(z)\mu^h p, \tag{13b}$$

$$\gamma(z) = \omega^h(z)\mu^h r(z). \tag{13c}$$

These, together with (7), (10), and (12c), imply

$$\frac{\eta^h}{\omega^h(z)} = u^h(\cdot) + v(\cdot) - \mu^h[c_1^h + pc_2^h + r(z)q^h(z)] = U_0^h - \mu^h E^h, \tag{14}$$

from which we can conclude that $\omega^h(z)$ must be constant for all $z \in Z^h$. That is, all identical consumers will be treated equally in this travel-time model

[8]Different from Wang (1990), this paper imposes no pecuniary transportation cost, but allows for an endogenous determination of the immobile goods consumption. Nevertheless, the procedure of the proof below is in analogy to Wang (1990, theorem 2).

regradless of their residential locations, contrary to the standard commuting time models [such as Wildasin (1986) and the survey by Wheaton (1979)]. Notice, however, that this 'equal treatment of equals' conclusion may not obtain if the utility functions are not separable in goods consumption and the location-dependent variable, leisure.

For convenience, we select and normalize the sizes of population: $(N^1, N^2) \in A = \{(N^1, N^2) | \eta^h/(U_0^h - \mu^h E^h) = 1, h = 1, 2\}$.

*Assumption 2* (Regularity condition). $A \neq \emptyset$.

Hence, for all $(N^1, N^2) \in A$, we can set $\omega^h(z) = 1$ for all $z$ and $h$ such that the associated equal-treatment social optimum can be supported by competitive equilibrium prices, in Negishi's (1960) sense.

## 5. Characterizing the relocation process

The competitive price supportability of an equal treatment social optimum enables us to circumvent an analytical difficulty associated with market equilibrium by focusing only on the social planner's problem. In this section we characterize the socially optimal allocation obtained.

First we define the agglomerative process of residential relocation. Consider the spatial economy as described above with a population initially distributed uniformly. Such a spatial economy is said to be *agglomerative with respect to the CBD* if the initial (uniform) population distribution is a mean-preserving spread of the optimal population distribution over the landscape $Z$.[9] A sufficient condition yielding relocation agglomeration is that $n^*(z)$ be monotone decreasing in $|z|$; the resulting optimal town is thus associated with a population distribution in which consumers relocate toward the CBD, $\{0\}$.

For $(N^1, N^2) \in A$, we have $\omega^h(z) = 1$ (for all $z$ and $h$) and thus combining (12a)–(12c) gives

$$u^h(\{c_i^{h*}\}, q^{h*}) + v(1 - |z|) - \sum_i u_{c_i}^h(\cdot) c_i^{h*} - u_q^h(\cdot) q^{h*} - \eta^{h*} = 0. \tag{15}$$

Notice that $\lambda$ and $\eta$ are location-independent. Given $\omega^h(z) = 1$, total differentiation of (4), (10), (12a), (12b) and (15) with respect to $z$ yields

$$\sum_i u_{c_i}^h \frac{dc_i^{h*}}{dz} = -u_q^h \frac{dq^{h*}}{dz} + v_l \frac{d|z|}{dz}, \tag{16a}$$

[9]Let $F(z)$ be the population distribution function [i.e. $F(z) = \int_{-1}^{z} n(s)\,ds$]. Suppose that $\int_{-1}^{1} G(z)\,dz = \int_{-1}^{1} F(z)\,dz$ and that $\int_{-1}^{b} [G(z) - F(z)]\,dz \geq 0$ for all $b \in [-1, 1]$ with strict inequality for some $b \in [-1, 1]$. Then $G$ is called a mean-preserving spread of $F$.

