Labor-Market Development, Institutions and Inequality

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A. Introduction

The labor market plays a key role in the process of economic development *(Porzio-Rossi-Santangelo 2022*). The topic covers many dimensions:

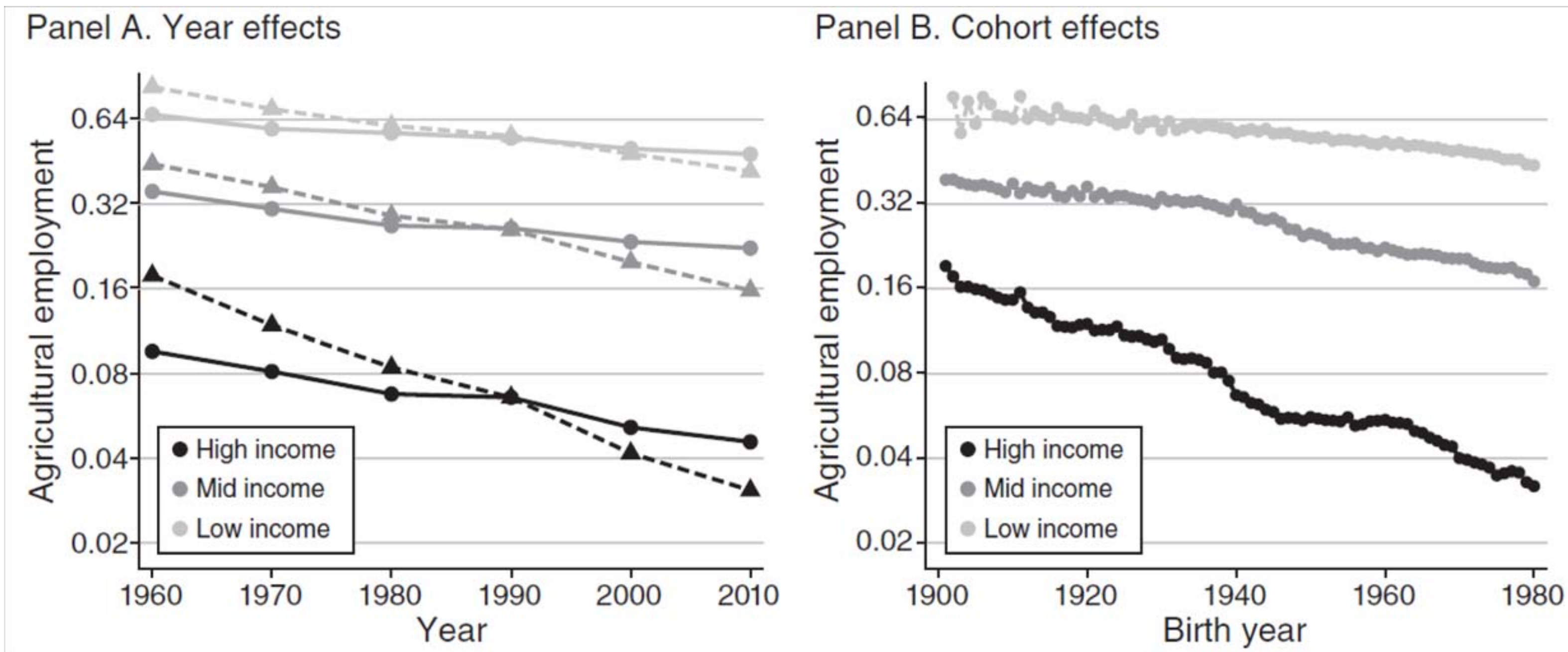
- **Educational Choice:**
  - **basics: Lucas (1988), Laing-Palivos-Wang (1995)**
  - **extensions: Glomm-Ravikumar (1992), Tsiddon (1992), LLoyd-Ellis (2000), Tamura (2001), Fender-Wang (2003), Chen-Chen-Wang (2011)**
- **Job training: Acemoglu (1997), Kuruscu (2006)**
- **OTJ learning: Lucas (1993), Laing-Palivos-Wang (1995, 2004)**
- **Occupational choice: Banerjee-Newman (1993), Grossman (2004), *Jiang-Wang-Wu (2010)*, Lu-Wang (2012), Alter-Lee-Wang (in progress)**
- **Locational choice:**
  - **stratification by human capital: Benabou (1996), Chen-Peng-Wang (2009)**
  - **rural-urban migration: *Lucas (2004)*, Bond-Riezman-Wang (2009), Garriga-Hedlund-Yang-Wang (2000), Liao-Wang-Wang-Yip (2000)**
- **Health: Acemoglu-Johnson (2007), *Bloom-Canning-Grahm (2010)*, Sholz-Seshadri (2010), Y. Wang (2012), Wang-Wang (2013), *Jones (2016)*, Chen-Wang-Yao (2017), *Eichenbaum-Rebelo-Trabandt (2020)* and *Acemoglu-Chernozhukov-Werning-Whinston (2020)*, Wang-Yao (2021)**
- **Misallocation/mismatch: Laing-Coulson-Wang (2004), Jovanovic (2014)**

Human capital is crucial for explaining economic growth, income inequalities and cross-/within-country productivity differences:

- **human capital accounts for a large % of growth (Krueger-Lindahl 2001):**
  - **measured by education enrollment (Benhabib-Spiegel 1994): 14-28%**
  - **measured by education attainment (Barro & Sala-i-Martin 1992): 20%**
  - **measured by differential attainment index (Tallman-Wang 1994): 20-45%**
- **cross-country/cross-region productivity differences: Lucas (1990, 2000), Caselli (2005), Prescott (1998), Basu-Weil (1998), Wang-Wong-Yip (2018)**
- **income inequality (skill premium, residual inequalities) and wealth inequality:**
  - **Aghion (2000), Violante (2002), Huggett-Ventura-Yaron (2007), Acemoglu-Dell (2009), Tang (2017)**
  - ***Jovanovic (2009, 2014)*, *Lucas-Moll (2014)*, *Burstein-Morales-Vogel (2015)*, Yang-Wang (2015), Cheng (2017)**
  - **Pikkety (2014), Jones-Kim (2014), *Gabaix-Lasry-Lions-Moll (2015)*, *Aghion-Akcigit-Bergeaud-Blundell-Hemous (2015)*, Benhabib-Bisin (2016), Kaymak-Poschke (2016), Lusardi-Michaud-Mitchell (2017), Wong (2018)**
  - **Economic institutions and inequalities: *Galor-Moav-Vollrath (2009)*, Cheng-Liu-Wang (in progress)**

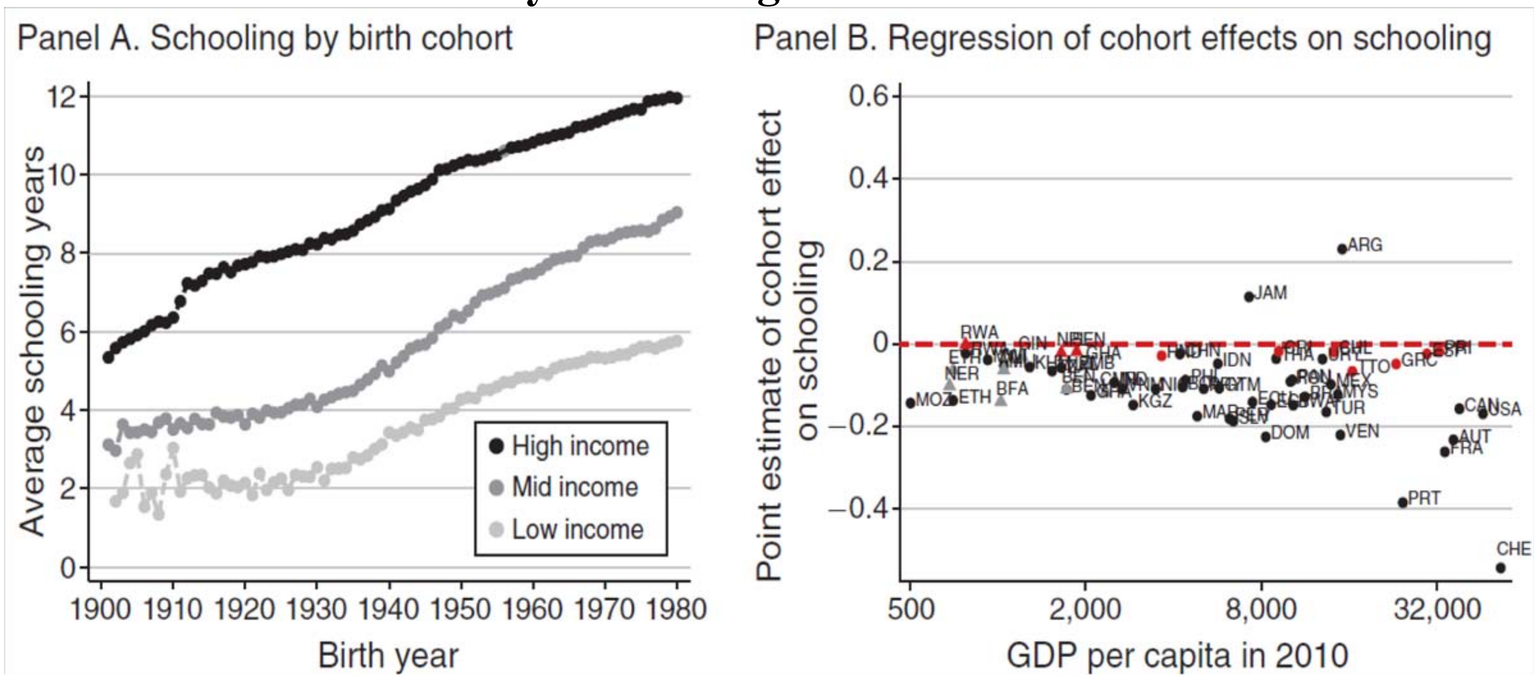
B. The Human Side of Structural Transformation: Porzio-Rossi-Santangelo (2022)

- **Caselli-Coleman (2001) stress that the supply of agricultural workers are important for structural change**
- **Based on IPUMS data (Integrated Public Use Microdata Series):**



  - ○ **aggregate structural transformation as a result of**
    - \- **over time labor shifting away from agriculture (year component)**
    - \- **younger cohorts less likely to enter agriculture (cohort component)**

- **more educated less likely to enter agriculture**



1. The Model

- **N+1 overlapping cohorts, indexed by c, each with a continuum workers of mass one working for N+1 periods over {c, c+1, ..., c+N}**
- **In agriculture, all workers have identical productivity**
- **In non-agriculture, workers supply efficiency units,** $h(c, \varepsilon) = h_c^{\gamma} \varepsilon^{1-\gamma}$**, depending on cohort specific productivity shifter h_c and individual ability ε**

- **Aggregate human capital with ε ~ β(ν,1), ν inversely related to within-cohort variability, and within-cohort distribution F(ε):** $H_t = \sum_{c=t-N}^{t} \int h(c,\varepsilon)dF(\varepsilon)$

- **Sectoral production (X = land, L = labor):**

$$p_t Y_{A,t} = p_t Z_{A,t} X^{\alpha} L_{A,t}^{1-\alpha},$$
$$Y_{M,t} = Z_{M,t} L_{M,t}.$$

- **Sectoral labor:**

$$L_{A,t} = \sum_{c=t-N}^{t} \int \omega_t(c,\varepsilon)dF(\varepsilon),$$
$$L_{M,t} = \sum_{c=t-N}^{t} \int h(c,\varepsilon)[1-\omega_t(c,\varepsilon)]dF(\varepsilon)$$

- **Labor demand:**

$$y_{A,t} = w_{A,t} = (1-\alpha)p_t Z_{A,t} X^{\alpha} L_{A,t}^{-\alpha}$$
$$y_{M,t}(c,\varepsilon) = w_{M,t} h(c,\varepsilon) = Z_{M,t} h(c,\varepsilon),$$

- **Occupational choice ω given ω_{c-1} = 1 (born in agriculture):**

$$\max_{\{\omega_t\}_{t=c}^{c+N}} \sum_{t=c}^{c+N} \beta^{t-c}\left[\omega_t y_{A,t} + (1-\omega_t) y_{M,t}(c,\varepsilon) - C(\omega_{t-1},\omega_t,y_{A,t},y_{M,t}(c,\varepsilon))\right]$$

where sectoral mobility involves a periodic cost *i* and a one-time fixed cost *f*

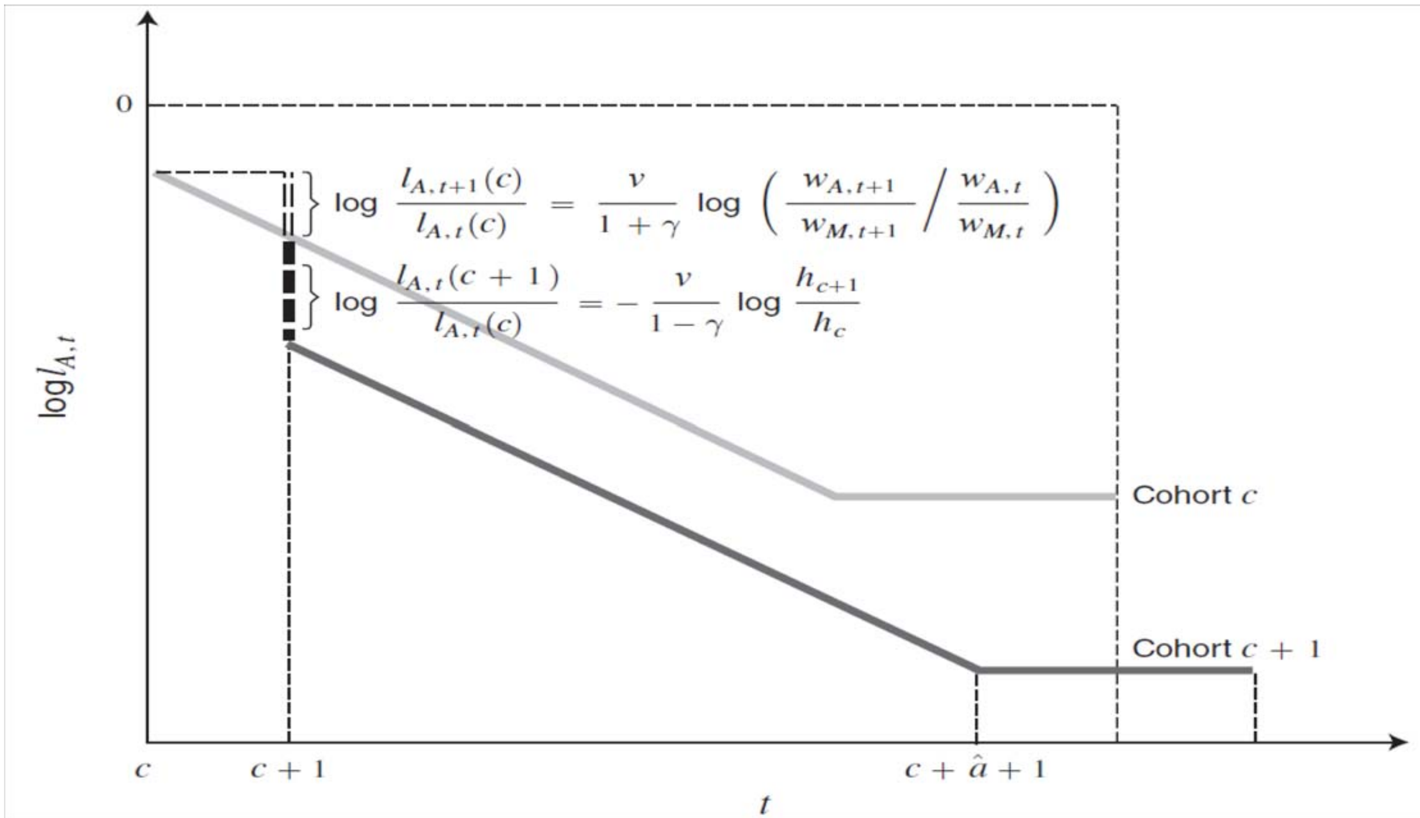
$$C(\omega_{t-1},\omega_t,y_{A,t},y_{M,t}(c,\varepsilon)) = \mathrm{I}(\omega_t = 0)\, i\, y_{M,t}(c,\varepsilon) + \mathrm{I}(\omega_t < \omega_{t-1}) f y_{M,t}(c,\varepsilon) + \mathrm{I}(\omega_t > \omega_{t-1}) f y_{A,t},$$

- **Cohort-level human capital depends on the relative value of wages:**
  $\log h_c^\gamma = \sigma \log \frac{V_{M,c}}{V_{A,c}} + \log \xi_c$, **where** $V_{x,c} = \sum_{t=c}^{c+N} \beta^t w_{x,t}$
- **Equilibrium relative price of agricultural good:**
  $\underbrace{\log p_t}_{\text{agr price}} = \eta \left(\underbrace{\log \theta_t}_{\text{demand}} + \underbrace{\eta_z \log z_t}_{\text{supply}} + \underbrace{\eta_H \log H_t}_{\text{human capital}} \right)$, **where** $z_t \equiv Z_{A,t}/Z_{M,t}$
  - **higher relative demand raises agr price ($\eta > 0$)**
  - **higher relative technology reduces agr price ($\eta_z < 0$)**
  - **higher H lowers agr price due to income and relative productivity effects**
- **Regularity conditions:**
  - **relative demand θ, relative technology z and cohort human capital shifter ξ_c all change at constant rates**
  - **i and f are bounded (nondegenerate worker masses of agriculture and switchers)**
  - **negative year component**
  - **human capital shifter never leads to more than proportional decrease in H**

- **Occupational choice gives labor allocation by cohort at time t, age $a_t(c)$=t-c:**

$$\log l_{A,t,c} = \begin{cases} \lambda(i,f) + \frac{v}{1-\gamma}\log\frac{w_{A,t}}{w_{M,t}} - \frac{v\gamma}{1-\gamma}\log h_c & \text{if } a_{t+1}(c) \leq \hat{a}(f) \\ \lambda(i,f) + \frac{v}{1-\gamma}\log\frac{w_{A,c+\hat{a}}}{w_{M,c+\hat{a}}} - \frac{v\gamma}{1-\gamma}\log h_c & \text{if } a_{t+1}(c) > \hat{a}(f) \end{cases}$$

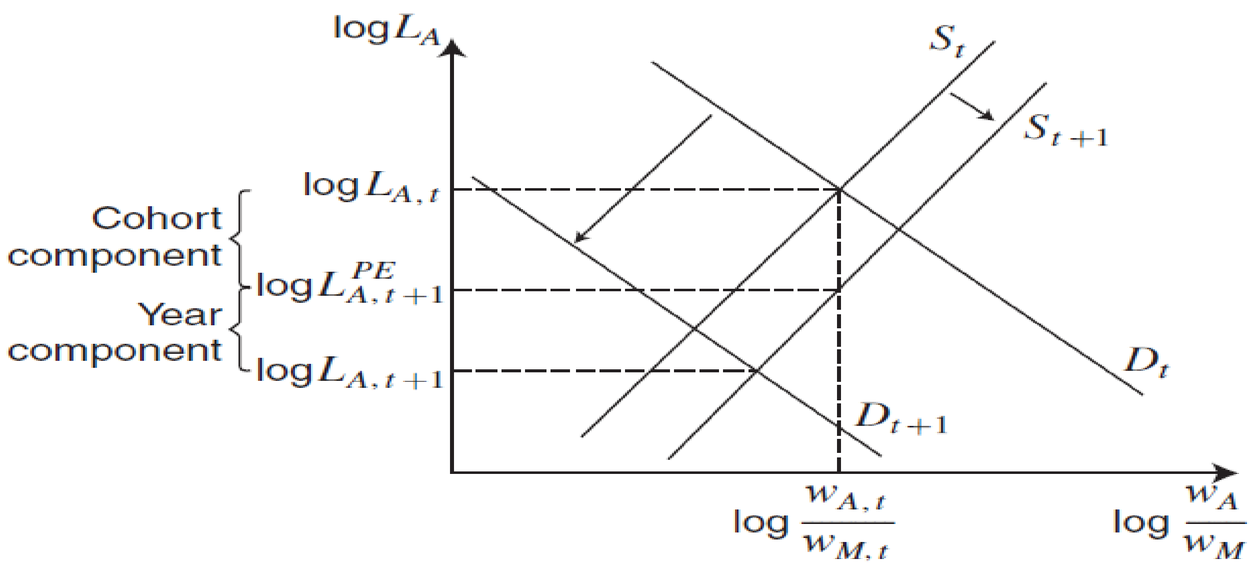
where after the threshold age, wages are flat



- **Labor reallocation out of agriculture:**

$$\log g_{L_A} = \frac{\nu\alpha}{1-\gamma+\alpha\nu}\underbrace{\left[\frac{1}{\alpha}(\log g_p + \log g_z)\right]}_{\text{Demand shift } (\mathbf{D})} + \frac{(1-\gamma)}{1-\gamma+\alpha\nu}\underbrace{\left[-\frac{\nu\gamma}{1-\gamma}\log g_h\right]}_{\text{Supply shift } (\mathbf{S})}$$

  - **demand for agricultural labor contains relative price & technology effects**
  - **agricultural labor supply depends on relative human capital growth**
- **Decompose into cohort and year components:**



- **Human capital growth:** $\gamma \log g_h = \frac{(1-\gamma+\alpha\nu)\log g_\xi - \sigma(1-\gamma)\log g_{\theta z}}{(1-\gamma)(1+\eta\eta_H\sigma)+\alpha\nu(1+\sigma)}$,

increases in cohort human capital shifter/decreases in relative demand shifter

C. Entrepreneurship: Jiang-Wang-Wu (2010)

- **Basic facts:**
  - **entrepreneurs measured by self-employed business owners (Survey of Consumer Finances) account for only 7.6% of the U.S. population but 1/3 of total net worth (Cagetti-De Nardi 2006)**
  - **entrepreneurs receive more than 20% of income of the entire US population (Mondragon-Velez 2006)**
  - **the International Social Survey Programme of 1989 shows about 63% of Americans, 48% of Britons and 49% of Germans desire to become entrepreneurs, but only 15% realize their dream**
- **Major factors affecting agent's choice (Kihlstrom-Laffont 1979):**
  - **entrepreneurial ability vs. labor skills**
  - **access to capital markets**
  - **individual attitude towards risk**
  - **preference bias toward "heroes" (Blanchflower-Oswald 1998)**
- **Barriers to entrepreneurship:**
  - **financial/liquidity constraints (Evans-Jovanovic 1989, Evans-Leighton 1989, Den Haan-Ramey-Watson 2003)**
  - **institutional barriers (risk-taking behavior, entrepreneurial rent)**

- **Literature:**
  - **Barnajree-Newman (1993): dual dynamics of wealth distribution and w**
  - **Piketty (1997): dual dynamics of wealth distribution & r**
  - **Aghion-Bolton (1997): dual dynamics of wealth distribution & composition of borrowers/lenders**
  - **Others studies: Galor-Zeira (1993), LLoyd-Ellis (2000), Ghatak-Jiang (2002), Fender-Wang (2003), Grossman (2004), and Mino-Shimomura-Wang (2004)**
- **Questions:**
  - **How would changes in the entrepreneurial ability affect occupational choice over time and macroeconomic performance in the long run?**
  - **Is there a positive relationship between entrepreneurship & economic growth?**
    - **conventionally – yes: financial development, entrepreneurs and growth are all positively related (King-Levine 1993)**
    - **new view – not necessarily: across OECD countries, no clear relationship between growth and the number of entrepreneurs (Blanchflower 2000)**

1. The Model

- **The Basic Environment**

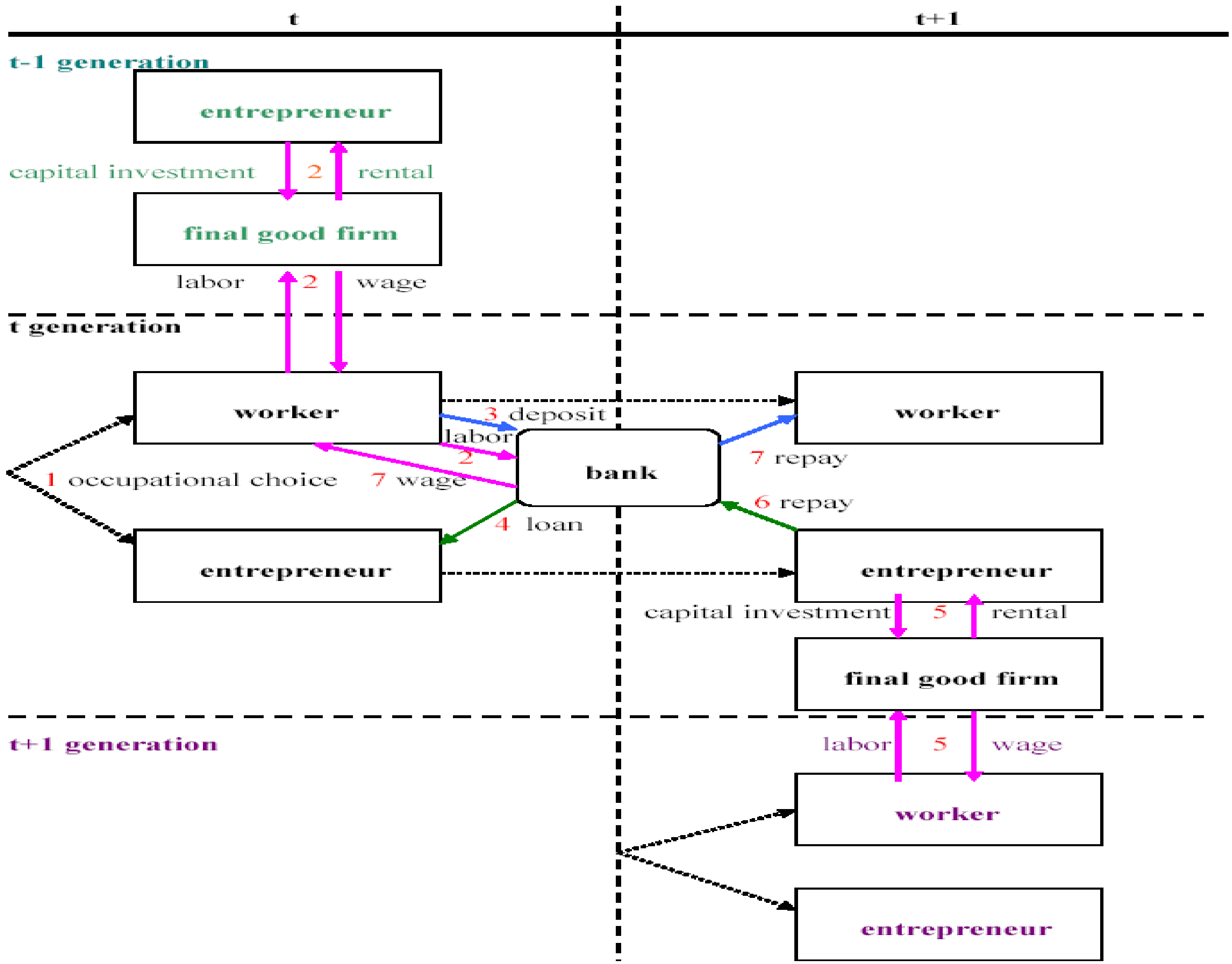
  - **all agents are born with same level of human capital h_t, which evolves according to:** $h_{t+1} = K_t^{\beta} h_t^{1-\beta}$
  - **agents have different entrepreneurial "implementation" ability τ (Lucas 1978) and different type of investment ideas if they are so implemented (Romer 1993), $\tau \sim F(\tau)$ over $[0,\tau^H]$ (τ^H may be ∞)**
  - **transformation of loan x into capital good k depends on own ability τ and average quality of entrepreneurs in the firm M_{t+1}:**

    $$k_{t+1}(\tau) = \tau \frac{M_{t+1}}{\Omega_{t+1}^{\zeta}} x_t$$

  - **Production function:** $Y_{t+1} = AK_{t+1}^{\alpha}\left(h_{t+1}L_{t+1}\right)^{1-\alpha}$ **where**

    - L_{t+1} **= (t+1)-generation workers' labor**
    - $K_{t+1} = \left(\int_0^{N_{t+1}} k_{t+1}(i)^{\theta}\, di\right)^{\frac{1}{\theta}}$ **= composite capital (variety matters)**

Figure 1: The Structure of the Economy

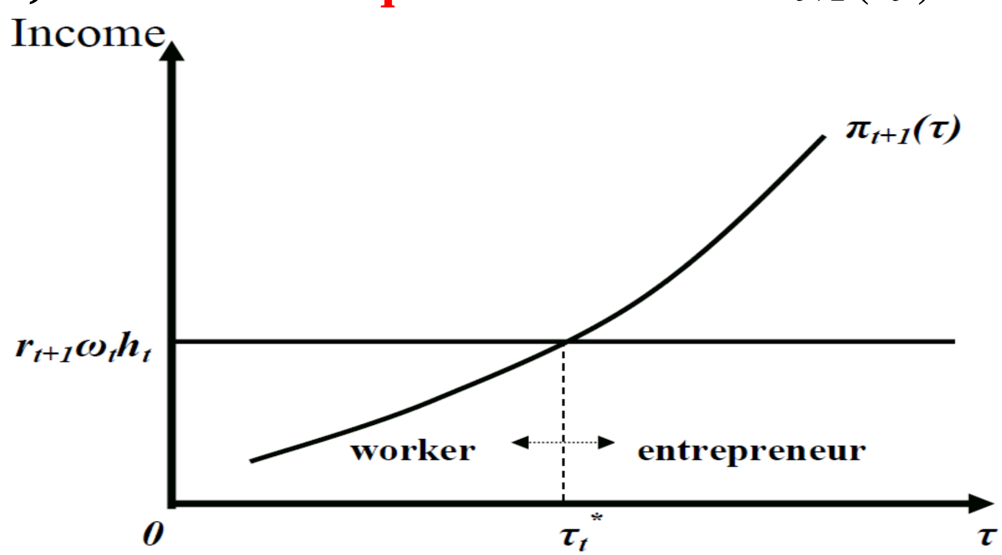


- **Optimization:**
  - **Production efficiency:** $MPK = P_{t+1}$, $MPL = \omega_{t+1}$
  - **Competitive profit condition:** $1 = \frac{1}{A\alpha^{\alpha}(1-\alpha)^{1-\alpha}} P_{t+1}{}^{\alpha} w_{t+1}{}^{1-\alpha}$
  - $k_{t+1}(i) = \left(\frac{P_{t+1}}{p_{t+1}(i)}\right)^{\frac{1}{1-\theta}} K_{t+1}$ **and** $P_{t+1} = \left(\int_0^{N_{t+1}} p_{t+1}(i)^{-\frac{\theta}{1-\theta}} di\right)^{-\frac{1-\theta}{\theta}}$
- **Entrepreneurial decision:**
  - **Optimization:** $\max_{x_t} \pi_{t+1}(\tau) = [p_{t+1}(\tau)\frac{\tau M_{t+1}}{\Omega_{t+1}{}^{\zeta}} - \delta_{t+1}]x_t$
  - **F.O.C.:** $p_{t+1}(\tau) = \frac{1}{\theta}\frac{\delta_{t+1}}{\tau M_{t+1}/\Omega_{t+1}{}^{\zeta}}$
  - **Loan demand:** $x_t(\tau) = \left(\frac{P_{t+1}}{p_{t+1}(\tau)}\right)^{\frac{1}{1-\theta}} \frac{K_{t+1}}{\tau M_{t+1}/\Omega_{t+1}{}^{\zeta}}$
  - **Profit:** $\pi_{t+1}(\tau) = (1-\theta)\left(\frac{P_{t+1}}{p_{t+1}(\tau)}\right)^{\frac{\theta}{1-\theta}} P_{t+1}K_{t+1}$

- **Overall quality index and variety adjustment index:**

$$M_{t+1} = \frac{\int_{\tau_t^*}^{\tau^H} \tau dF}{\left(\int_{\tau_t^*}^{\tau^H} dF\right)^{1-\eta}} \; ; \quad \Omega_{t+1} = \left(\int_{\tau_t^*}^{\tau^H} \tau^{\frac{\theta}{1-\theta}} dF\right)^{\frac{1-\theta}{\theta}}$$

- **Occupational choice:**
  - **a *t*-generation type-τ agent compares entrepreneurial profit $\pi_{t+1}(\tau)$ with worker's interest-included wage income $r_{t+1}\omega_t h_t$**
  - **since $\pi_{t+1}(\tau)$ strictly increases in τ and $r_{t+1}\omega_t h_t$ is independent of it, we have a unique cutoff τ^* s.t.** $\pi_{t+1}(\tau_t^*) = r_{t+1}\omega_t h_t$



  - **mass of entrepreneurs:** $\mathbf{N_{t+1} = \int_{\tau_t^*}^{\tau^H} dF = 1 - F(\tau_t^*)}$

- **Banking sector:**
  - **1 unit of deposit can be transformed into 1 unit of loan (zero bank reserves)**
  - **to make 1 unit of loan, a bank must use *1/φ* unit of effective labor (Ricardian technology)**
  - **competitive banking implies zero profit:** $\delta_{t+1} = r_{t+1}(1+\frac{\omega_t}{\phi})$
  - **financial markup is thus measured by:** $s_{t+1} \equiv \frac{\delta_{t+1} - r_{t+1}}{r_{t+1}} = \frac{\omega_t}{\phi}$

- **Market clearing conditions:**
  - **loan market:** $\int_{\tau_t^*}^{\tau^H} x_t(\tau) dF = \omega_t h_t \left[\left(1+\frac{\omega_t}{\phi}\right)^{-1} \int_0^{\tau_t^*} dF \right], \forall t \geq 0$

    (loan demand = deposit supply by manufacturing workers)
  - **labor market:** $L_{t+1} = \left(1+\frac{\omega_{t+1}}{\phi}\right)^{-1} \int_0^{\tau_{t+1}^*} dF, \forall t \geq 0$

    (labor demand = labor supply in the goods sector)

2. Equilibrium Characterization

- **Key features:**
  - **time-invariant cutoff ability *τ**, given by,** $\dfrac{\int_{\tau_t^*}^{\tau^H}\left(\frac{\tau}{\tau_t^*}\right)^{\frac{\theta}{1-\theta}}dF}{\int_0^{\tau_t^*}dF}=\dfrac{1-\theta}{\theta},\ \forall t\geq 0$
  - **simple dynamics:**
    - **transformed state variable (K_t/h_t) evolves according to**

$$\frac{K_{t+1}}{h_{t+1}}=[A(1-\alpha)]^{\frac{1}{\alpha}}M(\tau*)\left(\int_{\tau^*}^{\tau^H}\tau^{\frac{\theta}{1-\theta}}dF\right)^{\frac{1-\theta}{\theta}}(\omega_t)^{-\frac{1-\alpha}{\alpha}}\left(\frac{K_t}{h_t}\right)^{1-\beta}$$

    - **since K/h can be expressed as a function of ω_t alone, the model dynamics can be simply captured by:**

$$\omega_{t+1}^{\frac{1}{\alpha}}\left(1+\frac{\omega_{t+1}}{\phi}\right)^{-1}=[A(1-\alpha)]^{\frac{1+\beta}{\alpha}}[B(\tau*)]\omega_t^{\frac{\alpha-\beta}{\alpha}}\left(1+\frac{\omega_t}{\phi}\right)^{-(1-\beta)},$$

where $B(\tau*)=M(\tau*)^{1-\zeta}\left(\int_{\tau*}^{\tau^H}\tau^{\frac{\theta}{1-\theta}}dF\right)/\left(\int_0^{\tau*}dF\right)^{\beta}$ **is related to the cutoff ability as well as the average quality**

  - **unique stationary fixed point (Poincaré-Bendixson+Sarkovskii)**

- **Balanced growth path (BGP):**
  - **Wage:** $\omega^{\frac{1-\alpha+\beta}{\alpha}}\left(1+\frac{\omega}{\phi}\right)^{-\beta} = [A(1-\alpha)]^{\frac{1+\beta}{\alpha}} M(\tau*)\Omega(\tau*)^{1-\zeta} Z(\tau*)^{-\beta}$**,** $Z(\tau*) = \int_0^{\tau*} dF$
  - **economic growth:** $g = [A(1-\alpha)]^{\frac{1}{\alpha}} M(\tau*)\Omega(\tau*)^{1-\zeta}(\omega)^{-\frac{1-\alpha}{\alpha}} - 1$

- **Relationship between cutoff ability and economic growth**

$$\frac{dg}{d\tau*} \propto \left(1+\frac{\alpha}{(1-\alpha)(1+\omega/\phi)}\right)\frac{dM/d\tau*}{M} + \frac{\alpha}{(1-\alpha)(1+\omega/\phi)}\frac{d\Omega/d\tau*}{\Omega}$$

  - **selectivity effect via M(τ*): (+)**
  - **variety effect via Ω(τ*): (-)**
  - **thus, g and τ* need not be positively related**

- **A uniformly rightward shift of the ability distribution by λ > 0 over a compact support [λ, τ^H+ λ]**
  - **raises τ* less than proportionately => higher N**
  - **increases equilibrium wage**
  - **has an ambiguous effect on the balanced growth rate:**
    - **productivity effect: (+)**
    - **loanable fund supply effect: (-)**

- **Effects of financial & real productivity changes**

	Financial Improvement (higher ϕ)	**Technological Advancement (higher *A*)**
Effective Wage Rate (ω)	**-**	+
Financial Markup (*S*)	**-**	+
Mass of Entrepreneur (*N*)	**0**	**0**
Economic Growth Rate (*g*)	+	+

Thus, financial markup and growth need not be positively related

3. Future Work

- **Entrepreneur vs. managers: risk attitudes and abilities: Jiang-Loukas-Wang-Wu (2003)**
- **Entrepreneurship, market structure and firm size: Lee-Wang (2013)**
- **Venture capitalism: Lu-Wang (2012), Alter-Lee-Wang (in progress)**

D. Migration, Skill Accumulation and Earning Evolution: Lucas (2004)

- **Rural population in advanced economies fell significantly over the past century (e.g., uural population in UK dropped from 50% in 1850 to 10.5% in 2000, with the agriculture share of employment decreasing from 21% to less than 2% over the same period)**
- **Rural population in many less developed countries still remain sizable**

Country	Share of Rural Population (%) 1950 (1)	2000 (2)	Share of Largest City in Urban Population (%) 1950 (5)	2000 (6)
Argentina	35	10	47	38
Brazil	75	19	15	13
Egypt	68	55	39	34
India	84	72	5	6
Mexico	58	26	27	25
Philippines	74	41	29	24
South Korea	79	18	25	26
Thailand	90	78	66	55

- **Three important features of labor-market development:**
  - **continual migration from traditional agriculture**
  - **gradual migration process**
  - **locational equilibrium (net income equalization, or, no-arbitrage)**
- **The literature of rural-urban migration:**
  - **classic: Todaro (1969), Harris-Todaro (1970), with institutionally fixed urban wage**
  - **generalization:**
    - **intersectoral capital mobility: Bhagwati-Srinivasan (1974) and Corden-Finlay (1975)**
    - **endogenous urban wage determination: Calvo (1978), Quibria (1988)**
  - **trade and migration: Khan (1980, 1982), Batra-Naqvi (1987)**
  - **Migration and growth:**
    - **classic: Drazen-Eckstein (1988) and Glomm (1992, CE=PO)**
    - **low-growth trap with informational asymmetry:**
      - **Bencivenga-Smith (1997): adverse selection of workers to urban**
      - **Banerjee-Newman (1998): urban modern sector with lower credit availability due to higher agency costs**
    - **endogenous growth: Lucas (2004) cities enable new immigrants to accumulate skills to use modern technologies, thus inducing growth**

- **1. The Model**
- **Preference:** $U = \int_0^\infty \frac{c^{1-\sigma} - 1}{1-\sigma} e^{-\rho t}\, dt$
- **Fixed supply of time (1 unit)**
- **Agricultural economy:**
  - **land supply: 1 unit**
  - **farm production:** $F(x) = Ax^{\alpha}(1)^{1-\alpha}$
  - **competitive wage:** $w = \alpha A(x)^{\alpha-1} = \alpha A$
  - **land rent:** $F(1) - wx = (1-\alpha)A$
  - **real interest rate:** $r = \rho$
  - **consumption:** $c = F(1)$
- **Urban economy:**
  - **output linear in human capital: uh**
  - **human capital accumulation (HA):** $\dot{h} = \delta(1-u)h$
  - **urban wage: constant (as a result of linear technology), normalized to one**
  - **household optimization can be separated in two problems:**
    - **optimal time allocation: maximize lifetime time wage income,**

      $\int_0^\infty \exp\left[-\int_0^t r(s)\,ds\right] h(t)\,u(t)\,dt$, **subject to (HA)**

- FOC(u): $h(t) = \delta \int_t^{\infty} \exp\left[-\int_t^{\tau} r(s)ds\right] h(\tau)u(\tau)d\tau$, which with (HA) yields:

$$h(\tau) = h(t) \exp\left\{\delta \int_t^{\tau} [1 - u(s)]ds\right\}$$

- Fundamental Lemma of Calculus of Variation (CV) gives: r = δ

\- optimal consumption: maximize U s.t. $\int_0^{\infty} \exp\left[-\int_0^{t} r(s)ds\right] c(t)dt \leq a$

- Keynes-Ramsey (KR): $\frac{\dot{c}}{c} = \frac{r - \rho}{\sigma} = \frac{\delta - \rho}{\sigma}$

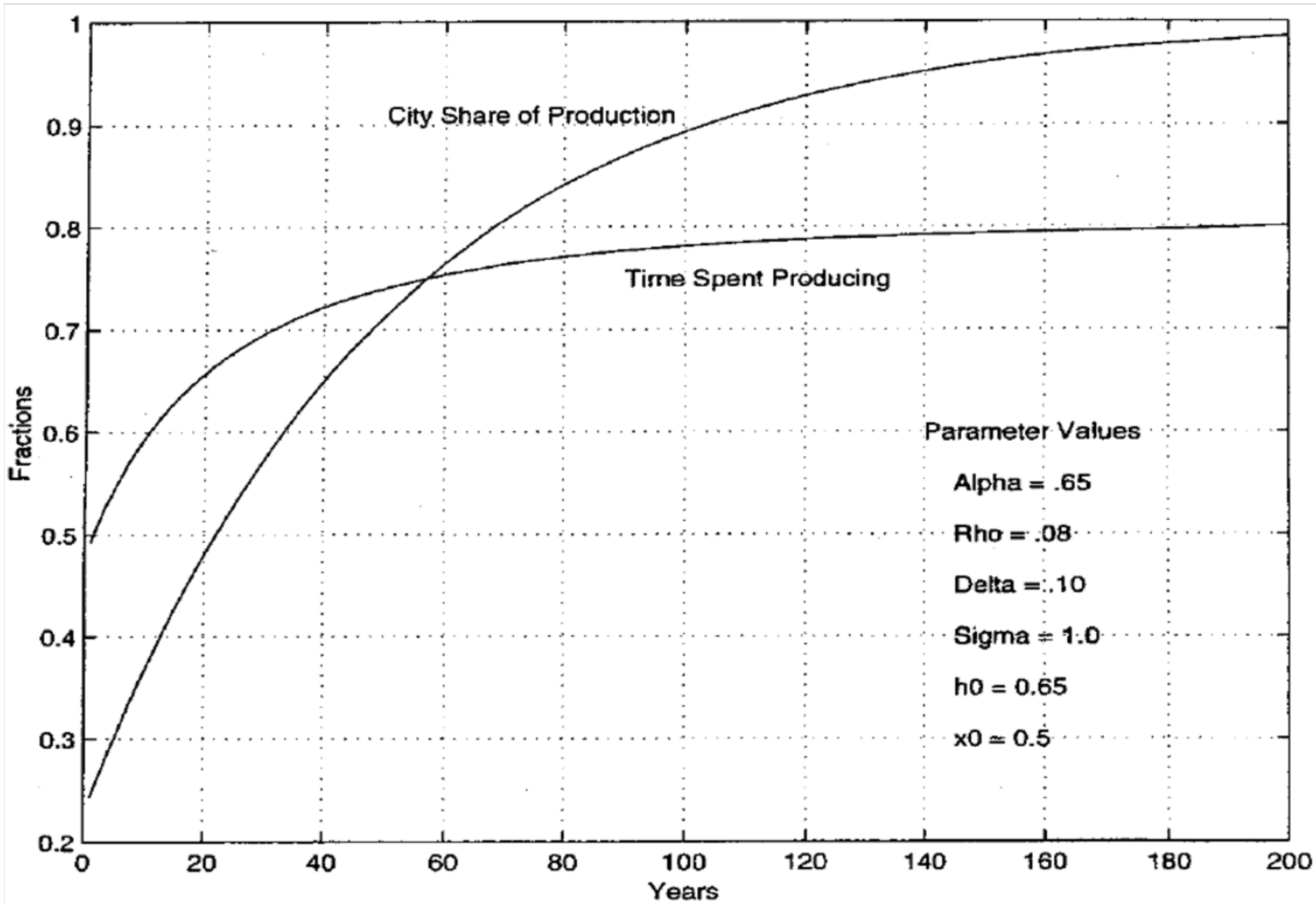
- (KR) & (HA) together imply: $u = 1 - [(\delta - \rho)/(\delta\sigma)]$ (requiring $\delta\sigma \geq \delta - \rho \geq 0$)