$$\frac{\mathrm{d}q^{h*}}{\mathrm{d}z} = -\frac{q^{h*}}{n^*}\frac{\mathrm{d}n^*}{\mathrm{d}z}, \tag{16b}$$

$$\sum_j u^h_{c_i c_j}\frac{\mathrm{d}c_j^{h*}}{\mathrm{d}z} + u^h_{c_i q}\frac{\mathrm{d}q^{h*}}{\mathrm{d}z} = 0, \tag{16c}$$

$$\frac{\mathrm{d}\gamma^*}{\mathrm{d}z} = \sum_i u^h_{c_i q}\frac{\mathrm{d}c_i^{h*}}{\mathrm{d}z} + u^h_{qq}\frac{\mathrm{d}q^{h*}}{\mathrm{d}z}, \tag{16d}$$

$$-\sum_j\left(u^h_{c_j q}q^{h*} + \sum_i u^h_{c_i c_j}c_i^{h*}\right)\frac{\mathrm{d}c_j^{h*}}{\mathrm{d}z} - \left(u^h_{qq}q^{h*} + \sum_i u^h_{c_i q}c_i^{h*}\right)\frac{\mathrm{d}q^{h*}}{\mathrm{d}z} = v_l\frac{\mathrm{d}|z|}{\mathrm{d}z}. \tag{16e}$$

Multiplying (16c) by $c_i^{h*}$ and summing it up for all $i$ leads to

$$\sum_i\sum_j u^h_{c_i c_j}c_i^{h*}\frac{\mathrm{d}c_j^{h*}}{\mathrm{d}z} + \sum_i u^h_{c_i q}c_i^{h*}\frac{\mathrm{d}q^{h*}}{\mathrm{d}z} = 0. \tag{17}$$

Define by $\varepsilon^h = -u^h_{qq}q^{h*}/u^h_q$ the elasticity of the marginal utility of land. To simplify the analysis, we assume

*Assumption 3* (Constant cross-elasticity). The cross-elasticity of marginal utilities of land and mobile goods, $k^h = u^h_{c_i q}q^{h*}/u^h_{c_i}$, is constant across locations.[10]

We then substitute (16a), (16b), and (17) into (16e) to obtain

$$u^h_q q^{h*}(k^h + \varepsilon^h)\frac{\mathrm{d}n^*}{\mathrm{d}z} = -n^*(1 + k^h)v_l\frac{\mathrm{d}|z|}{\mathrm{d}z}. \tag{18}$$

We next assume

*Assumption 4* (Weak Pareto complementarity). The cross-partial derivative, $u^h_{c_i q}$, is non-negative for $i = 1, 2$.

That is, mobile goods and land are assumed to be weakly Pareto complementary, implying that the marginal utilities of mobile good consumptions increase in consumption of space.[11]

We are led to the following proposition.

[10]Most utility functions used in the literature satisfy this assumption. For instance, all utility function forms specified in Examples 1–3 below have constant cross-elasticities.

[11]This condition can also be named after Edgeworth or Fisher [see Samuelson (1947)]. For example, a generalized Cobb–Douglas utility function or any homogeneous (of degree one) utility function will satisfy this condition.

*Proposition 1. Under Assumptions 1–4, an optimal city is agglomerative.*

*Proof.* Under Assumptions 1–3, an optimal city satisfies the relation as described by (18) above. Under Assumption 4, $k$ is non-negative and, by strict concavity of $u$, $\varepsilon^h > 0$ for all $h \in H$. Hence, (18) implies

$$\operatorname{sign}\left(\frac{\mathrm{d}n^*}{\mathrm{d}z}\right) = -\operatorname{sign}\left(\frac{\mathrm{d}|z|}{\mathrm{d}z}\right) = -\operatorname{sign}(z), \quad \text{for all } z \neq 0. \tag{19}$$

Therefore, $n^*(z)$ is monotone decreasing in $|z|$ and an agglomerative optimal town results. Q.E.D.

Consider a generalized version of the Wheaton (1974) conditions on the Rmu [i.e. marginal rate(s) of substitution]: $\mathrm{d}(u_q/u_{c_i})/\mathrm{d}c > 0$, $\mathrm{d}(u_{c_i}/u_q)/\mathrm{d}q > 0$, $i = 1, 2$, which essentially require all goods to be normal. These conditions ensure $k^h + \varepsilon^h > 0$. Nonetheless, the right-hand side of (18), especially $1 + k^h$, remains unsignable under Rmu conditions. In other words, Rmu conditions are not sufficient to lead to agglomeration in this more general spatial economy with multiple mobile goods and with location entering into the utility function directly.[12]

For completeness, we summarize all possible results concerning residential relocation process in the following proposition.