2. Transition from Agriculture to Urban Economy

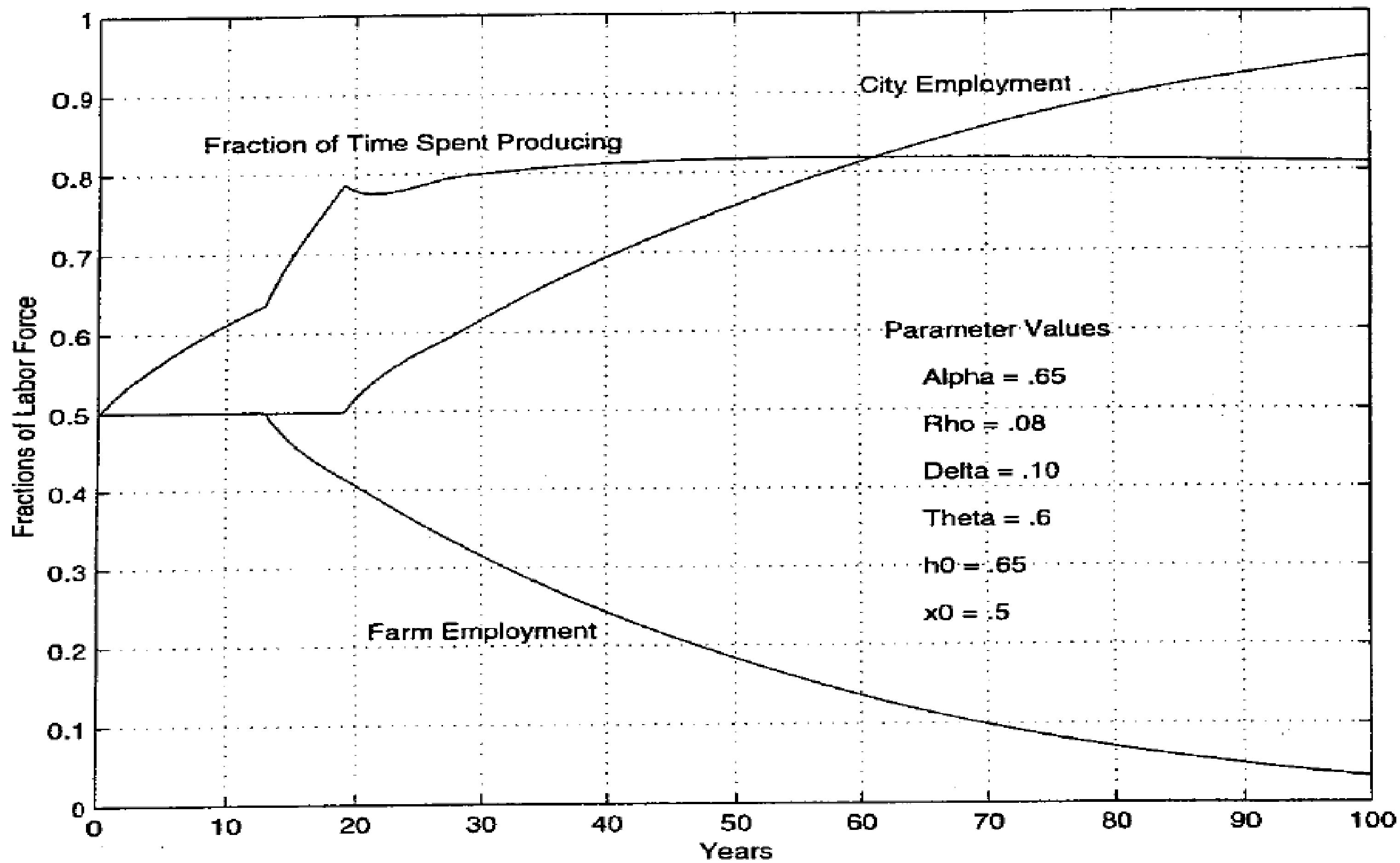
- Everyone initially in rural, with human capital h_0 and with human capital accumulation once moving to urban with its process given by (HA)
- (CV) and (KR) continue to hold
- Key: migration equilibrium, assuming no moving cost/no urban unemployment

- **Rural earning: E^R =** $\int_t^\infty \exp\left[-\delta(\tau - t)\right] F'(x(\tau))\,d\tau$
- **Urban earning:**
  - $E^U = \int_t^\infty \exp[-\delta(\tau - t)]u(\tau)h(\tau)\,d\tau = h_0 \int_t^\infty \exp\left[-\delta(\tau - t)\right] \exp\left\{\delta \int_t^\tau [1 - u(s)]ds\right\} u(\tau)d\tau$
- **Migration equilibrium requires $E^R = E^U$, or, simplifying (integration by parts),** $h_0 \int_t^\infty \exp\left[-\delta(\tau - t)\right] \exp\left\{\delta \int_t^\tau [1 - u(s)]ds\right\} u(\tau)d\tau = \frac{h_0}{\delta}$**, implying:**
  - **all paths of u(t) lead to the same lifetime earning**
  - $F'(x(t)) = h_0$
    - **$x(t) = x_0$ ∀ t**
    - **if migration occurs, it takes place at t = 0**
- **Goods market clearing:** $c(t) = F(x_0) + (1 - x_0)h(t)u(t)$
- **Equilibrium time allocation and human capital allocation:**
  - $u(t) = \frac{c(0)}{(1 - x_0)h(t)} \exp\left(\frac{\delta - \rho}{\sigma} t\right) - \frac{F(x_0)}{(1 - x_0)h(t)}$ **→ $1 - [(\delta - \rho)/(\delta\sigma)]$**
  - $\dot{h} = \delta h(t) - \delta \frac{c(0)}{1 - x_0} \exp\left(\frac{\delta - \rho}{\sigma} t\right) + \delta \frac{F(x_0)}{1 - x_0}$**, with $h(0) = h_0$ and asymptotic rate of accumulation at $(\delta - \rho)/\sigma$**

- **Calibration: the transition process of rural-urban migration**



- **Extension: incorporation of uncompensated positive human capital spillovers**



E. Health and Life-Cycle Savings: Bloom-Canning-Graham (2010)

- **Health improvements raise longevity, which in turn promote savings over the life cycle**
- **High saving rates in Asia vs. low saving rates in Africa**

1. The Model

- **Time allocation: leisure (*l*), work (1-*l*)**
- **Lifetime utility:** $\int_0^T e^{-\delta t} U(c_t, l_t, h_t)dt$

  Assumption:
  - **well-behaved utility**
  - **both *c* and *l* are normal goods**
- **Budget constraint (wealth accumulation):**

$$\frac{dW_t}{dt} = rW_t + (1-l_t)w_t - c_t$$

- **Euler equation governing the optimal consumption path:**

$$\frac{dc}{dt} = \frac{(r-\delta)\frac{dU}{dc} + \frac{d^2U}{dcdl}\frac{dl}{dt} + \frac{d^2U}{dcdh}\frac{dh}{dt}}{-\frac{d^2U}{dc^2}}$$

- **Consumption-leisure trade-off:**

$$\frac{dU}{dl_t} = w_t \frac{dU}{dc_t} \quad if\; 1 < l < 0$$

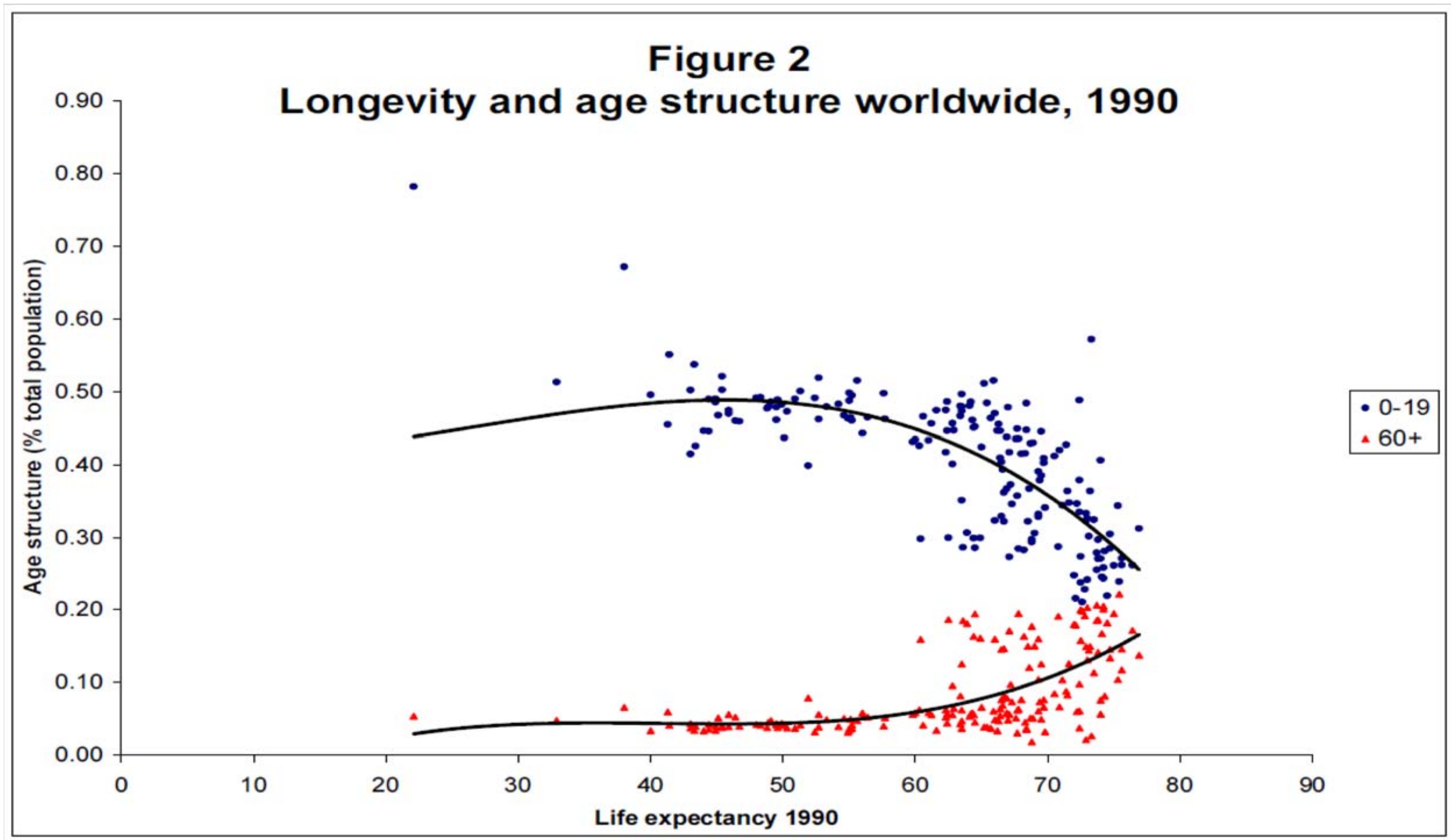
- **Saving rate:**

$$s_t = \frac{y_t - c_t}{y_t},\; where\; y_t = rW_t + (1-l_t)w_t$$

  - **positive wealth assumption: r is sufficient large such that W > 0 for all t**
  - **key property: if $W_0 = 0$, then an increase in T rises saving rate s**

2. Empirical Evidence

- **Longevity and age structure:**



- **Dependent = average saving rate (over cohort i):** $\bar{s} = \sum_{i=0}^{T} s_i \frac{Y_i}{Y} = \sum_{i=0}^{T} s_i \frac{P_i}{P} \frac{y_i}{\bar{y}}$

Variable	Specification (1)	(2)	(3)	(4)
Life expectancy	0.462* (0.088)	0.455* (0.087)	0.410* (0.093)	0.459* (0.094)
Growth rate of income per capita during preceding decade	0.713* (0.130)	0.717* (0.126)	0.716* (0.125)	0.735* (0.141)
Youth share of population		-80.41* (28.38)		-89.48* (11.07)
Elderly share of population		-48.58 (59.16)		-153.0* (22.44)
Age effect: linear term		13.28 (16.45)	33.72* (14.25)	
Age effect: square term		-1.234 (2.223)	-1.409 (2.260)	
Age effect: cubic term		0.006 (0.096)	-0.049 (0.097)	

(constant term and country fixed effect/year dummies not reported)

F. Life and Growth: Jones (2016)

- **Innovations in medicine, green energy, safety devices, etc., save lives.**
- **Key decisions:**
  - **how to allocate scientists to the conventional consumption sectors and to the lifesaving sectors**
  - **how to allocate production workers to these sectors**

1. The Model

- **Production of standard consumption good C using a variety of intermediate inputs x up to the range of available technologies captured by A:**

$$C_t = \left[\int_0^{A_t} x_{it}^{1/(1+\alpha)}\, di\right]^{1+\alpha}$$

  - **elasticity of substitution = (1+α)/α > 1 as in standard variety models**
- **Production of lifesaving good H using intermediate inputs z up to range B:**

$$H_t = \left[\int_0^{B_t} z_{it}^{1/(1+\alpha)}\, di\right]^{1+\alpha}$$

- **Upon discoveries of the technologies, each variety can be produced by one unit of production labor**
- **Production labor allocation:** $$\underbrace{\int_0^{A_t} x_{it}\,di}_{\equiv L_{ct}} + \underbrace{\int_0^{B_t} z_{it}\,di}_{\equiv L_{bt}} \le L_t$$
- **Technology evolution driven by scientists S:** $\dot{A}_t = S_{at}^{\lambda} A_t^{\phi}$ and $\dot{B}_t = S_{bt}^{\lambda} B_t^{\phi}$
- **Scientists allocation:** $S_{at} + S_{bt} \le S_t$
- **Labor constraint:** $S_t + L_t \le N_t$
- **Mortality at time varying rate δ:** $M_t = e^{-\int_0^t \delta_s ds}$
    - **so** $\dot{M}_t = -\delta_t M_t, \quad M_0 = 1$
    - **mortality rate is reduced with higher h = H/N:** $\delta_t = h_t^{-\beta}$
- **Expected lifetime utility (Murphy-Topel 2006):** $$U = \int_0^{\infty} e^{-\rho t} u(c_t) M_t\, dt$$
    - **flow utility:** $u(c_t) = \bar{u} + \frac{c_t^{1-\gamma}}{1-\gamma}, \quad c_t \equiv C_t/N_t$
    - $\bar{u}$ **= flow value of life (vs. death)**
    - **thus, lifesaving activities lengthen life but do not enhance the quality of life**

- **Population evolution:** $\dot{N}_t = \bar{n}N_t$
  - $\bar{n}$ **= fertility rate - mortality rate, assumed to be constant**
- **Key decisions:**
  - $s_t \equiv S_{at}/S_t$ **(fraction of scientists in consumption innovation)**
  - $\ell_t \equiv L_{ct}/L_t$ **(fraction of workers in consumption production)**
  - $\sigma_t \equiv S_t/N_t$ **(scientist share)**

2. Equilibrium

- **Symmetry, (asymptotic) BGP equilibrium**
- **BGP under a "rule of thumb allocation" with constant (s, ℓ, σ):**
  - **technology growth:** $g_A^* = g_B^* = \dfrac{\lambda\bar{n}}{1-\phi}$
  - **consumption/lifesaving goods growth:** $g_c^* = g_h^* = \alpha g_A^* = \alpha g_B^* = \bar{g} \equiv \dfrac{\alpha\lambda\bar{n}}{1-\phi}$
  - **asymptotic mortality rate:** $g_\delta^* = -\beta\bar{g},\ \delta_t \to 0$**, implying ever-increasing life expectancy**
  - **common sectoral growth is counterfactual, thus motivating further consideration of time-varying decisions on (s, ℓ, σ) and then the dynamics toward (asymptotic) BGP**

- **Optimal allocation under symmetry (benevolent Mill social planner):**

$$\max_{\{s_t,\ell_t,\sigma_t\}} U = \int_0^\infty M_t u(c_t) e^{-\rho t} dt$$

s.t.
$$c_t = A_t^\alpha \ell_t (1-\sigma_t)$$
$$h_t = B_t^\alpha (1-\ell_t)(1-\sigma_t)$$
$$\dot{A}_t = s_t^\lambda \sigma_t^\lambda N_t^\lambda A_t^\phi$$
$$\dot{B}_t = (1-s_t)^\lambda \sigma_t^\lambda N_t^\lambda B_t^\phi$$
$$\dot{M}_t = -\delta_t M_t, \quad \delta_t = h_t^{-\beta}$$

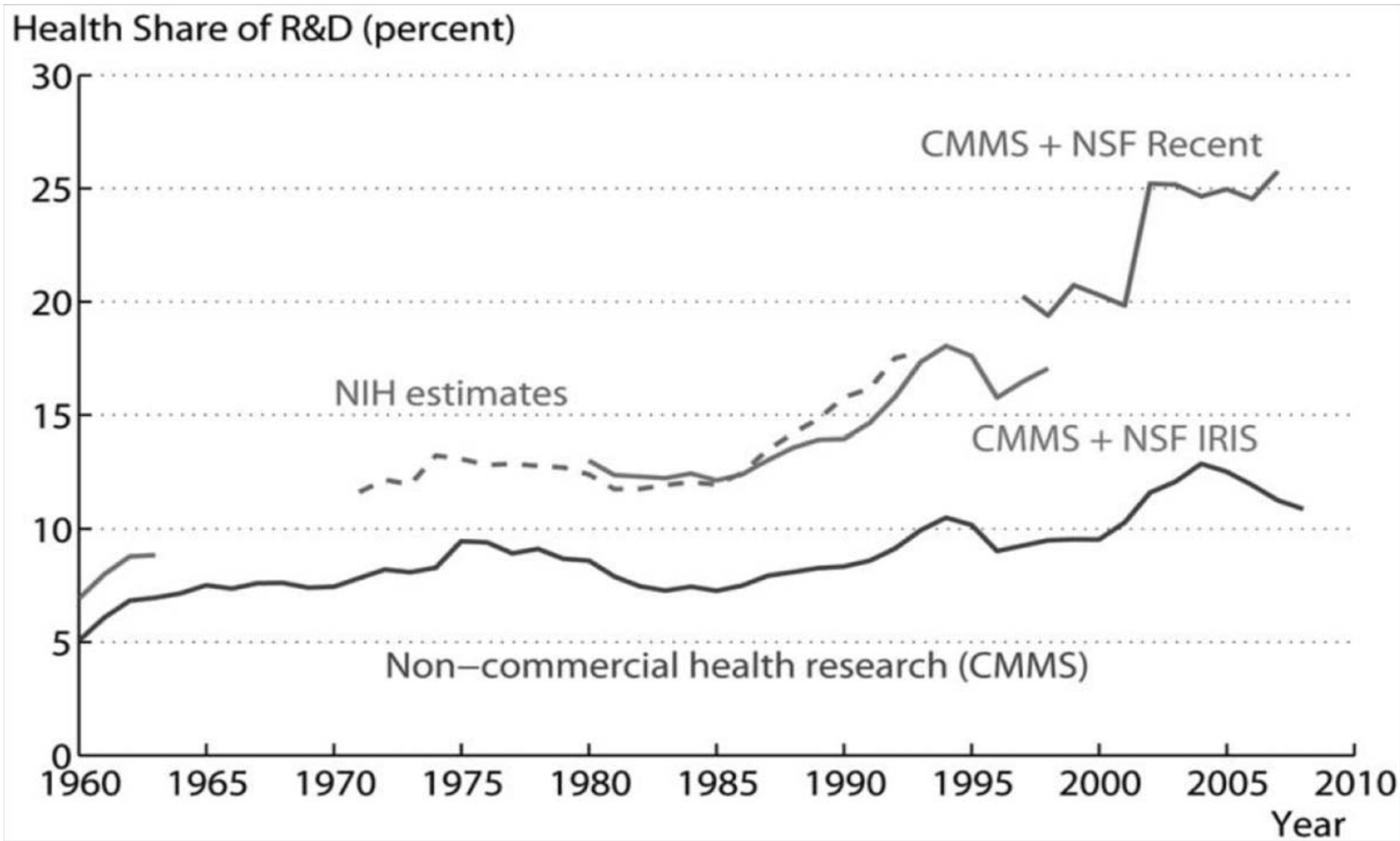
- **Optimal growth with $\gamma > 1 + \beta$ (strong diminishing return in MU_c):**
  - $$g_s^* = g_\ell^* = \frac{-\bar{g}(\gamma - 1 - \beta)}{1 + (\gamma-1)\left(1 + \frac{\alpha\lambda}{1-\phi}\right)} < 0, \quad \bar{g} \equiv \frac{\alpha\lambda\bar{n}}{1-\phi}$$
  - $$g_A^* = \frac{\lambda(\bar{n} + g_s^*)}{1-\phi}, \quad g_B^* = \frac{\lambda\bar{n}}{1-\phi} > g_A^*$$
  - $$g_\delta^* = -\beta\bar{g}, \quad g_h^* = \bar{g}$$
  - $$g_c^* = \alpha g_A^* + g_\ell^* = \bar{g} \cdot \frac{1 + \beta\left(1 + \frac{\alpha\lambda}{1-\phi}\right)}{1 + (\gamma-1)\left(1 + \frac{\alpha\lambda}{1-\phi}\right)} < \bar{g}$$

- **Main findings: when life is a luxury good relative to consumption:**
  - **reallocation toward lifesaving (s, ℓ decreasing over time)**
  - **lifesaving technology outgrows consumption technology**
  - **lifesaving outgrows consumption**

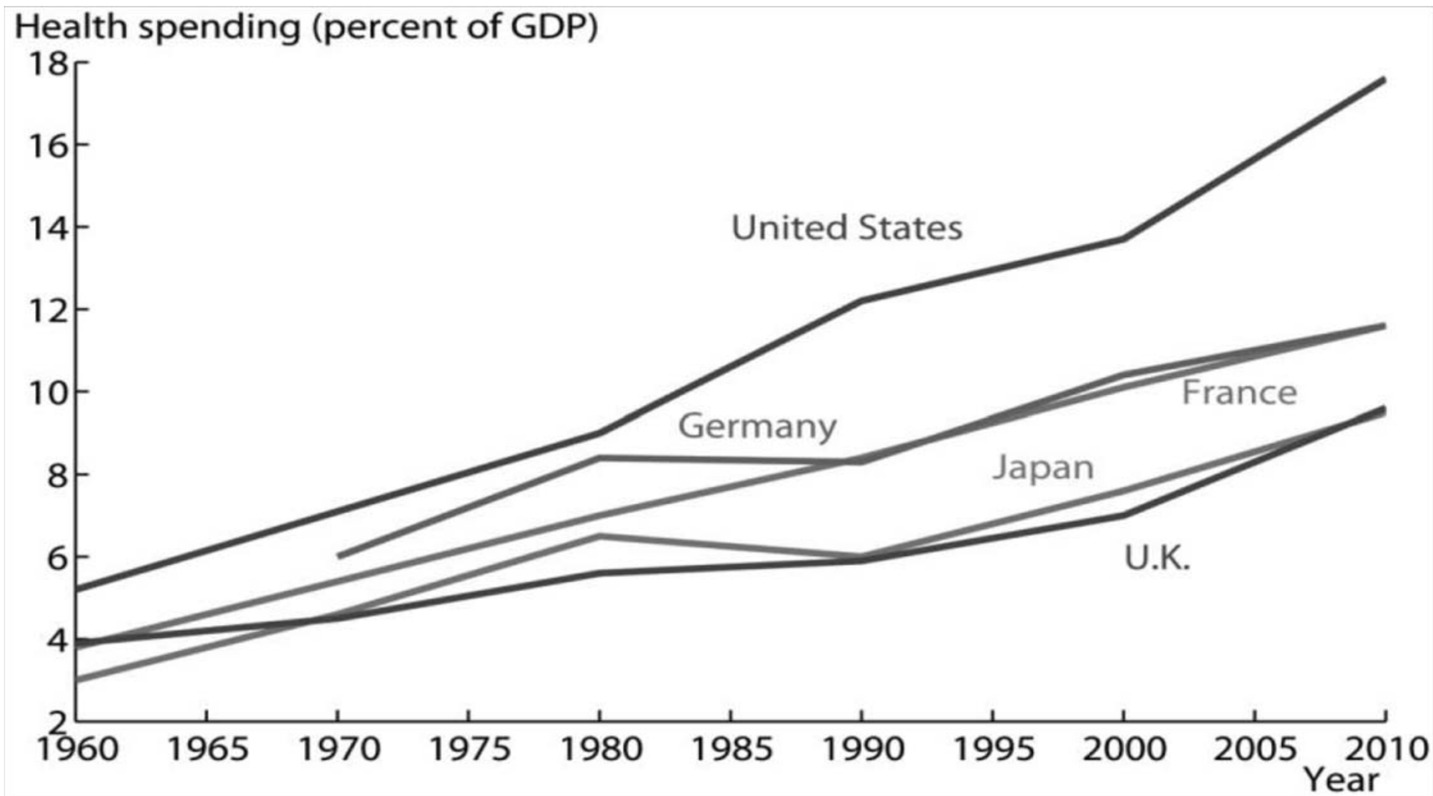
- **Optimal growth with γ < 1 + β (weak diminishing return in MU_c):**
  - $g^*_{\tilde{s}} = g^*_{\tilde{\ell}}$ **< 0**
  - $g^*_A = \frac{\lambda \bar{n}}{1-\phi}, \quad g^*_B = \frac{\lambda(\bar{n} + g^*_{\tilde{s}})}{1-\phi} < g^*_A$ **(consumption tech grows faster)**
  - $g^*_c = \bar{g} \equiv \frac{\alpha\lambda\bar{n}}{1-\phi}, \quad g^*_\delta = -\beta g^*_h$, g^*_h < $\bar{g}$ **(consumption grows faster)**
- **Optimal growth with γ = 1 + β (a knife edge case):**
  - $g^*_A = g^*_B = \frac{\lambda\bar{n}}{1-\phi}$ **(common technology growth)**
  - $g^*_c = g^*_h = \frac{\alpha\lambda\bar{n}}{1-\phi} = \bar{g}, \quad g^*_\delta = -\beta\bar{g}$ **(common sectoral growth)**

3. Empirical Evidence

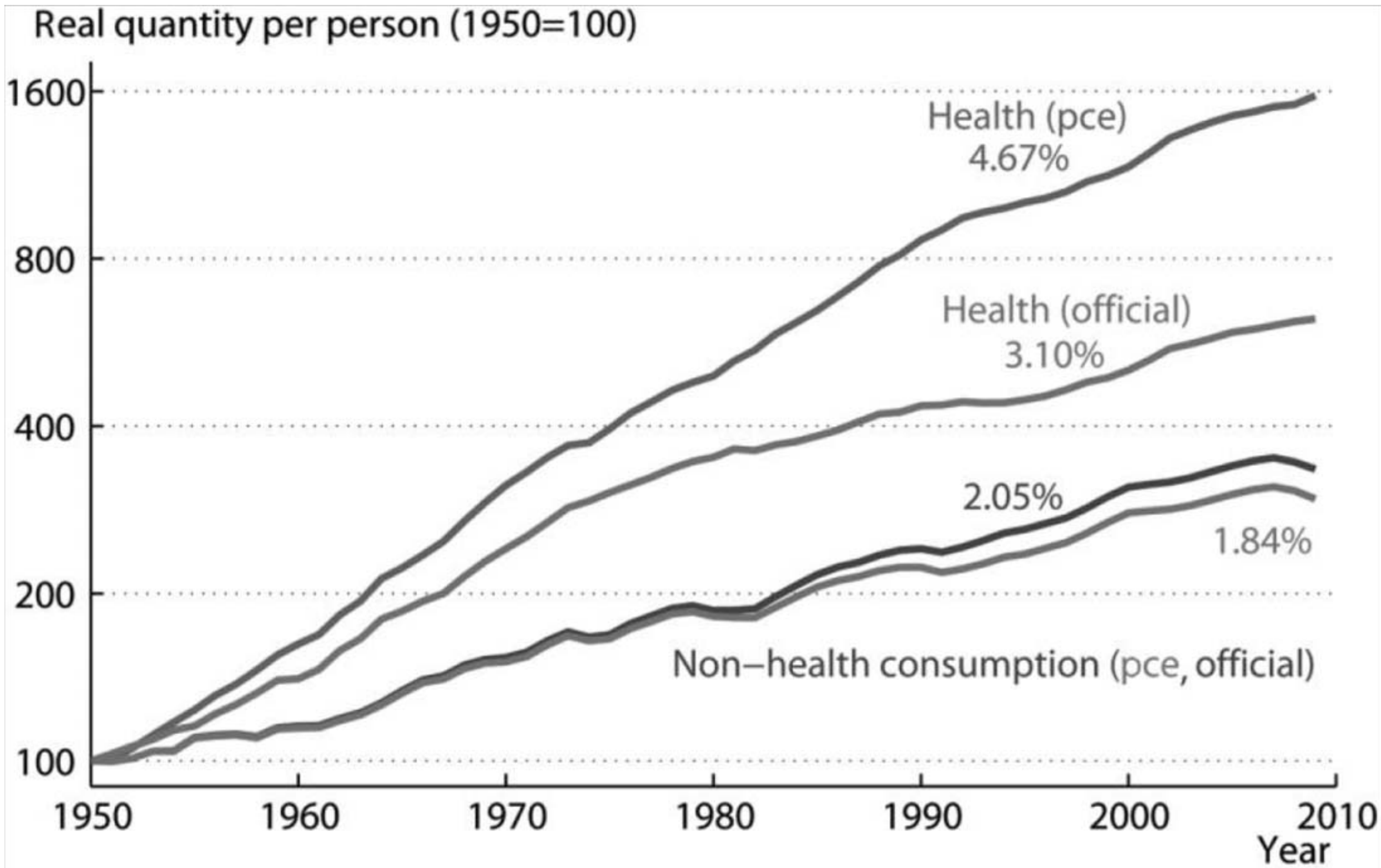
- **U.S. NIH/NSF data: rising R&D allocation to health (similar trend in OECD)**



- **Estimated health parameter in reducing mortality: $\beta = 0.291$**
- **Conventional estimation of the elasticity of intertemporal substitution => γ is mostly between 1.5 and 4, so $\gamma > 1 + \beta$ most likely**
- **This may explain why**
    - **the health sector expands (OECD Health Data):**

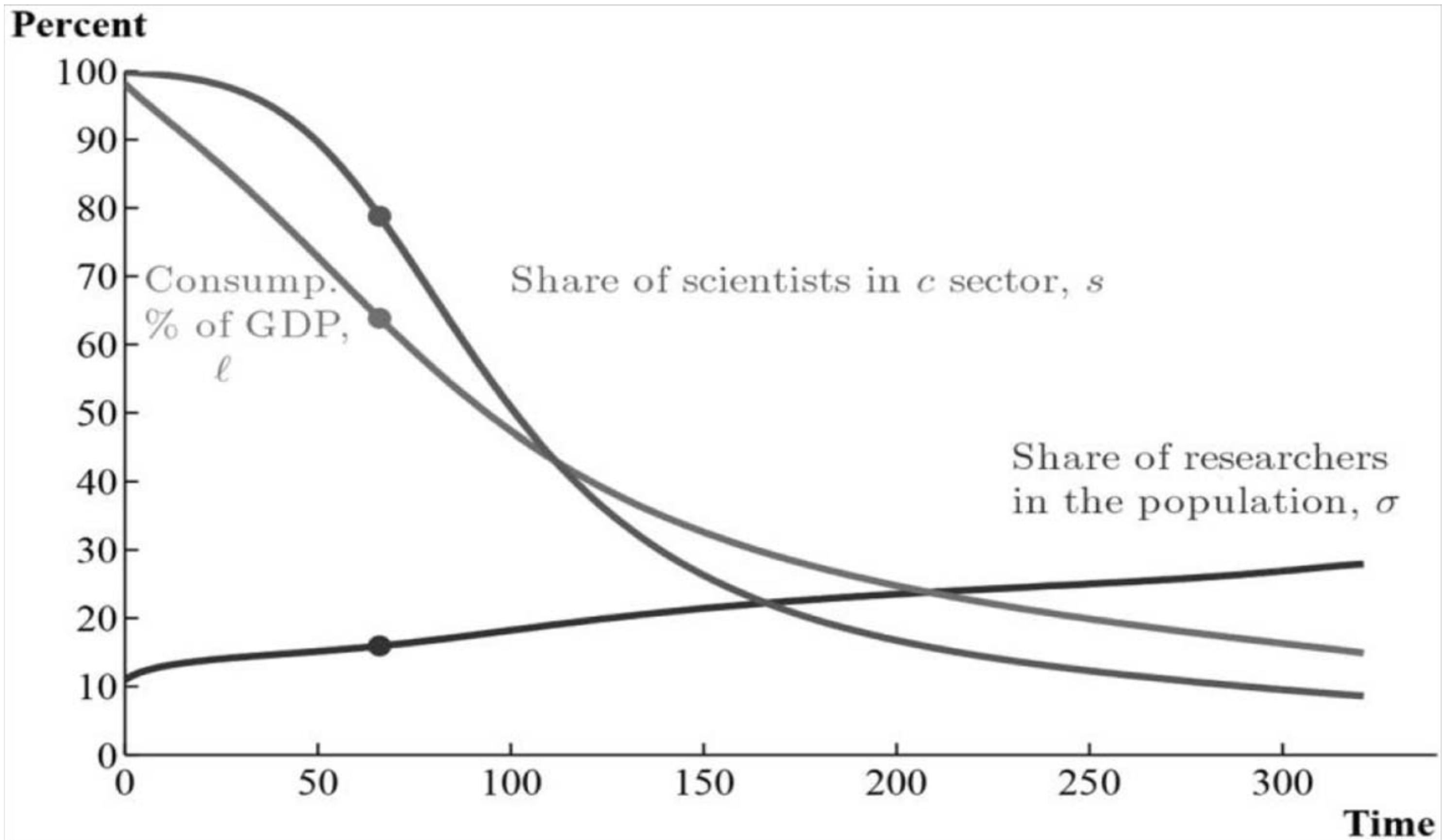


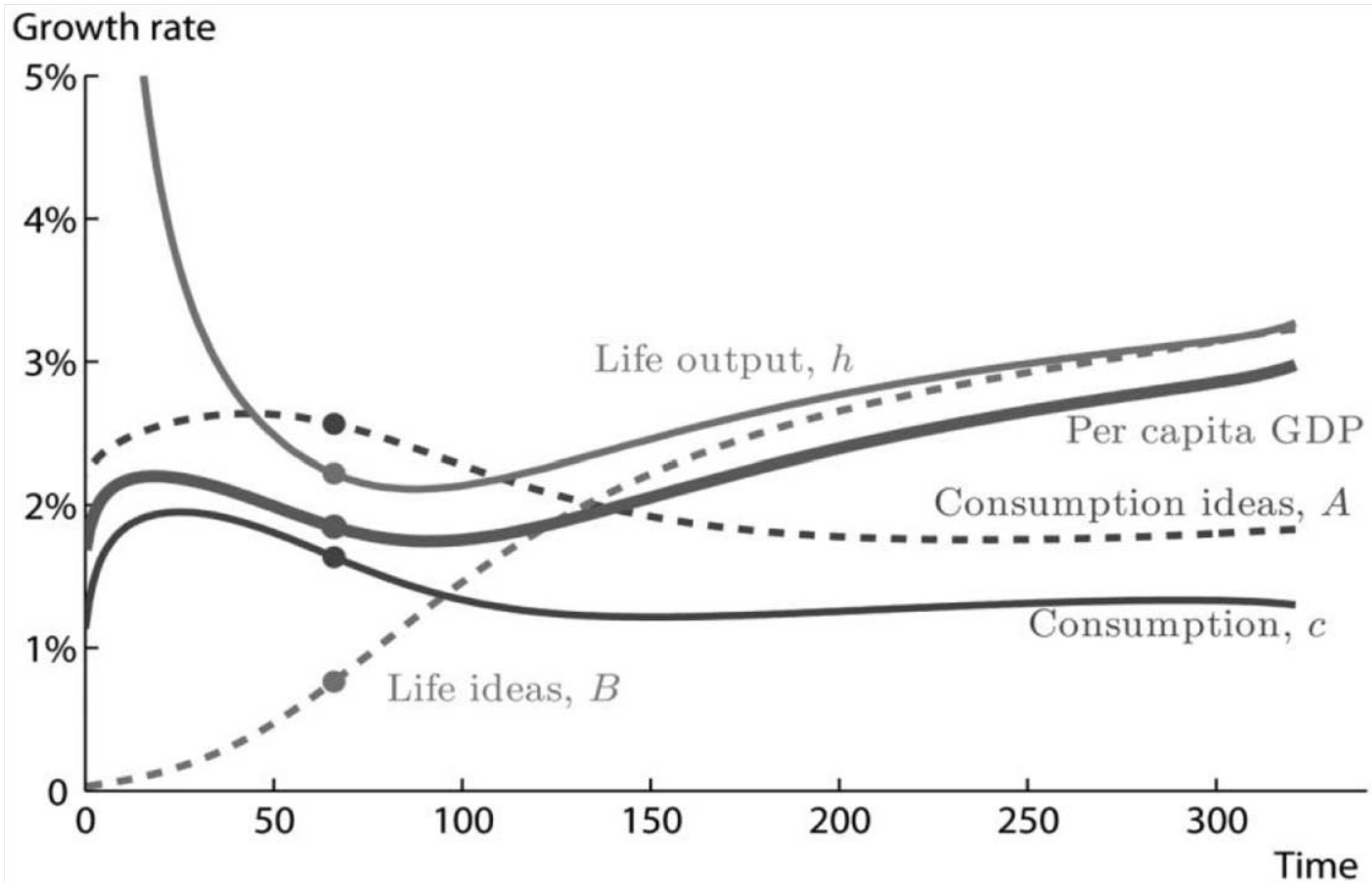
○ **health outgrows non-health consumption:**



4. Calibration Analysis

- **Key parameters: $\gamma = 2.6953$, $\beta = 0.6006$, $\lambda = 0.5377$, $\delta = 0.0002$, $\varphi = 5/6$**





- **Open issues:**
  - **endogenous quality of life**
  - **endogenous net population growth with general social welfare function**

G. Technology Cycle and Inequality: Jovanovic (2009)

- **Key: Technology trade**
- **Arrow (1962) + Lucas (1988) + Matsuyama (2002)**

1. The Model

- **Final good:** $$y = \left(\int_0^N x_i^{1/2} di \right)^2$$
- **Intermediate goods:** $$x_i = z_i H_i$$
- **Labor allocation:** $$H = \int_{N_{\min}}^{N} H_i di + H^{\mathrm{R}} + H^{\mathrm{A}}$$

(intermediate + research + human capital investment)

- **Labor available for intermediate production:** $H^{\mathrm{I}} \equiv H - H^{\mathrm{R}} - H^{\mathrm{A}}$
- **Labor allocation to i: choose** $(H_i)_{i=N-1}^{N}$ **to maximize Lagrangian,**

$$L = \left(\int_{N-1}^{N} z_i^{1/2} H_i^{1/2} di \right)^2 + \eta \left(H^{\mathrm{I}} - \int_{N-1}^{N} H_i di \right) \Rightarrow H_i = \left(\frac{H^{\mathrm{I}}}{\int_{N_{\min}}^{N} z_i di} \right) z_i$$

- **Final producer optimization:** $\max_{(x_i)_0^N} \left\{ y - \int_0^N P_i x_i di \right\}$ => $y^{1/2} x_i^{-1/2} - P_i = 0$
- **Intermediate producer optimization:**
  - **production:** $x_i = z_i s$ **(linear production)**
  - **skill efficiency unit:** $s = u_P h$ **(production time * human capital)**
  - **revenue:** $P_i x_i = y^{1/2} z_i^{1/2} s^{1/2}$
- **Labor market clearing:** $u_P + u_R + u_I = 1$ **(production+research+investment)**

- **Technology adoption:**
  - **an intermediate producer adopt technology to maximize profit given by,** $\pi(s) = \max\{y^{1/2} z^{1/2} s^{1/2} - p(z)\}$
  - **(FOC)** $\frac{1}{2}\left(\frac{sy}{z}\right)^{1/2} - p'(z) = 0$
    - **a first-order differential equation solvable depending on the functional of s (e.g., linear functional)**

2. Equilibrium

- **Key concept: positive assignment equilibrium**

- **A *positive-assignment equilibrium* in the market for technology is a price function *p(z)* and an assignment *s* = ψ*(z)* of skill to technology such that:**
  - **ψ is increasing (*positive* assignment),**
  - **z is the optimal choice of agent ψ(z), i.e.,** $\frac{1}{2}\left(\frac{\psi(z)y}{z}\right)^{1/2} - p'(z) = 0$ **(FOC)**
  - **technology market clears, i.e., for all active technologies z,**

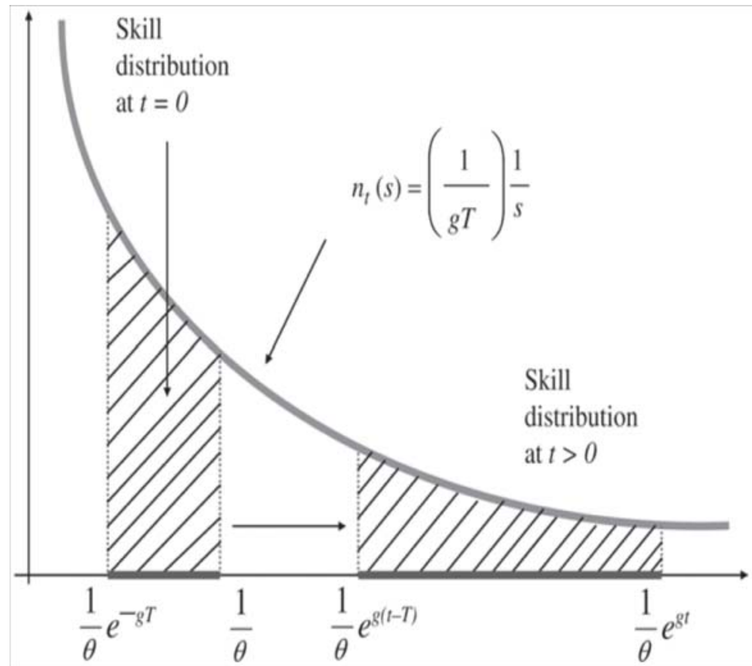
    $$\int_{z}^{1} n(v)dv = \int_{\psi(z)}^{s_{\max}} m(s)ds$$ **(equating skill-technology matching)**

    where date-zero frontier z_{max} = 1 and date-zero maximal skill is s_{max}

- **With the worst technology z_{min} and unit mass of agents operating different technologies, we have:**
  - **all active technologies satisfy:** $\int_{z_{\min}}^{1} n(z)\,dz = 1$
  - **inactive technologies satisfy:** $p(z) = 0 \text{ for } z < z_{\min}$ **and** $p(z_{\min}) = 0$
    - **no licensing fee paid to badly obsolete technologies, which become inactive**

- **Suppose $z_{\max}(t) \equiv e^{gt}$ and each technology is retired after a time-invariant T period of life. Then ln(z) is uniform over [g(t-T), gt] which implies the density of z, given by,**

$$n_t(z) = \left(\frac{1}{gT}\right)\frac{1}{z}, \quad \text{for } z \in [e^{g(t-T)}, e^{gt}]$$

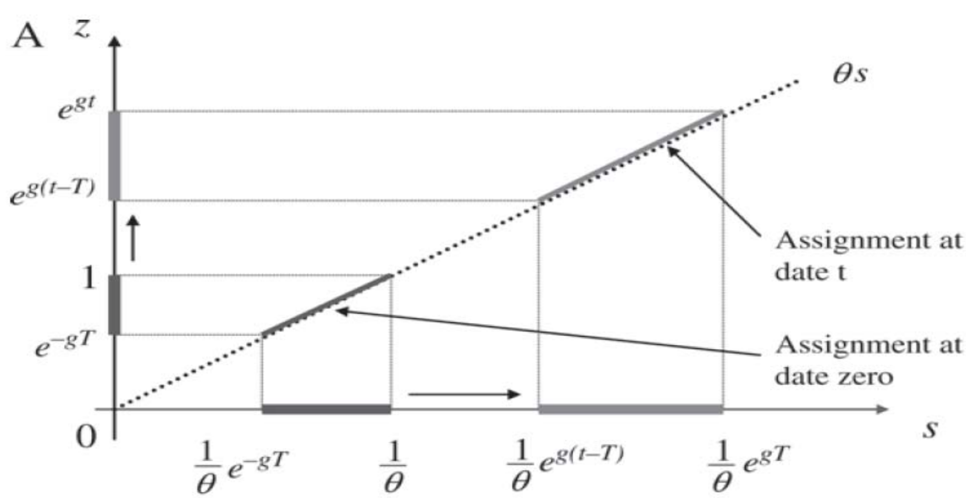


- **Conjectured market-clearing assignment and license fee p(z):**
  - **conjectured time-invariant assignment:** $\psi(z) = \frac{1}{\theta}z$ **(FOC solvable)**
  - **license fee:** $p_t(z) = \frac{1}{2}\left(\frac{y_t}{\theta}\right)^{1/2}\left(z - e^{g(t-T)}\right) \quad \text{for } z \in [e^{g(t-T)}, e^{gt}]$
  - **density of skill:** $m_t(s) = \left(\frac{1}{gT}\right)\frac{1}{s}, \quad \text{for } s \in [\frac{1}{\theta}e^{g(t-T)}, \frac{1}{\theta}e^{gt}]$
    - **taking the same form as n**

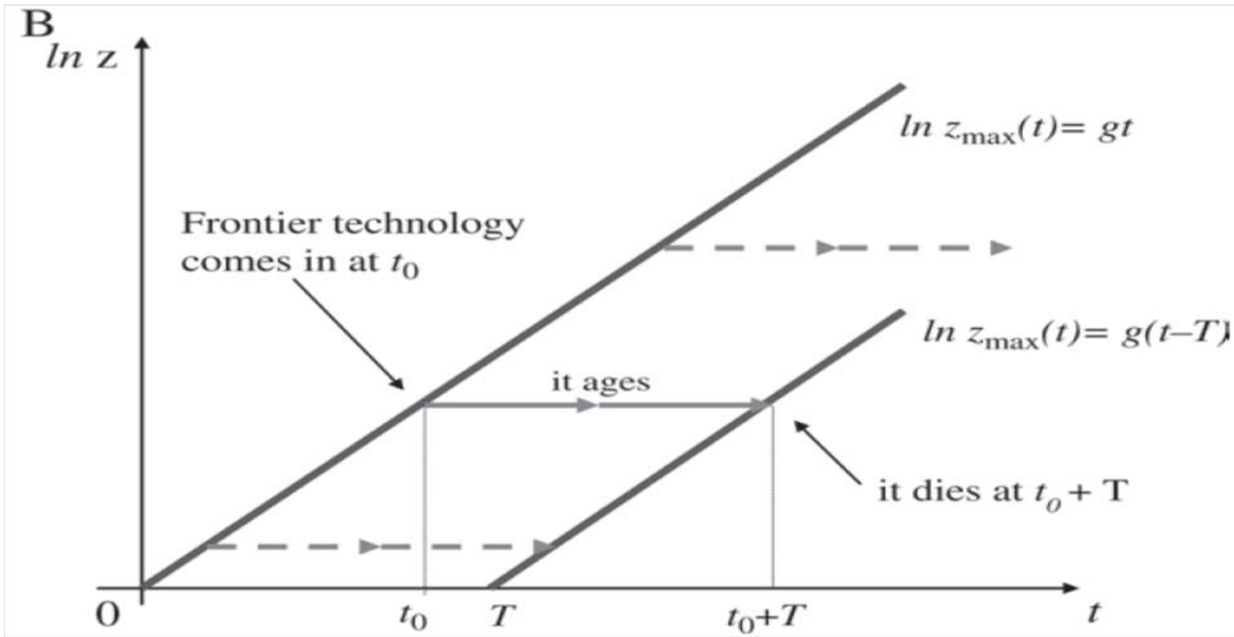
- **Verifying the conjecture:**
  - **market clearing at t holds:** $\int_{z}^{e^{gt}} n_t(v)dv = \int_{z/\theta}^{e^{gt}/\theta} m_t(s)ds$
  - **(FOC)** $\frac{1}{2}\left(\frac{\psi(z)y}{z}\right)^{1/2} - p'(z) = 0$ **, holds for all z over [g(t-T), gt]**
  - **boundary condition** $p(z_{\min}) = 0$ **holds at** $z_{\min} = e^{g(t-T)}$ **(tolerance for obsolescence)**