*Proposition 2. Assuming Assumptions 1–3, we have the following:*

(a) *when mobile and immobile goods are either sufficiently substitutable so that* $k^h < \min\{-1, -\varepsilon^h\}$ *or sufficiently complementary so that* $k^h > \max\{-1, -\varepsilon^h\}$, *an optimal city is agglomerative.*

(b) *when mobile and immobile goods are moderately substitutable so that* $\min\{-1, -\varepsilon^h\} < k^h < \max\{-1, -\varepsilon^h\}$, *an optimal city is deglomerative.*

*Proof.* It is an immediate consequence of (18). Q.E.D.

We have shown that a linear city can be agglomerative in a competitive general equilibrium framework under proper assumptions; neither positive interaction externalities nor increasing returns are required. Intuitively, those consumers residing in the outer areas generally need to consume more goods in order to balance out the associated disutility of travel time. Weak Pareto complementarity implies that they would consume more of both mobile and immobile goods. Thus, the outer consumers end up with higher land

[12]Also note that Rmu conditions might not be implied by the quasi-concavity assumption on preferences if there is more than one mobile good. See theorem 1 in Wheaton (1974) for a discussion of the single mobile good case.

consumption. This leads to relocation agglomeration since the population measure at each location is negatively related to per capita land consumption for given (fixed) land supply at each location. Consequently, a standard monocentric city structure emerges, in which the optimal population density is monotone decreasing in $|z|$. It arises due to gains from trade rather than interaction externalities or market imperfections. It is also noteworthy that the magnitude of $\varepsilon$ may vary across locations. Thus, a polycentric city may obtain depending on individual's preference structure, in particular, the cross-elasticity of marginal utilities of mobile and immobile goods.

We next provide three useful examples with well-behaved utility functions: the first shows agglomeration under weak Pareto complementarity; the second demonstrates that weak Pareto complementarity is, though sufficient, not necessary; and the last one presents a case of deglomeration even when the Rmu conditions are satisfied.

*Example 1* (Agglomeration under weak Pareto complementarity). Let all consumers have the same utility function given by $U=u(c_1,c_2,q)+v(l)=c_1^a c_2^b q^{1-a-b}+ml^\delta$, with $a,b,m>0$, $a+b\in(0,1)$ and $\delta\in(0,1)$. In this case, $k^h=1-a-b$ and $\varepsilon^h=a+b$. Thus, (18) implies $\mathrm{d}n^*/\mathrm{d}z=-n^*(2-a-b)[v_l/(u_q q^*)](\mathrm{d}|z|/\mathrm{d}z)$. So $\mathrm{sign}(\mathrm{d}n^*/\mathrm{d}z)=-\mathrm{sign}(z)$ and an agglomerative linear city emerges.

*Example 2* (Agglomeration without weak Pareto complementarity). Let all consumers have the same utility function given by $U=u(c_1,c_2,q)+v(l)=\ln[(c_1^a c_2^{1-a})^\rho+bq^\rho]^{1/\rho}+ml^\delta$, with $b,m>0$, $\rho<1$, $a\in(0,1)$ and $\delta\in(0,1)$. Straightforward derivations lead to $k^h=-\rho bq^\rho/B$ and $\varepsilon^h=[(1-\rho)(c_1^a c_2^{1-a})^\rho+bq^\rho]/B$, where $B=(c_1^a c_2^{1-a})^\rho+bq^\rho$. Since $k^h<0$, we learn that mobile goods and land are Pareto substitutes. However, from (18), one can get $\mathrm{d}n^*/\mathrm{d}z=-[n^*/(1-\rho)]\{[(c_1^a c_2^{1-a})^\rho+(1-\rho)bq^\rho]/B\}[v_l/(u_q q^*)](\mathrm{d}|z|/\mathrm{d}z)$. Thus, sign $(\mathrm{d}n^*/\mathrm{d}z)=-\mathrm{sign}(z)$ and we still obtain an agglomerative linear city without Assumption 4.