3. Main Findings

- **Skills (s) and technologies (z) co-move over time (tech-skill complementarity)**



- **Technology cycle: assignment hold at t only if products move down the skill distribution**



  - **dynamic complementarity between skills and technologies with vintage driven technology cycles (obsolescence of aged technologies)**
  - **association of co-existence of different technologies with skill (and hence wage income) inequalities, depending on the distribution (tail index) and the arrival and aging process**
- **Closing the model: consumption-saving decision, human capital accumulation**

H. Search, Knowledge Creation and Growth: Lucas-Moll (2014)

- **Key idea: by devoting time to productive knowledge creation (s) can lead to sustained growth**
- **Jovanovic-Rob (1992)/Berliant-Reed-Wang (2006) + Laing-Palivos-Wang (1995) + Violante (2002)**

1. The Model

- **Infinitely lived agents, each**
  - **is indexed by a production cost $\tilde{z}$ drawn from** $F(z,t) = \Pr\{\tilde{z} \leq z \text{ at date } t\}$, **with density f(z,t)**
  - **produces** $\tilde{a} = \tilde{z}^{-\theta}$ **units of a single consumption good (Pareto with tail 1/θ)**
- **Productivity distribution:** $G(a,t) = \Pr\{\tilde{z}^{-\theta} \leq a\} = \Pr\{\tilde{z} \geq a^{-1/\theta}\} = 1 - F(a^{-1/\theta}, t)$
- **Aggregate production:** $Y(t) = \int_0^\infty [1 - s(z,t)]\, z^{-\theta} f(z,t) dz$
- **Value function:** $V(z,t) = \mathbb{E}_t \left\{ \int_t^\infty e^{-\rho(\tau - t)} [1 - s(\tilde{z}(\tau), \tau)]\, \tilde{z}(\tau)^{-\theta} d\tau \,\middle|\, z(t) = z \right\}$
- **Learning by creation of productive knowledge upon search and meeting (Jovanovic-Rob 1989, Laing-Palivos-Wang 1995, Berliant-Reed-Wang 2006)**

- **Meeting-learning process follows Kortum (1997), with arrival α(s):**
  - **F(z,t) measures fraction of agents with cost below z at t, so it evolves as:**

$$\begin{aligned}1 - F(z, t + \Delta) &= \Pr\{\text{cost above } z \text{ at } t \text{ and no lower cost found in } [t, t+\Delta)\} \\ &= \int_z^\infty f(y,t) \Pr\{\text{no lower cost found in } [t, t+\Delta)\} dy \\ &= \int_z^\infty f(y,t) \left[1 - \alpha(s(y,t))\Delta + \alpha(s(y,t))\Delta \left(1 - F(z,t)\right)\right] dy \\ &= 1 - F(z,t) - F(z,t) \int_z^\infty \alpha(s(y,t)) f(y,t) \Delta dy.\end{aligned}$$

  - **Dividing by Δ and letting Δ →∞ ,**
    - $$\frac{\partial F(z,t)}{\partial t} = F(z,t) \int_z^\infty \alpha(s(y,t)) f(y,t) dy$$
    - $$\frac{\partial f(z,t)}{\partial t} = -\alpha(s(z,t)) f(z,t) \int_0^z f(y,t) dy + f(z,t) \int_z^\infty \alpha(s(y,t)) f(y,t) dy$$

$$= - \left.\frac{\partial f(z,t)}{\partial t}\right|_{\text{out}} + \left.\frac{\partial f(z,t)}{\partial t}\right|_{\text{in}}$$

      - **that is, a *birth-death process* with inflows net of outflows**
      - **faster arrival => faster replacement (α(s) affects both in and out)**

- **Bellman equation (flow value = flow benefit + 2 expected value changes):**

$$\rho V(z,t) = \max_{s\in[0,1]} \left\{ (1-s)z^{-\theta} + \frac{\partial V(z,t)}{\partial t} + \alpha(s)\int_0^z [V(y,t) - V(z,t)] f(y,t)dy \right\}$$

s.t. birth-death process

- **This turns out to be a mean-field game a la Lasry-Lions (2007)**

2. Equilibrium

- **An equilibrium is a triple (f, s, V) such that, given the initial distribution f(z,0),**
  - **given s, f satisfies the birth-death process for all (z,t)**
  - **given f, V satisfies the Bellman**
  - **z attains the maximum for all (z,t)**
- **A BGP is a growth rate γ and a triple (φ,σ,v) such that, all (z,t),**
  - **trend normalization (non-common growth):**

    $f(z,t) = e^{\gamma t}\phi(ze^{\gamma t})$, $V(z,t) = e^{\theta\gamma t}v(ze^{\gamma t})$, $s(z,t) = \sigma(ze^{\gamma t})$
  - **is an equilibrium given the initial condition:** $f(z,0) = \phi(z)$
- **Thus, along a BGP,**
  - **the cost cdf satisfies:** $F(z,t) = \Phi(ze^{\gamma t})$
  - **the q-th quantile satisfies:** $z_q(t) = e^{-\gamma t}\Phi^{-1}(q)$ **(for computing inequality)**

- **Rewriting in unit cost** $x = ze^{\gamma t}$ **leads to:**
  - **population evolution (PE): LHS = ∂*f*/∂*t* (applying detrending formula)**

$$\phi(x)\gamma + \phi'(x)\gamma x = \phi(x)\int_x^\infty \alpha(\sigma(y))\phi(y)dy - \alpha(\sigma(x))\phi(x)\int_0^x \phi(y)dy$$

  - **Bellman equation (BE): LHS includes detrended ρ*v* and ∂*v*/∂*t***

$$(\rho - \theta\gamma)\, v(x) - v'(x)\gamma x = \max_{\sigma\in[0,1]} \left\{(1-\sigma)x^{-\theta} + \alpha(\sigma)\int_0^x [v(y) - v(x)]\phi(y)\, dy\right\}$$

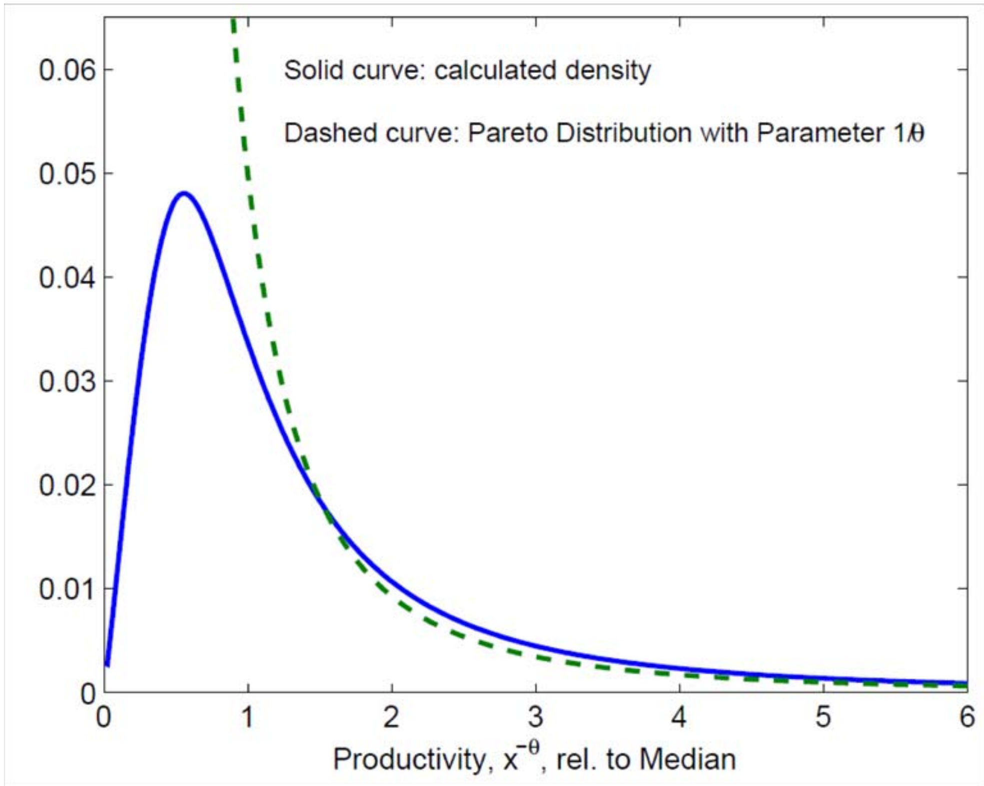
  - **BGP output:** $Y(t) = e^{\theta\gamma t}\int_0^\infty [1-\sigma(x)]x^{-\theta}\phi(x)dx$
  - **balanced growth rate (BG):** $\gamma = \int_0^\infty \alpha(\sigma(x))\phi(x)dx$ **(Poisson)**

3. Computation

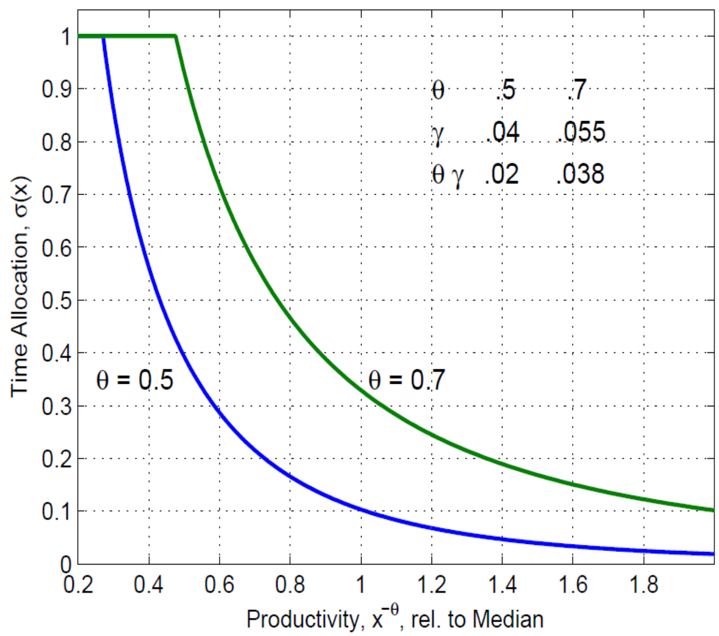
- **Specification of meeting arrival rate:** $\alpha(s) = ks^\eta, \quad \eta \in (0,1)$
- **Calculate BGP in 3 steps:**
  - **let initial guess be (φ_0, γ_0)**
  - **given (φ_n, γ_n), use (BE) to calculate (v_n, σ_n)**
  - **given σ_n, use (PE) and (BG) to obtain (φ_{n+1}, γ_{n+1})**
  - **iterate until j^{th} and $(j+1)^{th}$ becomes sufficiently close (i.e., reaching BGP)**

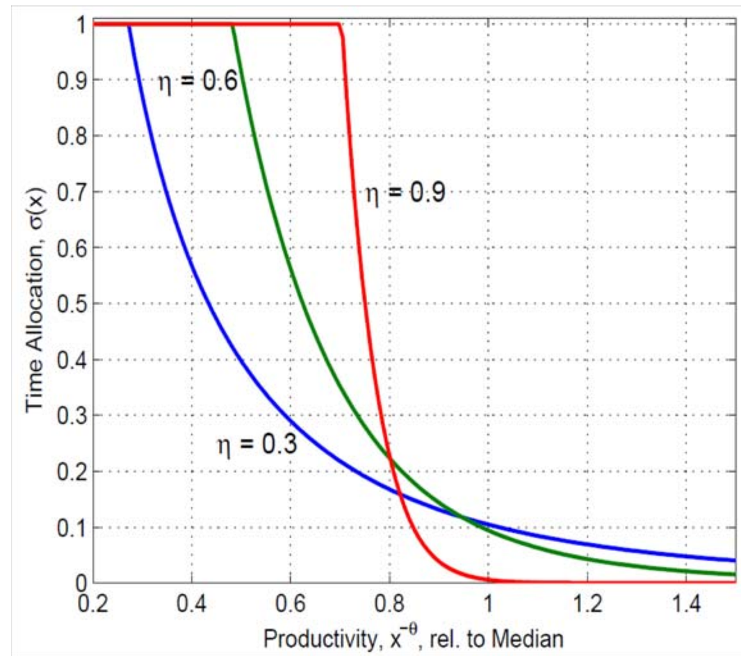
4. Main Findings

- **Productivity distribution for θ = 0.5:**



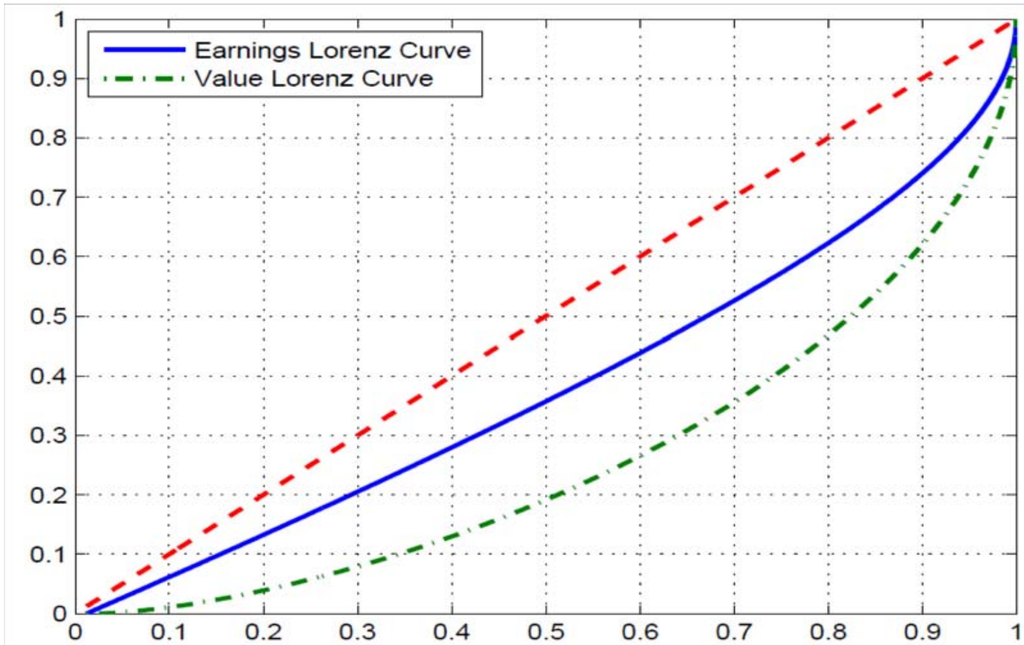
- **Time allocation to knowledge creation (σ) under different tail shape (θ) and search intensity elasticity (η):**





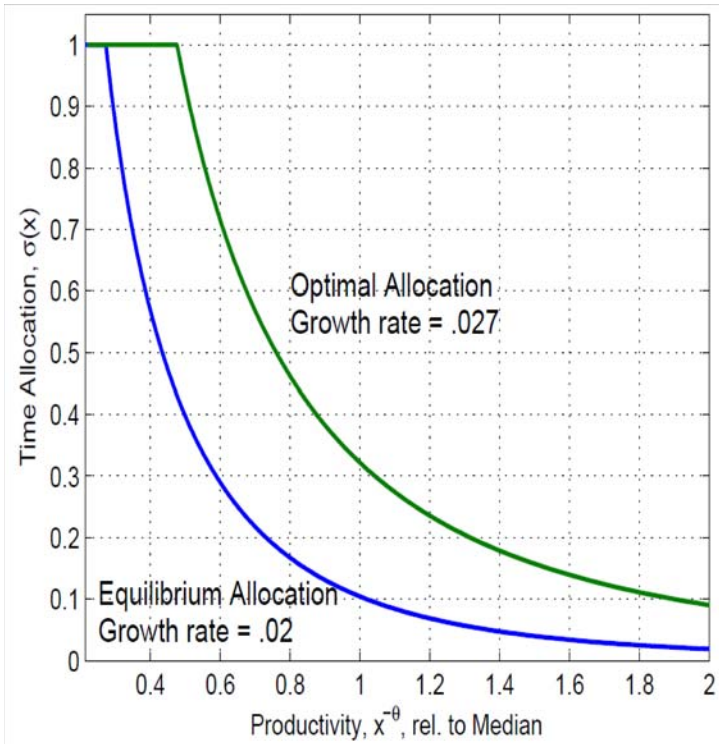
  - **thicker tail (low θ) encourages allocation of time toward working, thus leading to lower knowledge creation (σ) –** ***possibly counterfactual***
  - **more elastic search intensity (high η) discourages knowledge creation of the more productive but encourages that of the less productive – i.e., time or effort can make up for talent**

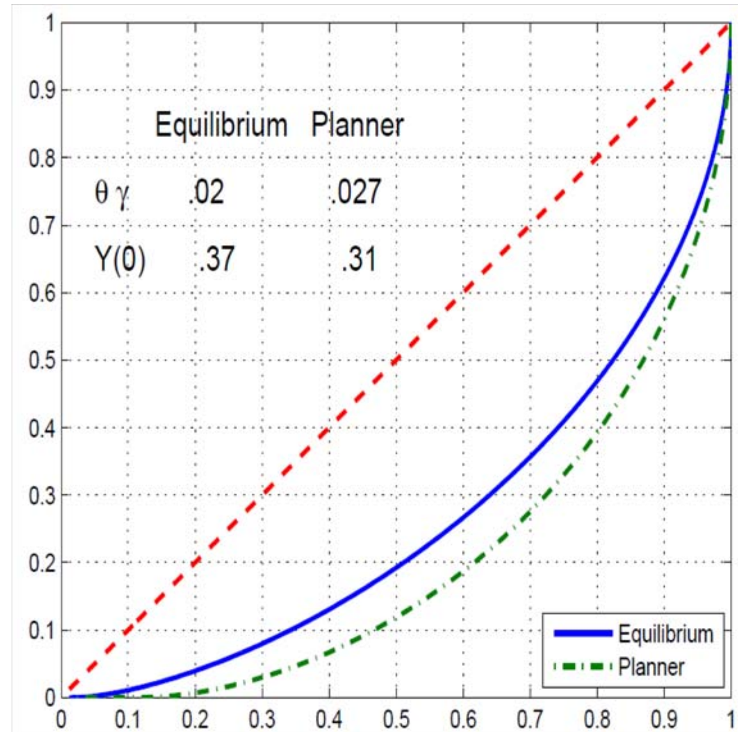
- **Inequality with benchmark (θ, η) = (0.5, 0.3):**



  - **earned income inequality (green dash) is more unequal than the value inequality (blue solid)**
  - **reduced inequality due to mobilizing between states by re-drawing z**
  - **caveat: value inequality here differs from standard wealth inequality because *intergenerational mobility, occupational choice,* and *portfolio management* are not considered**

- **Optimus versus Equilibrium**





  - **under-invest in knowledge creation in equilibrium due to the presence of positive search/matching externality as in Laing-Palivos-Wang (1995)**
  - **such *under-investment mitigate inequality***

I. Matching, Growth and Inequality: Jovanovic (2014)

- **Key: growth via OJL where learning is a by-product of production in a micro-matching setting with heterogeneous firms and workers**
  - **OJL-LBD:**
    - **in optimal growth: Lucas (1990)**
    - **with search-matching: Laing-Palivos-Wang (1995), Lucas-Moll (2014)**
  - **micro-matching:**
    - **in static games: Crawford-Knoer (1981)**
    - **with technical progress: Chen-Mo-Wang (2012)**
- **Observations:**
  - **substitutability in production and training:**
    - **Lentz-Mortensen (2010): cov(w, firm quality) < 0 if worker quality private information (survey data)**
    - **Lazear-Shaw-Stanton (2011): cov(worker quality, firm CEO quality)>0**
  - **LBD:**
    - **Lucas (1990): learning elasticity=0.2, delay rapidly with $d\mu/dt=0.25$**
    - **Gottschalk (1997), Jeong-Kim-Manovski (2011): experience premium about 2-5% per year**
  - **Gibrat's Law (reversion to common trend in firm TFP and output)**

1. **The Model**

- **2-period OLG, with skill captured by (y, x) for (young, old)**
- **Lifetime utility: $c_y + \beta c_o$**
- **Each firm hires one old and one young worker:**
  - **output:** $q_t = f(x_t, y_t)$ **(team work)**
  - **training:** $x_{t+1} = g(x_t, y_t)$ **(apprenticeship)**
  - **both are well-behaved CES**

2. **Homogeneous Skill Endowment**

- **Skill formation of a young with innate talent ε:**
  - $y = b\varepsilon\bar{x}$ **(the old are the same, so** $\bar{x} = x$ **)**
  - **under homogeneity, if g is *linear*, then** $x_{t+1} = Ax_t$
- **LR growth:** $\frac{q_{t+1}}{q_t} = \frac{x_{t+1}}{x_t} = \phi\left(1, \frac{1}{x_t} b\varepsilon x_t\right) = \phi(1, b\varepsilon) \equiv \Gamma$ **(state evolution)**

3. **Heterogeneous Skill in the Absence of Frictions**

- **Skill of an old $x \sim H(x)$; Skill of a young $y \sim G^H(y)$ with external effect**
- **One-for-one matching: $H(x) = G^H(y)$**

- **Assignment: y = α(x) =>** $\alpha^{\mathrm{H}}(x) = \left(G^{\mathrm{H}}\right)^{-1}(H[x])$
- **Evolution of firm productivity:** $x' = \xi^{\mathrm{H}}(x) = \phi\left(x, \alpha^{\mathrm{H}}(x)\right)$
- **Aggregate law of motion of H:** $H' = \chi(H) = H \cdot \left(\xi^{\mathrm{H}}\right)^{-1}$
- **Lifetime value of the young:** $V^{\mathrm{H}}(y) = w^{\mathrm{H}}\left(\left(\alpha^{\mathrm{H}}\right)^{-1}(y)\right) + \beta\pi^{\chi(\mathrm{H})}\left(\phi\left[\left(\alpha^{\mathrm{H}}\right)^{-1}(y), y\right]\right)$
- **Firm's decision:** $\pi^{\mathrm{H}}(x) = \max_{w,y}\{f(x,y) - w\}$ **s.t.** $w + \beta\pi^{\chi(\mathrm{H})}(\phi[x,y]) \geq V^{\mathrm{H}}(y)$,

  implying: $\pi^{\mathrm{H}}(x) = \max_{y}\left\{f(x,y) - V(y) + \beta\pi^{\chi(\mathrm{H})}(\phi[x,y])\right\}$
- **Equilibrium assignment:** $\alpha^{\mathrm{H}}(x) = \arg\max_{y}\{f(x,y) - V(y) + \beta\pi^{\chi(\mathrm{H})}(\phi[x,y])\}$
- **Spillover (externality):** $y = b\bar{x}_{\mathrm{H}}\varepsilon$ **with** $\bar{x}_{\mathrm{H}} = \int x dH(x)$
- **Heterogeneity: ε ~** $\hat{G}(\varepsilon)$ **=> y is distributed according to** $G^{\mathrm{H}}(y) = \hat{G}\left(\frac{y}{b\bar{x}_{\mathrm{H}}}\right)$
- **Along a BGP, defining** $\tilde{x} = \Gamma x$ **and its distribution by** $H^{\Gamma}(\tilde{x}) = H\left(\frac{\tilde{x}}{\Gamma}\right)$**, then:**
  - **assignment and its distribution:** $\alpha(x) = b\bar{\varepsilon}x$ **and** $\alpha^{H^{\Gamma}}(x) = \Gamma\alpha^{H}\left(\frac{x}{\Gamma}\right)$
  - **skill of the old:** $H(x) = \hat{G}\left(\frac{\varepsilon}{C}\right)$**, with** $H_0 = \hat{G}\left(\frac{\varepsilon}{C_0}\right)$**,** $H_t = \hat{G}\left(\frac{\varepsilon}{C_0\Gamma^t}\right)$
  - **balanced growth factor:** $\Gamma = \phi(1, b\bar{\varepsilon})$
  - **wage and profit:** $w(x) = \omega f(1, b\bar{\varepsilon})x$ **and** $\pi(x) = (1-\omega) f(1, b\bar{\varepsilon})x$ **where *ω* is the *output share of the young***