*Example 3* (Deglomeration under the Rmu conditions). Let all consumers have the same utility function given by $U=u(c_1,c_2,q)+v(l)=-(c_1c_2)^{-a}q^{-(1+b)}+ml^\delta$, with $m>0$, $a>b>0$ and $\delta\in(0,1)$. This type of utility function is uncommon in the urban economics literature; however, it has been used in the uncertainty literature with the degree of relative risk aversion being greater than unity. Under this preference structure, $k^h=-(1+b)$ and $\varepsilon^h=2+b$. Thus, $\mathrm{sign}[\mathrm{d}(u_{c_i}/u_q)/\mathrm{d}q]=\mathrm{sign}[k^h+\varepsilon^h]=\mathrm{sign}(1)>0$ and $\mathrm{sign}[\mathrm{d}(u_q/u_{c_i})/\mathrm{d}c_i]=\mathrm{sign}(a-b)>0$; the Rmu conditions hold. We then use (18) to derive $\mathrm{d}n^*/\mathrm{d}z=n^*b[v_l/(u_q q^*)](\mathrm{d}|z|/\mathrm{d}z)$ and, consequently, $\mathrm{sign}(\mathrm{d}n^*/\mathrm{d}z)=\mathrm{sign}(z)$. A deglomerative linear city, therefore, forms even under the Rmu conditions.

Next we characterize the shadow price of land, $\gamma^*(z)$. Notice that this shadow price differs from the bid-rent function, $r^e(z)$, only by a constant multiplier, $\mu$ [see (13c) with $\omega^h(z)=1$].

*Proposition 3. Under Assumptions 1–3, the shadow price of land associated with the equal-treatment social optimum is a decreasing function of the distance of a location from the CBD.*

*Proof.* Under Assumptions 1–3, an optimal city satisfies (16)–(18). Multiplying (16d) by $q^{h*}$ and manipulating it using (16a), (16b), and (18), one obtains

$$\frac{\mathrm{d}\gamma^*}{\mathrm{d}z}=\frac{-n^*v_l}{q^{h*}}\frac{\mathrm{d}|z|}{\mathrm{d}z}. \tag{20}$$

Hence, we have

$$\operatorname{sign}\left(\frac{\mathrm{d}\gamma^*}{\mathrm{d}z}\right)=-\operatorname{sign}\left(\frac{\mathrm{d}|z|}{\mathrm{d}z}\right),$$

which completes the proof. Q.E.D.

In Proposition 3 we have shown that the closer a location is to the CBD, the higher is the shadow price of land. Such a relation holds up even when a city is not agglomerative. This provides theoretical support to the empirical evidence that in a city, population density and shadow rent need not be positively correlated.

Finally, when the location-dependent factor (leisure) does not enter the utility function, $v_l=0$ and using (18) and (20), a uniform populaton distribution results and the shadow price of land is flat, even with spatial heterogeneity. On the other hand, if a positive geographical factor that partially offsets the disutility of travel time is introduced, one may obtain a very complex population distribution, but shadow rent is always higher toward the CBD as long as the net marginal utility of location-dependent factors is not nil. These observations seem richer than those obtained from a simple commuting-cost model and can help explain why cities with different relocation processes may occur in the real world.

## 6. Conclusions

This paper constructs a general equilibrium model with consumers heterogeneous in endowments and tastes that generates an agglomerative city. An equal-treatment social optimum is found to have a competitive price support under a regularity condition. The socially optimal relocation process is agglomerative if the immobile good and mobile goods are weakly Pareto complementary. In contrast with the existing literature, neither increasing returns in production nor interdependence of preferences is imposed. More surprisingly, a city may be deglomerative even under standard conditions used in the monocentric city models (such as the Rmu conditions). Moreover, the shadow price of land is found to be higher toward the CBD, independent of the conditions that generate agglomeration.

This paper has characterized the socially optimal and competitively supportable residential relocation process. The analysis developed relies crucially on the separability of the utility function, an assumption that is not normally imposed in the general equilibrium literature. Along these lines, it seems worthwhile for future work to seek a set of regularity conditions such that this analysis can be applied to models with more general, well-behaved utility functions.

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