- **lifetime utility:** $V(y) = \left(\omega f\left(\frac{1}{b\bar{\varepsilon}}, 1\right) + \beta(1-\omega)\frac{\Gamma}{b\bar{\varepsilon}}\right) y$ **, where the output share of the young is:** $\omega = \dfrac{\frac{f_y(1,b\bar{\varepsilon})}{f(1,b\bar{\varepsilon})} + \beta\phi_y(1,b\bar{\varepsilon}) - \frac{1}{b\bar{\varepsilon}}\beta\Gamma}{\frac{1}{b\bar{\varepsilon}} + \beta\phi_y(1,b\bar{\varepsilon}) - \frac{1}{b\bar{\varepsilon}}\beta\Gamma}$

- **Example:** $b\bar{\varepsilon} = 1$, $f(x,y) = x^{1-\rho}y^{\rho}$, $\phi(x,y) = Ax^{1-\theta}y^{\theta}$ **=>** $\omega = \dfrac{\rho - \beta(1-\theta)A}{1-\beta(1-\theta)A}$
  - **(SR) *ω* ↑ in ρ (young's output elasticity) and θ (young's learning share)**
  - **(SR) *ω* ↓ in A (MP of learning)**
  - **∃ a *compensating differential* with wage share ω < ρ as long as β > 0**
    - **this is to compensate the old for the *apprenticeship***
    - **it is possible that *ω* < 0, implying that the young is paying tuition**
  - **(LR) wage growth young-old:** $\dfrac{1-\omega}{\omega}A = \dfrac{1-\rho}{\rho-(1-\theta)\beta A}A$**, ↓ in (ρ,θ), ↑ in A**
  - **(LR) experience premium:** $\dfrac{1-\omega}{\omega}A - A = \dfrac{1-2\rho+(1-\theta)\beta A}{\rho-(1-\theta)\beta A}A$**, ↓ in (ρ,θ), ↑ in A**

- **BGP under private information about y**
  - **wage:** $w(x) = w_0\bar{x} - wx$
  - **value:** $\hat{V}(y) = \max_x \{w_0\bar{x} - wx + \beta(-w_0\bar{x}\Gamma + [w + f(1,b\bar{\varepsilon})]\phi(x,y))\}$
  - **output share of the young:** $w = \dfrac{\beta f(1,b\bar{\varepsilon})\phi_x(1,b\bar{\varepsilon})}{1-\beta\phi_x(1,b\bar{\varepsilon})}$

- **compensating differential:** $w'(x) = -w$
- **which, under the special example, becomes:** $w = \dfrac{\beta(1-\theta)A}{1-\beta(1-\theta)A}$,

which is increasing in A and decreasing in θ (BGP => LR > SR)

4. Heterogeneous Skill in the Presence of Assignment Frictions

- **Assignment frictions: as a result of imperfect information about a young worker's ability due to imperfect public signal s of y (rather than private information)**
- **Publicly observed signal (say, GPA) distributed as:** $\Pr(\tilde{s} \leq s \mid y) = F(s \mid y)$
    - **so the signal distribution is given by,** $\Phi(s) = \int F(s \mid y)\, dG(y)$
    - **with assignment,** $s = \Phi^{-1}(H(x)) = \alpha(x)$
    - **posterior** $\pi(y \mid s)$ **and conditional probability of y given x** $\pi(y \mid \alpha^{-1}(x))$
    - **thus,** $\Pr(\tilde{x} \leq x' \mid x) = \Pr(Ax^{1-\theta}y^{\theta} \leq x' \mid x) = \pi\left(\left(\frac{x'}{Ax^{1-\theta}}\right)^{1/\theta} \mid \alpha(x)\right)$
- **Under log normal skill and signal distribution (for** $\hat{G}, H_0,$ *and* F**) and letting** $\hat{s} = \hat{y} + \eta$**, we can set** $r^2 = \dfrac{\sigma_{\hat{y}}^2}{\sigma_{\hat{y}}^2 + \sigma_{\eta}^2} = \dfrac{\sigma_{\hat{\varepsilon}}^2}{\sigma_{\hat{\varepsilon}}^2 + \sigma_{\eta}^2}$ **as in *signal extraction* models**

- **Along a BGP, we have:**
  - **balanced growth factor:** $\Gamma = A\left(b\bar{\varepsilon}\right)^{\theta} = Ab^{\theta}\exp\left(\theta\mu_{\hat{\varepsilon}} + \frac{\theta}{2}\sigma_{\hat{\varepsilon}}^{2}\right)$**, or, taking log to obtain the growth rate,** $\hat{\Gamma} = \hat{A} + \theta\left(\ln b + \mu_{\hat{\varepsilon}} + \frac{\theta}{2}\sigma_{\hat{\varepsilon}}^{2}\right)$
  - **evolution of skills of the old:**

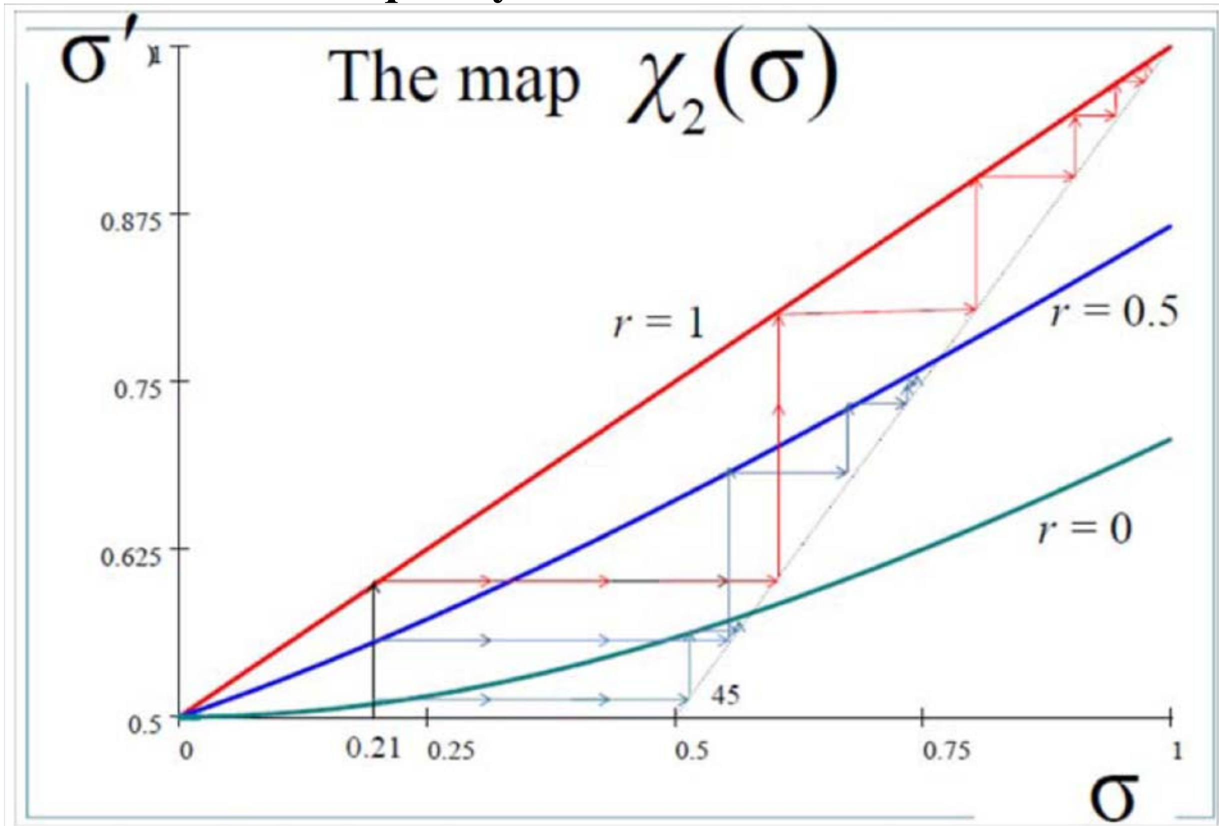
$$H_{t+1}\left(x'\right) = \int \pi\left(\left(\frac{x'}{Ax^{1-\theta}}\right)^{1/\theta} \mid \Phi_t^{-1}\left(H_t\left(x\right)\right)\right) dH_t\left(x\right)$$

    - **since H_0 is log normal, H becomes log normal for all t**
    - **so the distribution is fully summarized by two parameters – mean and standard deviation (sd)**
      - **mean:** $\mu' = \chi_1\left(\mu, \sigma\right) = \mu + \hat{A} + \theta\left(\ln b + \mu_{\hat{\varepsilon}} + \frac{1}{2}\sigma^{2}\right)$
      - **sd:** $\sigma' = \chi_2\left(\mu, \sigma\right) = \sqrt{\left(1-\theta\right)^{2}\sigma^{2} + 2\theta\left(1-\theta\right) r\sigma_{\hat{\varepsilon}}\sigma + \theta^{2}\sigma_{\hat{\varepsilon}}^{2}}$
    - **stationarity leads to a unique fixed point** $\sigma = \chi_2\left(\sigma\right)$**, given by,**
      $\sigma\left(r\right) = \frac{\sigma_{\hat{\varepsilon}}}{2-\theta}\left(r\left(1-\theta\right) + \sqrt{1-\left(1-r^{2}\right)\left(1-\theta\right)^{2}}\right)$**, satisfying:**

$$\frac{\partial\sigma}{\partial r} \geq \left.\frac{\partial\sigma}{\partial r}\right|_{r=0} = \frac{1-\theta}{2-\theta}\sigma_{\hat{\varepsilon}}$$

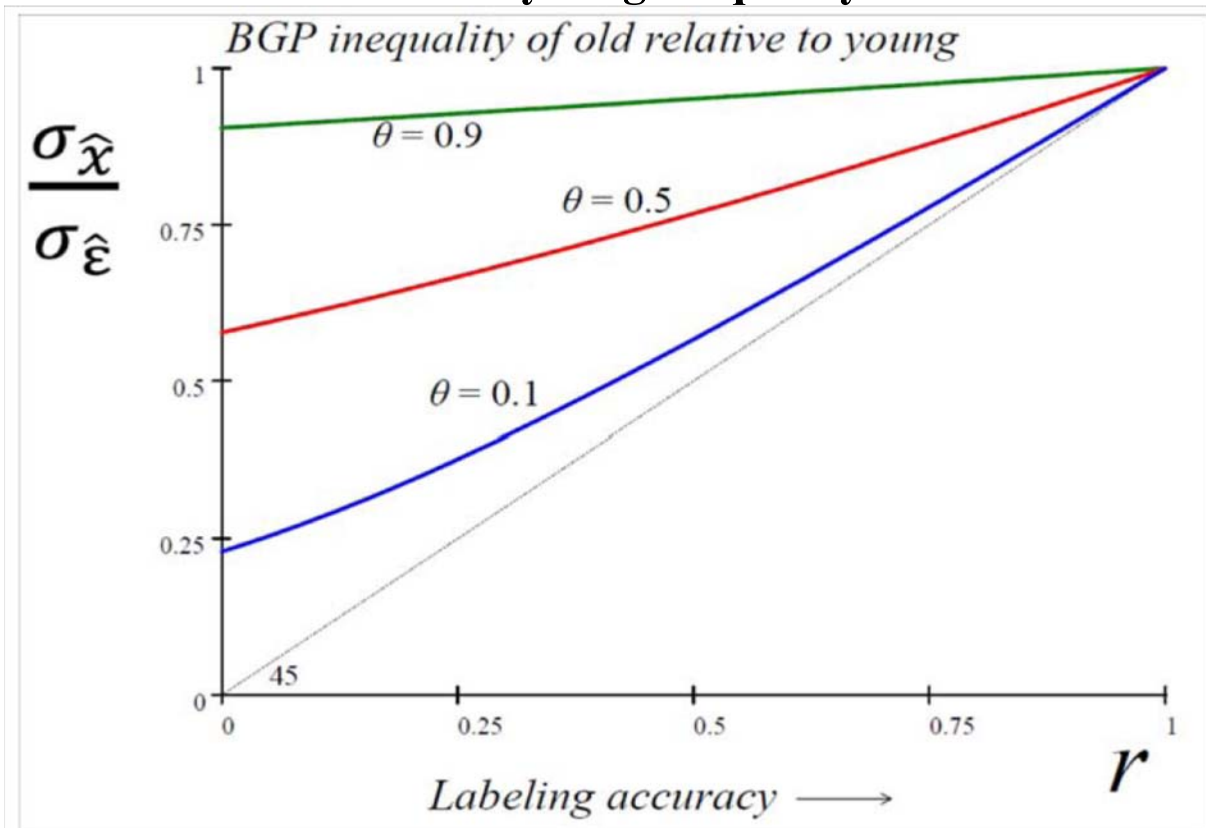
$$\frac{\partial\sigma}{\partial r} \leq \left.\frac{\partial\sigma}{\partial r}\right|_{r=1} = \left(1-\theta\right)\sigma_{\hat{\varepsilon}}.$$

- **within-the-old-inequality:**



- **when signal is noisier (r lower), the dispersion of the (old) skill distribution (an inequality measure) features slower rise in with-group inequality**
- **when the *signal quality is perfect* (r = 1), the economy sees the *fastest rise in within-group inequality***

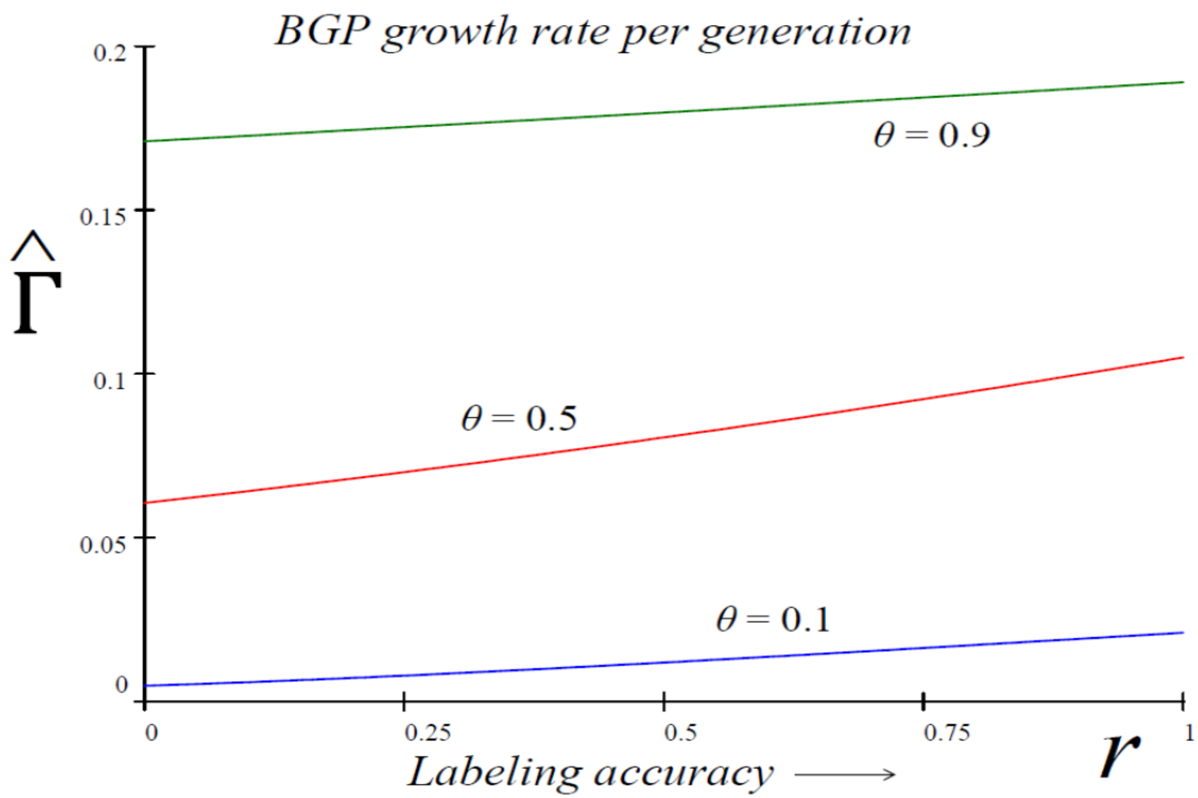
- **between-the-old-and-the-young inequality:**



- **a rise in the young's learning share (higher θ) reduces between-group inequality**
- ***better signal quality* (higher r) mitigates imperfect information about the young, which also *reduces between-group inequality* despite it *raises within-the-old-group inequality***

- **charactering the balanced growth rate:**

$$\hat{\Gamma} = \hat{A} + \theta\left(\ln b + \mu_{\hat{\varepsilon}} + \frac{1}{2}\left(\frac{\sigma_{\hat{\varepsilon}}}{2-\theta}\left(r\left(1-\theta\right) + \sqrt{1-\left(1-r^2\right)\left(1-\theta\right)^2}\right)\right)^2\right)$$



- **increasing in the young's learning share and signal quality**
- **if θ sufficiently high at low development stage without much training, then Kuznets' curve emerges**

- **firms' TFP growth:**

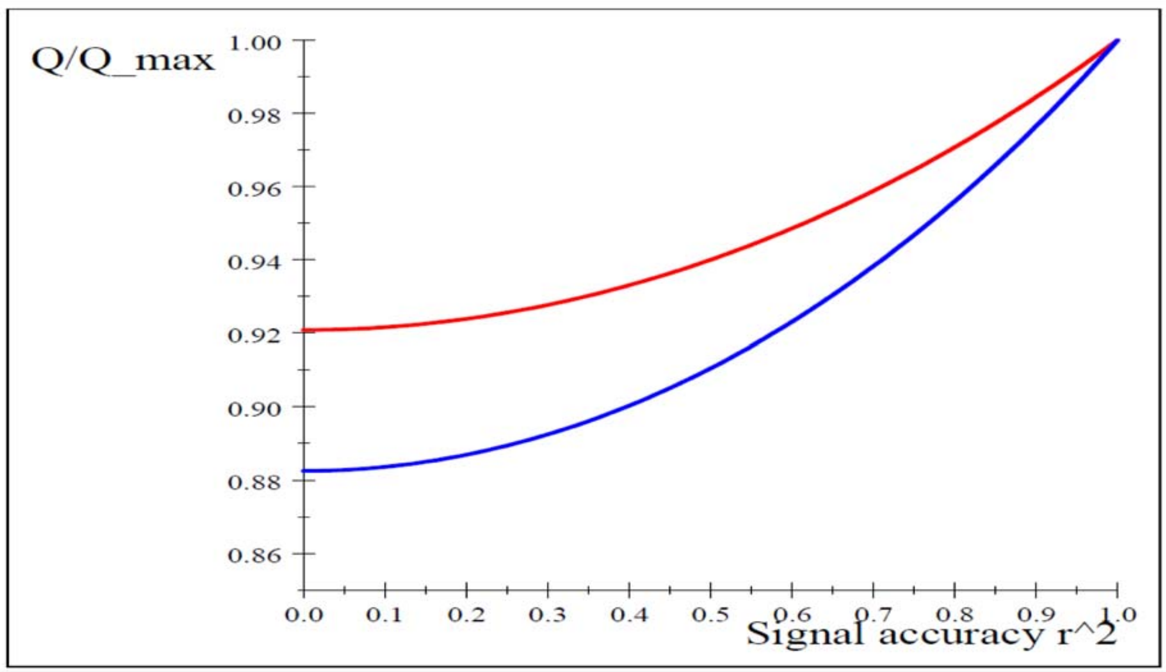
$$\hat{x}' - \hat{x} = \underbrace{\hat{A} + \theta\left(\mu_{\hat{y}} - r\mu_{\hat{x}}\frac{\sigma_{\hat{\varepsilon}}}{\hat{\sigma}_{\hat{x}}}\right)}_{\text{independent of } \hat{x}\text{, and constant on BGP}} - \underbrace{\theta\left(1 - r\frac{\sigma_{\hat{\varepsilon}}}{\hat{\sigma}_{\hat{x}}}\right)}_{>0 \text{ on BGP}}\hat{x} + \underbrace{\theta\sqrt{1-r^2}\sigma_{\hat{\varepsilon}}\zeta}_{\text{zero-mean}}$$

- **taking average, then firms' TFP growth depends on its initial state negatively (convergence)**
- **the speed of convergence depends positively on the young's learning share but negatively on signal quality – *Great Moderation as a result of better information***
- **the BGP stationary x is given by,**

$$\hat{x}' = \hat{A} + \theta\left(\ln b + \mu_{\hat{x}} + \mu_{\hat{\varepsilon}} + \frac{1}{2}\left(\sigma^2 + \sigma_{\hat{\varepsilon}}^2\right) - r\frac{\sigma_{\hat{\varepsilon}}}{\sigma}\mu_{\hat{x}}\right) + (1-\theta)\hat{x} + \theta r\frac{\sigma_{\hat{\varepsilon}}}{\sigma}\hat{x} + \theta\sqrt{1-r^2}\sigma_{\hat{\varepsilon}}\zeta$$

- **thus, with random growth of individual firms (old-young pairs), the law of large numbers implies that the aggregate economy evolves deterministically**

- **Consider two economies with identical distribution but different r (r_1 =1, $r_2 \leq 1$)**
  - **take** $\sigma^2_{\hat{\varepsilon}} = 0.33$ **(cf. Heathcote-Storesletten-Violante 2005)**
  - **compute cross-country output gap:** $\frac{Q_1}{Q_2} = \exp\left(\rho\left(1-\rho\right)\sigma_{\hat{\varepsilon}}\sigma_{\hat{x}}\left[r_1 - r_2\right]\right)$
  - **in the LR,** $\sigma_{\hat{x}} = r\sigma_{\hat{\varepsilon}}$ **, so** $\lim_{t\to\infty}\frac{Q_{1,t}}{Q_{2,t}} = \exp\left(\rho\left(1-\rho\right)\sigma^2_{\hat{\varepsilon}}\left[r_1^2 - r_2^2\right]\right)$ **(plot in Red)**



  - **the *cross-country output gap shrinks* as signal quality in 2 improves**
  - **as** $\sigma^2_{\hat{\varepsilon}}$ **rises to 0.50 (plot in Blue), such output gap is widened.**

J. Accounting for Between Group Inequality: Burstein-Morales-Vogel (2015)

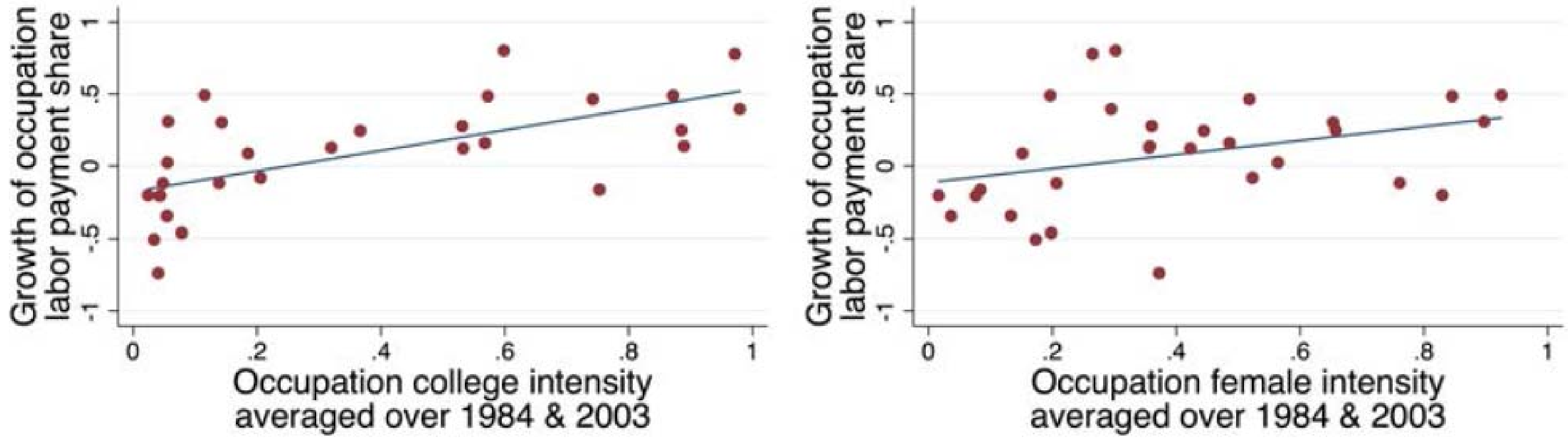
- U.S. between group inequality over 1984-2003
- Basic idea: assignment model with heterogeneous labor and machine
- Key drivers of changes in between group inequality: changes in
    - workforce composition
    - occupation demand
    - computerization
    - labor productivity

1. Empirical Observations

- Shares of hours worked with computer has been doubled, with even higher growth for female and college educated:

		1984	1989	1993	1997	2003
All		27.4	40.1	49.8	53.3	57.8
Gender	Female	32.8	47.6	57.3	61.3	65.1
	Male	23.6	34.5	43.9	47.0	52.1
Education	College degree	45.5	62.5	73.4	79.8	85.7
	No college degree	22.1	32.7	41.0	43.7	45.3

- **Education and female-intensive occupations have grown faster:**



2. The Model

- **A single final good is a CES aggregator of occupation (indexed by ω) outputs:**

$$Y_t = \left(\sum_{\omega} \mu_t(\omega)^{1/\rho} Y_t(\omega)^{(\rho-1)/\rho} \right)^{\rho/(\rho-1)}$$

  - **occupation demand is captured by exogenous share μ**
- **The output of this final good is used for consumption and capital equipment:**

$$Y_t = C_t + \sum_{\kappa} p_t(\kappa) Y_t(\kappa)$$

  - **capital depreciates fully every period with exogenous price p(κ), driven by computerization**

- **Occupation outputs: produced by perfectly competitive production units, each hiring k units of equipment type κ and *l* efficiency units of labor of group λ:**

  $Y_t(\omega)$ = $k^{\alpha}\left[T_t(\lambda,\kappa,\omega)\,l\right]^{1-\alpha}$

  - **T = productivity**
  - **λ′ is better matched with κ′ if** $T_t(\lambda',\kappa',\omega)/T_t(\lambda',\kappa,\omega) \geq T_t(\lambda,\kappa',\omega)/T_t(\lambda,\kappa,\omega)$
- **Each worker** $z \in \mathcal{Z}_t(\lambda)$ **provides** $\epsilon(z) \times \varepsilon(z,\kappa,\omega)$ **efficiency units of labor when paired with equipment κ in occupation ω**
  - $\epsilon(z)$ **is entirely worker specific, with** $\epsilon(z) \in (\underline{\epsilon}_{\lambda}, \bar{\epsilon}_{\lambda})$ **and mean normalized to one in each group λ**
  - $\varepsilon(z,\kappa,\omega)$ **is drawn independently from a Fréchet distribution with tail index 1/θ:** $G(\varepsilon) = \exp\left(\varepsilon^{-\theta}\right)$
- **Wage per efficiency unit:** $v_t(\lambda,\kappa,\omega) = \bar{\alpha} p_t(\kappa)^{\frac{-\alpha}{1-\alpha}} p_t(\omega)^{\frac{1}{1-\alpha}} T_t(\lambda,\kappa,\omega)$
- **Probability for a randomly sampled worker z to use κ in ω (Fréchet):**

$$\pi_t(\lambda,\kappa,\omega) = \frac{\left[T_t(\lambda,\kappa,\omega) p_t(\kappa)^{\frac{-\alpha}{1-\alpha}} p_t(\omega)^{\frac{1}{1-\alpha}}\right]^{\theta}}{\sum_{\kappa',\omega'}\left[T_t(\lambda,\kappa',\omega') p_t(\kappa')^{\frac{-\alpha}{1-\alpha}} p_t(\omega')^{\frac{1}{1-\alpha}}\right]^{\theta}}$$

  - $\frac{T_t(\lambda',\kappa',\omega)}{T_t(\lambda',\kappa,\omega)} \Big/ \frac{T_t(\lambda,\kappa',\omega)}{T_t(\lambda,\kappa,\omega)} = \left(\frac{\pi_t(\lambda',\kappa',\omega)}{\pi_t(\lambda',\kappa,\omega)} \Big/ \frac{\pi_t(\lambda,\kappa',\omega)}{\pi_t(\lambda,\kappa,\omega)}\right)^{1/\theta}$ **=> better matched more likely to be assigned together**

- **Average wage of workers in λ teamed with (κ, ω):**

$$w_t(\lambda,\kappa,\omega) = \bar{\alpha}\gamma T_t(\lambda,\kappa,\omega) p_t(\kappa)^{\frac{-\alpha}{1-\alpha}} p_t(\omega)^{\frac{1}{1-\alpha}} \pi_t(\lambda,\kappa,\omega)^{-1/\theta}$$

  - $\gamma \equiv \Gamma(1-1/\theta)$ **is a Gamma function depending on the tail index**
- **Average wage of group λ:** $w_t(\lambda) = \bar{\alpha}\gamma \left(\sum_{\kappa,\omega} \left(T_t(\lambda,\kappa,\omega) p_t(\kappa)^{\frac{-\alpha}{1-\alpha}} p_t(\omega)^{\frac{1}{1-\alpha}} \right)^{\theta} \right)^{1/\theta}$

2. Equilibrium

- **Occupation goods prices:** $\mu_t(\omega) p_t(\omega)^{1-\rho} E_t = \frac{1}{1-\alpha} \zeta_t(\omega)$
  - **total income:** $E_t \equiv (1-\alpha)^{-1} \sum_\lambda w_t(\lambda) L_t(\lambda)$
  - **total labor income in ω:** $\zeta_t(\omega) \equiv \sum_{\lambda,\kappa} w_t(\lambda) L_t(\lambda) \pi_t(\lambda,\kappa,\omega)$
  - **expenditure on ω good = total income earned in ω**
- **Goods market equilibrium:** $Y_t = E_t$ **=> aggregate equipment investment in κ,**

$$Y_t(\kappa) = \frac{1}{p_t(\kappa)} \frac{\alpha}{1-\alpha} \sum_{\lambda,\omega} \pi_t(\lambda,\kappa,\omega) w_t(\lambda) L_t(\lambda)$$

- **Assumption:** $T_t(\lambda,\kappa,\omega) \equiv T_t(\lambda) T_t(\kappa) T_t(\omega) T(\lambda,\kappa,\omega)$
  - **productivity of labor, equipment and occupation each can change, but not their interactions**

- **Define composite occupation demand shifter & composite capital price shifter:** $a_t(\omega) \equiv \mu_t(\omega) T_t(\omega)^{(1-\alpha)(\rho-1)}$ **and** $q_t(\kappa) \equiv p_t(\kappa)^{\frac{-\alpha}{1-\alpha}} T_t(\kappa)$

- **Changes in average wage of group λ:** $\hat{w}(\lambda) = \hat{T}(\lambda) \left[\sum_{\kappa,\omega} (\hat{q}(\omega) \hat{q}(\kappa))^{\theta} \pi_{t_0}(\lambda,\kappa,\omega) \right]^{1/\theta}$

  - **transformed occupation good price:** $q_t(\omega) \equiv p_t(\omega)^{1/(1-\alpha)} T_t(\omega)$**, whose rate of change solves the following system of equations**

    - $\hat{\pi}(\lambda,\kappa,\omega) = \dfrac{(\hat{q}(\omega)\hat{q}(\kappa))^{\theta}}{\sum_{\kappa',\omega'} (\hat{q}(\omega')\hat{q}(\kappa'))^{\theta} \pi_{t_0}(\lambda,\kappa',\omega')}$

    - $\hat{a}(\omega)\hat{q}(\omega)^{(1-\alpha)(1-\rho)} \hat{E} = \dfrac{1}{\zeta_{t_0}(\omega)} \sum_{\lambda,\kappa} w_{t_0}(\lambda) L_{t_0}(\lambda) \pi_{t_0}(\lambda,\kappa,\omega) \hat{w}(\lambda) \hat{L}(\lambda) \hat{\pi}(\lambda,\kappa,\omega)$

  - **4 key drivers:**
    - **workforce composition:** $\hat{L}(\lambda)$
    - **occupation demand:** $\hat{a}(\omega)$
    - **computerization:** $\hat{q}(\kappa)$
    - **labor productivity:** $\hat{T}(\lambda)$

3. Calibration Analysis

- **Data: CPS (May Outgoing Rotation Group, MORG) and October Supplement**

- **Key parameters:**

Parameter	Time Trend?	Estimate
(θ, ρ)	NO	(1.78, 1.78)
(θ, ρ)	YES	(1.13, 2.00)

- **Decomposition of *skill premium*:**

	Data	Labor comp.	Occ. shifters	Equip. prod.	Labor prod.
1984 - 1989	0.057	-0.031	0.026	0.052	0.009
1989 - 1993	0.064	-0.017	-0.009	0.045	0.046
1993 - 1997	0.037	-0.023	0.044	0.021	-0.005
1997 - 2003	-0.007	-0.043	-0.011	0.042	0.006
1984 - 2003	0.151	-0.114	0.049	0.159	0.056

  - ***computerization*** **is the most important positive driver and skilled labor force expansion the most important negative driver**
  - **the negative effect of skill expansion is almost offset by occupation and labor productivity shifts**

- **Decomposition of gender gap:**

	Data	Labor comp.	Occ. shifters	Equip. prod.	Labor prod.
1984 - 1989	-0.056	0.012	-0.009	-0.016	-0.044
1989 - 1993	-0.052	0.013	-0.035	-0.014	-0.016
1993 - 1997	-0.003	0.006	0.015	-0.005	-0.020
1997 - 2003	-0.021	0.012	-0.038	-0.012	0.019
1984 - 2003	-0.133	0.042	-0.067	-0.047	-0.061

  - **gap narrowed due to occupation shift, computerization and labor productivity, despite higher female labor expansion**
- **Variance decomposition:**

	1984-2003	1984-1989	1989-1993	1993-1997	1997-2003
Equipment Productivity	52.93%	47.91%	33.15%	27.70%	51.38%
Occupation Shifter	23.80%	24.57%	10.54%	48.50%	14.92%
Labor Productivity	23.27%	27.52%	56.30%	23.80%	29.19%

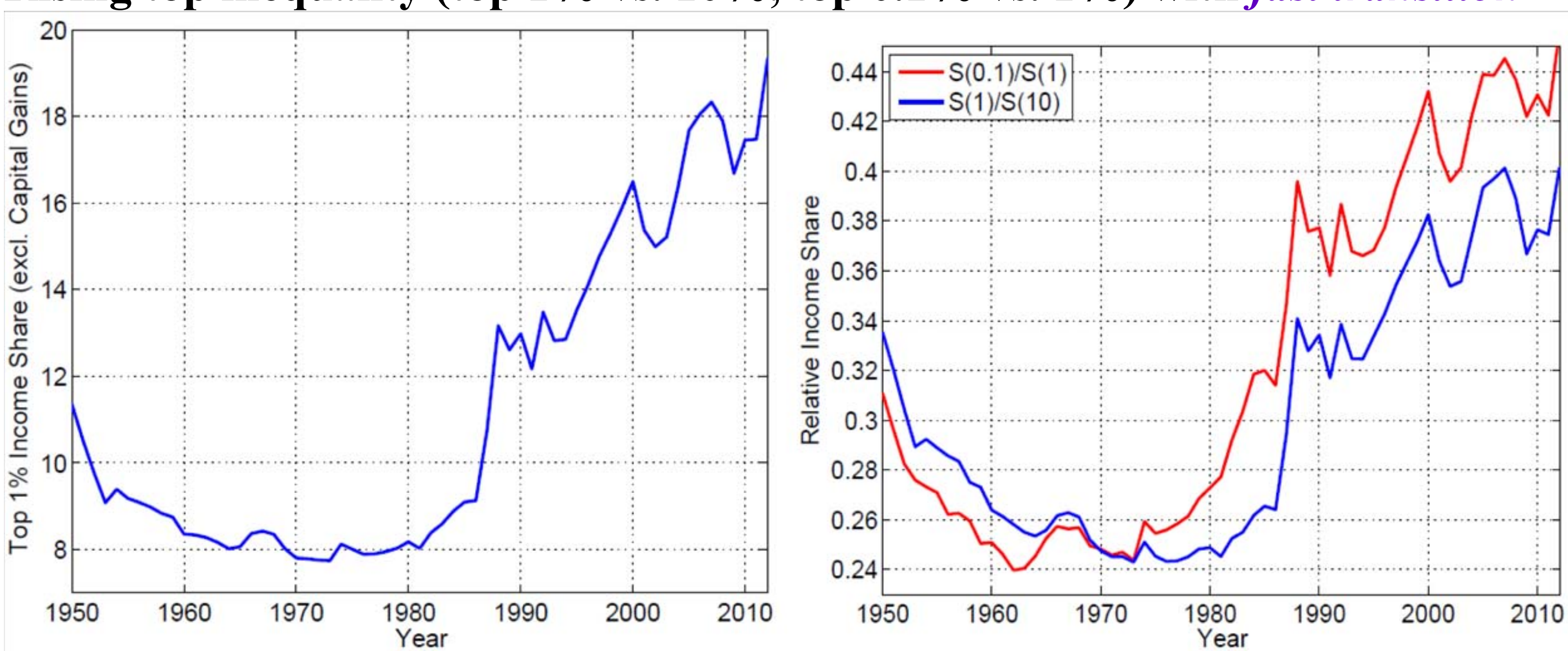
- **Open issues:**
  - **forces underlying shifts in occupation**
  - **forces underlying worker-machine matches**

K. Superstar Entrepreneurs and Inequality: Gabaix-Lasry-Lions-Moll (2015)

- **Cagetti-De Nardi (2006): over the past 3 decades in the U.S., top 1% own 1/3 of national wealth, top 5% more than 1/2**
- **Piketty (2014): sharp rise in top inequality based on the r > g theory**

1. Facts

- **Rising top inequality (top 1% vs. 10%, top 0.1% vs. 1%) with *fast transition***



2. Can Random Growth Theory Generate Fast Transition?

- **Continuous time with a continuum of workers who die/retire at rate δ**
- **Income dynamics: worker i's wage is given by, $w_{it} = \omega h_{it}$:**
  - **ω = exogenous skill price**
  - **h = human capital, evolving according to $dh_{it} = g(I_{it}, h_{it})dz_{it}$:**
    - **ability shocks are iid, following a geometric Brownian motion $dz_{it} = \bar{z}dt + \tilde{\sigma}dZ_{it}$**
    - **g is CRS in human capital and human capital investment, $I_{it} = \theta h_{it}$**
    - **initially identical human capital: $h_{i0} = \bar{h}_0$**
  - **so wage dynamics are governed by: $dw_{it} = \bar{z}g(\theta, 1)w_{it}dt + \tilde{\sigma}g(\theta, 1)w_{it}dZ_{it}$**
- **Wealth dynamics: $d\tilde{w}_{it} = (1-\tau)\tilde{w}_{it}dR_{it} + (y_t - c_{it})\,dt$**
  - **τ = capital income tax rate**
  - **rate of return on wealth: $dR_{it} = \tilde{r}dt + \tilde{\nu}dZ_{it}$**
  - **fixed consumption-wealth ratio: $c_{it} = \tilde{\theta}\tilde{w}_{it}$**
  - **identical labor income growing at rate g: $y_t = ye^{gt}$**
  - **so detrended wealth ($w_{it} = \tilde{w}_{it}e^{-gt}$) evolves according to:**

    $$dw_{it} = [y + (r - g - \tilde{\theta})w_{it}]dt + \sigma w_{it}dZ_{it}$$

    - $r = (1-\tau)\tilde{r}$ **= after-tax average rate of return on wealth**
    - $\sigma = (1-\tau)\tilde{\nu}$ **= after-tax wealth volatility**

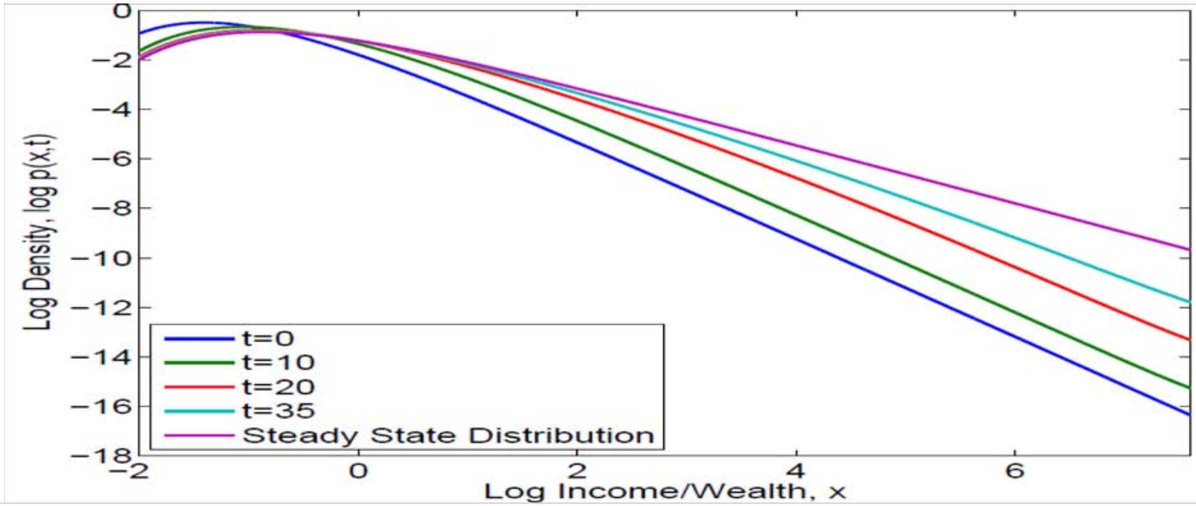
- **Aside from an additive term ydt for wealth, income and wealth have common reduced form dynamics:** $dw_{it} = \bar{\gamma} w_{it} dt + \sigma w_{it} dZ_{it}$
  - **to avoid the cross-sectional variance of w_{it} to grow unboundedly, consider:**
    - **(A1) death at δ accompanied by birth/injection at w_0, with $w_0 = 1$**
    - **(A2) a reflecting barrier $\underline{w}$, with $\underline{w} = 1$**
    - **(A3) both A1 and A2**
  - **taking log (**$x_{it} = \log w_{it}$**) and applying Ito's lemma, we have:**

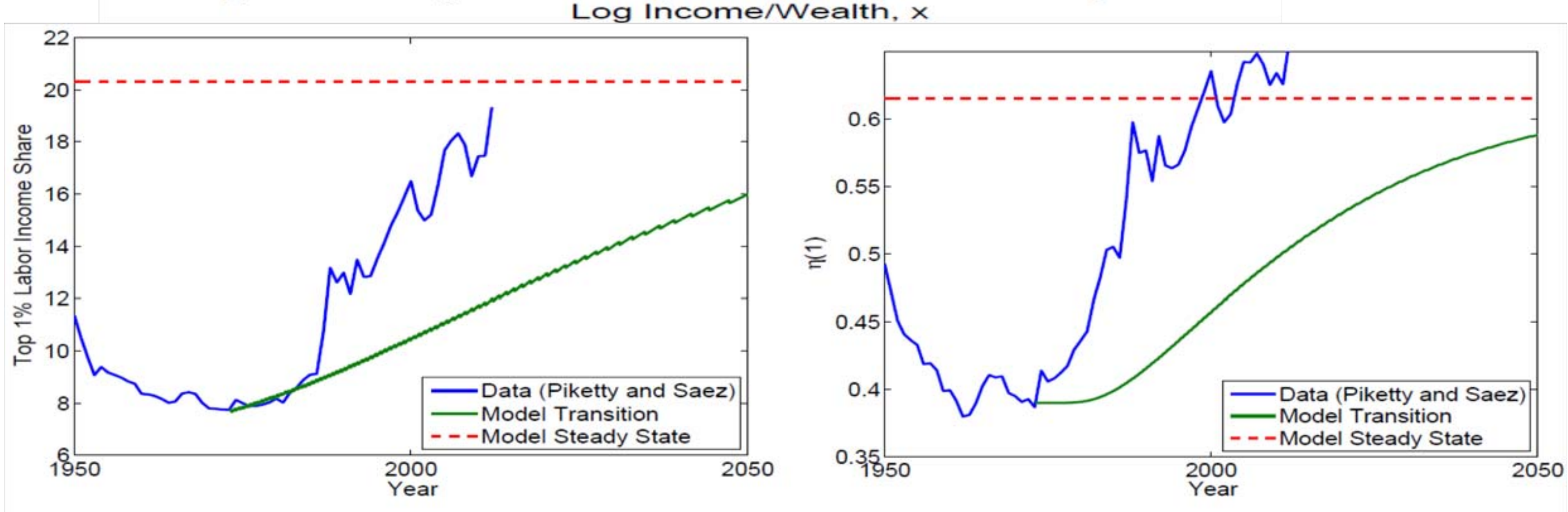
$$dx_{it} = \mu dt + \sigma dW_{it}, \quad \text{where} \quad \mu = \bar{\gamma} - \frac{\sigma^2}{2}$$

- **The stationary distribution of w has a Pareto tail:** $\mathbb{P}(w_{it} > w) \sim C w^{-\zeta}$
  - **under (A1) (for the case of income dynamics),** $\zeta = \dfrac{-\mu + \sqrt{\mu^2 + 2\sigma^2 \delta}}{\sigma^2}$
  - **the dispersion measure** $\eta = 1/\zeta$ **is related to top inequality:**
    - **if w has a Pareto tail above p%, then the share of top p%/10% relative to the share of the p% is simply** $\frac{S(p/10)}{S(p)} = 10^{\eta - 1}$
    - **this ratio only depends on η (due to power law)**

- The cross-sectional distribution of x is denoted by p(x,t), satisfying the Kolmogorov Forward equation: $p_t = -\mu p_x + \frac{\sigma^2}{2} p_{xx} - \delta p + \delta \delta_0$
  - $p_t = \partial p(x,t)/\partial t$, $p_x = \partial p(x,t)/\partial x$, $p_{xx} = \partial^2 p(x,t)/\partial x^2$
  - δ_0 is the point mass (Dirac delta function) at x = 0, capturing new born reinjection
  - (A4) its stationary distribution $p_\infty(x)$ is unique
- Converging rate to the stationary distribution:
  - converging exponentially $||p(x,t) - p_\infty(x)|| \sim ke^{-\lambda t}$
  - the converging rate $\lambda = -\lim_{t \to \infty} \frac{1}{t} \log ||p(x,t) - p_\infty(x)||$ is
    - $\lambda = \delta$ without a reflecting barrier but under (A1): fast death leads to more churning and faster convergence
    - $\lambda = \frac{1}{2}\frac{\mu^2}{\sigma^2} \mathbf{1}_{\{\mu<0\}} + \delta$ with a reflecting barrier ($\mu < 0$)
      - without death (A2): $\zeta = 2\mu/\sigma^2$, so $\lambda = (1/8)(\sigma^2/\eta^2)$ => converging *slowly* with large top inequality measured by η
      - with death (A3): combining both arguments

- **Results:**





(a) Top 1% Labor Income Share (b) Empirical Inverse Power Law Exponent

  - **transition too slow, even so with jumps added to the stochastic process**

3. Augmented Random Growth Model

- **Key additions:**
  - **J growth regimes indexed by j = 1, ..., J, with different mean growth μ and standard deviation σ**
  - **deviations from Gibrat's law captured by an arbitrary process S with finite mean: superstar shocks**
  - **jumps dN, following a Poisson process with parameter φ**
- **Income dynamics:**

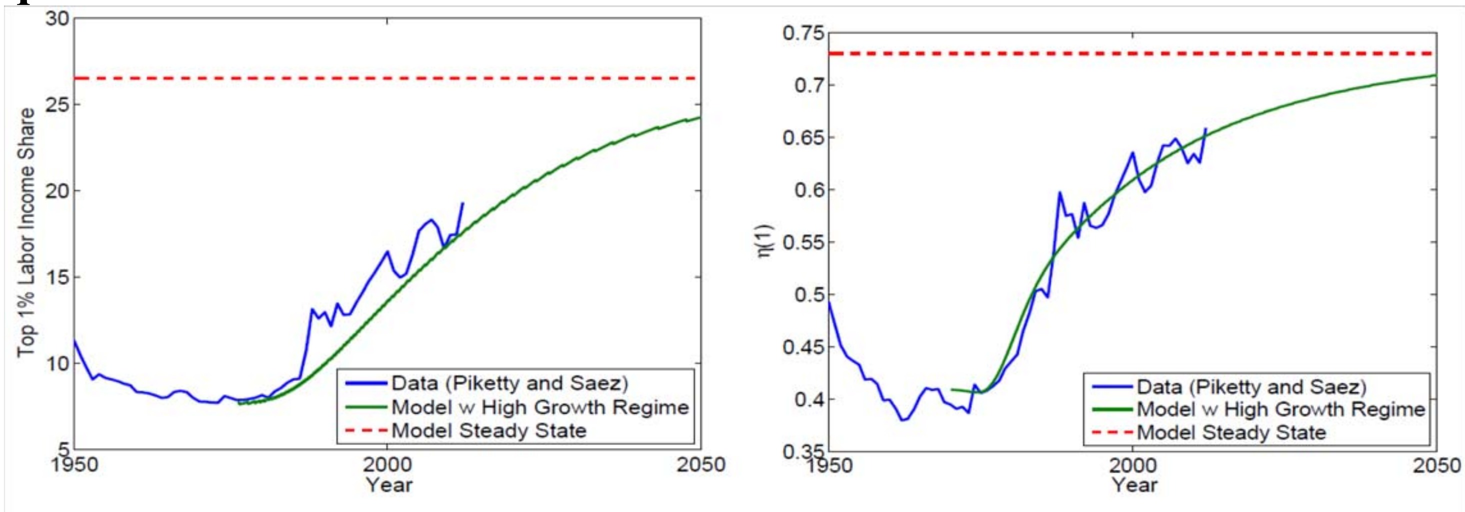
$$x_{it} = e^{b_j S_t} y_{it},$$

$$dy_{it} = \mu_j dt + \sigma_j dZ_{it} + g_{jit} dN_{jit} + \text{Injection} - \text{Death}$$

  - **g is a random variable with distribution f**
  - **superstar shocks affect top income earners disproportionately (shock to log income is multiplicative in log income rather than additive)**
  - **thus,** $dx_{it} = \tilde{\mu}_{jt} dt + \tilde{\sigma}_{jt} dZ_{it} + b_j x_{it} dS_t + g_{jit} dN_{jit} + \text{Injection} - \text{Death}$
    - $\tilde{\mu}_{jt} = \mu_j e^{b_j S_t}, \tilde{\sigma}_{jt} = \sigma_j e^{b_j S_t}$ and $d\tilde{N}_{jit} = dN_{jit} e^{b_j S_t}$
    - **when** $J = 1$ and $dS_t = dN_{jit} = 0$**, it reduces to the basic random growth model in 2**

- **The case of two regimes: high- and low-growth (without deviations or jumps)**
  - **cross-sectional logged wage distribution:** $p(x,t) = p^H(x,t) + p^L(x,t)$
    - $$p_t^H = -\mu_H p_x^H + \frac{\sigma_H^2}{2} p_{xx}^H - \psi p^H - \delta p^H + \beta_H \delta_0$$
      $$p_t^L = -\mu_L p_x^L + \frac{\sigma_L^2}{2} p_{xx}^L + \psi p^H - \delta p^L + \beta_L \delta_0,$$
    - **$\beta_H = \theta\delta$, $\beta_L = (1-\theta)\delta$ = regime specific birth rates, adjusted to deaths**
    - **ψ = rate of switching from the high to the low-growth regime**
  - **convergence rate in H-regime dominates in the SR (transition), that in L-regime determines the LR dynamics**
  - **heterogeneous mean growth in different regimes can generate *fast transition dynamics* of top inequality if**
    - **the *mean growth in the H-regime is sufficiently high* (high μ_H)**
    - **the *H-regime lasts only in short duration* (high switching rate ψ)**

- **quantitative result:**



(a) Top 1% Labor Income Share (b) Empirical Inverse Power Law Exponent

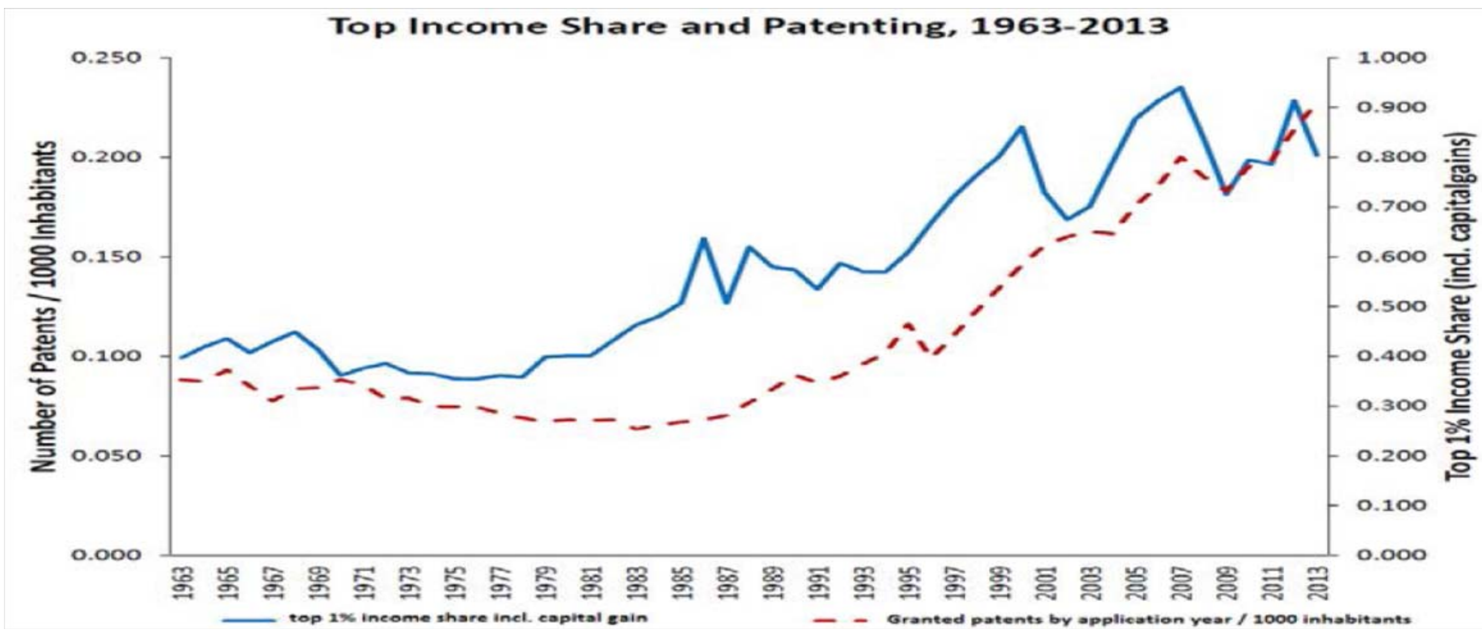
- **Deviations from Gibrat's law: superstar shocks**
  - **Pareto tail parameters satisfy:** $\zeta_t^x = e^{-S_t}\zeta_t^y$
  - **if the distribution of y does not change (constant ζ_t^y), then the distribution of x changes immediately when there is a shock to S, i.e., a fast process**
  - **so *superstars models can potentially generate fast transition***
- **Wealth dynamics: add ydt, then repeat similar analyses**
- **Overall finding: to account for fast transition dynamics of top inequality,**
  - **heterogeneous growth regimes with sufficiently high-regime mean growth over a short duration**
  - **superstar shocks affecting disproportionately top earners**

L. Innovation and Inequality: Aghion-Akcigit-Bergeaud-Blundell-Hemous (2015)

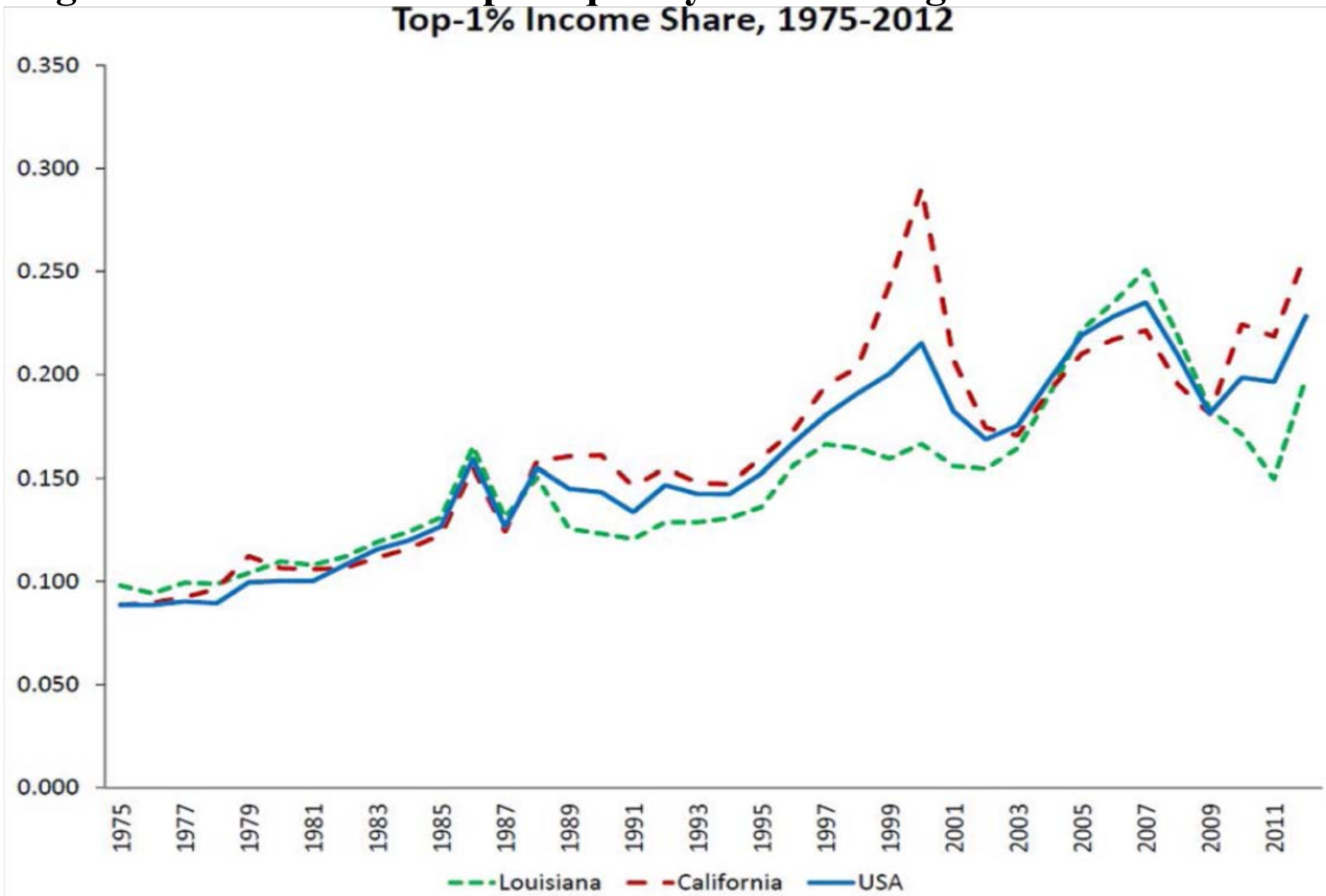
- Top inequality is likely to be associated with entrepreneurs and superstars, but what are the driving forces? A plausible one is innovation with imperfect imitation and innovation barrier.

1. Facts

- **Innovativeness and top inequality**

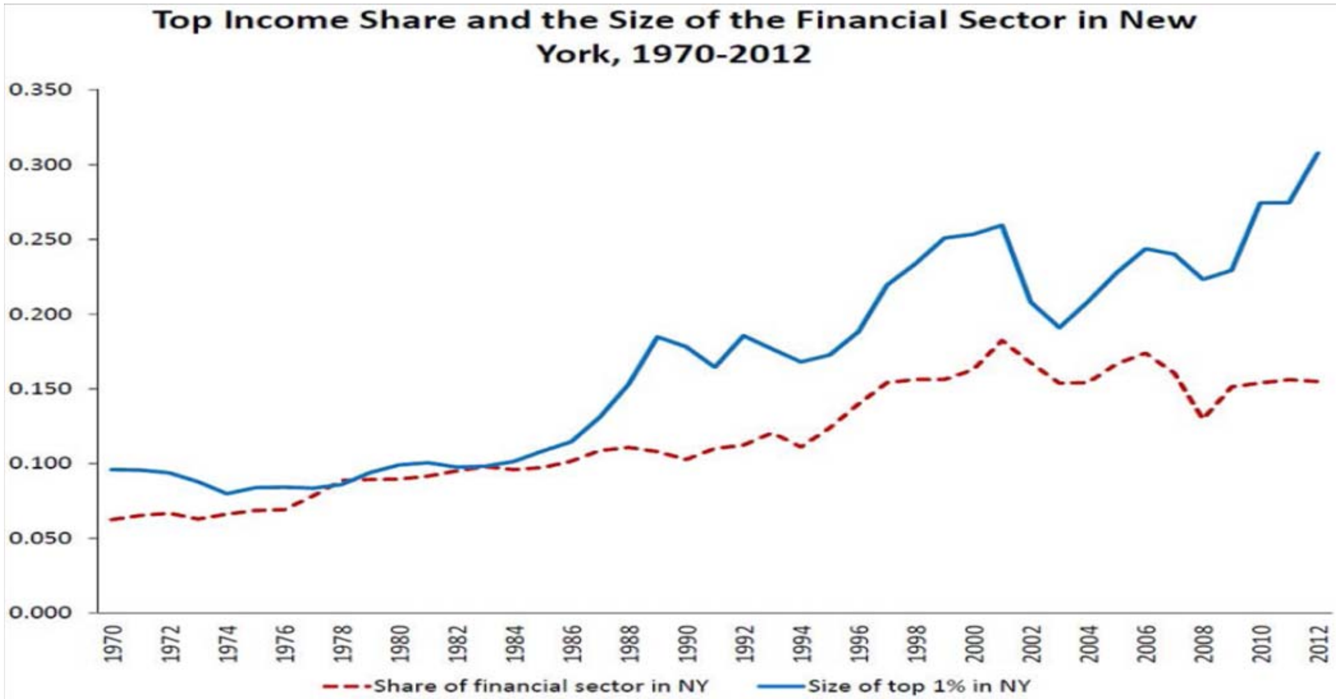


- **Regional differences in top inequality become larger since the end of 1980s**



- **Innovations are geographically concentrated**

- **The role of the financial sector**



2. A Baseline Schumpeterian Model with Exogenous Innovation

- **A measure of 2 agents:**
  - **a measure of 1 firm owners**
  - **a measure of 1 workers**
  - **all living for 1 period**
  - **agents born to current firm owners inherit the firm**

- **Final good production:** $\ln Y_t = \int_0^1 \ln y_{it} di$
- **Intermediate goods production:** $y_{it} = q_{it} l_{it}$
- **Arrival of a new innovation in period t:**
  - **improves the quality by η_H:** $q_{i,t} = \eta_H q_{i,t-1}$ **(technology leader)**
  - **makes previous technology $q_{i,t-1}$ publicly available**
  - **at the end of t, other firms partly imitate with $\eta_L < \eta_H$**
- **An incumbent not innovated can resort to lobbying to block entry of outside innovators:**
  - **lobbying succeeds with exogenous probability z**
  - **when succeeding, the implementation of new innovation is terminated**
- **Both potential entrants (K=E) and incumbents (K=I) may innovate with probability x at cost:** $C_{K,t}(x) = \theta_K \frac{x^2}{2} Y_t$
  - **for now, x is exogenous**
  - **when x is endogenized, this is a quadratic cost function on innovation effort (R&D)**
  - **the cost is in units of the final good**

- **Timing of events in each period:**
  - **in line i, a potential entrant pays cost C_E and the offspring of the incumbent pays C_I**
  - **patent race:**
    - **with probability $(1-z)x_i$, the entrant succeeds, overtaking the incumbent with η_H**
    - **with probability $\tilde{x}_i$, the incumbent succeeds moving from η_L to η_H**
    - **with probability $1-(1-z)x_i-\tilde{x}_i$, no success innovation with η_L**
  - **production and consumption take place**
- **Bertrand competition in innovation:**
  - **marginal cost of intermediate good production $MC_i = w/q_i$**
  - **Bertrand => markup = size of technological lead η**
  - **thus, limit pricing:** $p_{i,t} = \dfrac{w_t \eta_{it}}{q_{i,t}}$
- **Unit elastic intermediate goods demand by final producer =>** $Y_t = p_{i,t} y_{it}$
- **Combining intermediate production and demand and limit pricing,**
  - **labor demand:** $l_{it} = \dfrac{Y_t}{w_t \eta_{it}}$

- maximized profit: $\pi_{it} = (p_{it} - MC_{it})y_{it} = \frac{\eta_{it} - 1}{\eta_{it}} Y_t$
  - assumed to exceed wage so entrepreneurs earn more (need η high)
  - differential markups: $\pi_{H,t} = \underbrace{\frac{\eta_H - 1}{\eta_H}}_{\equiv \pi_H} Y_t > \pi_{L,t} = \underbrace{\frac{\eta_L - 1}{\eta_L}}_{\equiv \pi_L} Y_t$

- Labor market clearing: $1 = \int l_{it} di = \int \frac{Y_t}{w_t \eta_{it}} di = \frac{Y_t}{w_t} \left[\frac{\mu_t}{\eta_H} + \frac{1 - \mu_t}{\eta_L} \right]$
  - μ = share of high-markup sectors
- Income shares:
  - wage income share: $wages\_share_t = \frac{w_t}{Y_t} = \frac{\mu_t}{\eta_H} + \frac{1 - \mu_t}{\eta_L}$
  - Entrepreneur income share:
    $entrepreneur\_share_t = \frac{\mu_t \pi_{H,t} + (1 - \mu_t) \pi_{L,t}}{Y_t} = 1 - \frac{\mu_t}{\eta_H} - \frac{1 - \mu_t}{\eta_L}$
  - in high-markup sectors, income shifts from workers to entrepreneurs
- Law of large numbers => share of production lines with new innovation:
  $\mu_t = \tilde{x}_t + (1 - z) x_t$
  - less than one-for-one increase in successful entrants due to barriers induced by incumbents' lobbying (z)

- **Upward social mobility, measured by the probability for the offspring of a worker to become a firm owner:** $\Psi_t = x_t (1 - z)$
  - **independent of innovation by incumbents**
- **Equilibrium wage:** $w_t = \frac{Q_t}{\eta_H^{\mu_t} \eta_L^{1-\mu_t}}$
  - **overall quality index** $Q_t = \exp \int_0^1 \ln q_{it} di$ $= Q_{t-1} \eta_H^{\mu_t}$
    - **depending only on new innovation**
  - **thus,** $w_t = \eta_L^{\mu_t - 1} Q_{t-1}$
    - **wage income depending positively on η_L**
    - **this is due to technology diffusion via imitation**
    - **thus, there is a trickle down effect from the rich to the poor**
- **BGP growth rate:** $g^* = \eta_H^{(1-z)x^* + \tilde{x}^*} - 1$
  - **only new innovation matters**
    - **increasing in new technology size**
    - **increasing in new innovation**
  - **decreasing in entry barrier**

3. Endogenous Innovation

- **Optimization by the offspring of last period's incumbent given potential entrant's innovation effort x*:**

$$\max_{\tilde{x}} \left\{ \tilde{x}\pi_H Y_t + (1 - \tilde{x} - (1-z)\,x^*)\,\pi_L Y_t + (1-z)\,x^* w_t - \theta_I \frac{\tilde{x}^2}{2} Y_t \right\}$$

  - **optimal innovation effort:** $\tilde{x}_t^* = \tilde{x}^* = \frac{\pi_H - \pi_L}{\theta_I} = \left(\frac{1}{\eta_L} - \frac{1}{\eta_H}\right)\frac{1}{\theta_I}$
    - **increasing in the net technology gain (new innovation over imitation), which translates into the markup differential**

- **Optimization by the offspring of last period's worker given $\tilde{x}$ (potential entrant):** $\max_{x} \left\{ (1-z)\,x\pi_H Y_t + (1 - x\,(1-z))\,w_t - \theta_E \frac{x^2}{2} Y_t \right\}$

  - **optimal innovation effort:** $x_t^* \equiv x^* = \left(\pi_H - \left[\frac{\mu_t}{\eta_H} + \frac{1-\mu_t}{\eta_L}\right]\right)\frac{(1-z)}{\theta_E}$
    - **increasing in technology gain (the high-markup) and the share of high-markup sectors (μ)**
    - **decreasing in entry barrier (z)**

- in equilibrium, $x^* = \dfrac{\left(\pi_H - \frac{1}{\eta_L} + \left(\frac{1}{\eta_L} - \frac{1}{\eta_H}\right)\tilde{x}^*\right)(1-z)}{\theta_E - (1-z)^2\left(\frac{1}{\eta_L} - \frac{1}{\eta_H}\right)}$
  - positive spillover of incumbent's R&D on new entrant's R&D due to a larger share of high-markup sectors
- **Main results:**
  - higher new innovation size =>
    - increases new innovator markup and the share of high-markup sectors, thus raising entrepreneur income share
    - increases innovation by both incumbents and new entrants, so further raising entrepreneur income share
    - these may create large effects on top inequality
  - higher existing innovation size =>
    - increases noninnovator markup, thus raising entrepreneur income share
    - discourages incumbent's innovation, thus offsetting partly the positive effect on entrepreneur income share

4. Empirical Analysis

- **Panel estimation using state data with various controls including gdppc, financial sector share, government size and others**
- **Main findings:**
  - **innovativeness (patent per capita) affects top 1% inequality positively (mostly significant at 5% level)**
  - **effects stronger and more significant using citations or lagged innovations**
  - **financial share, infrastructure (highways) and knowledge spillovers usually significantly positive; government size negative**

5. Open Issues

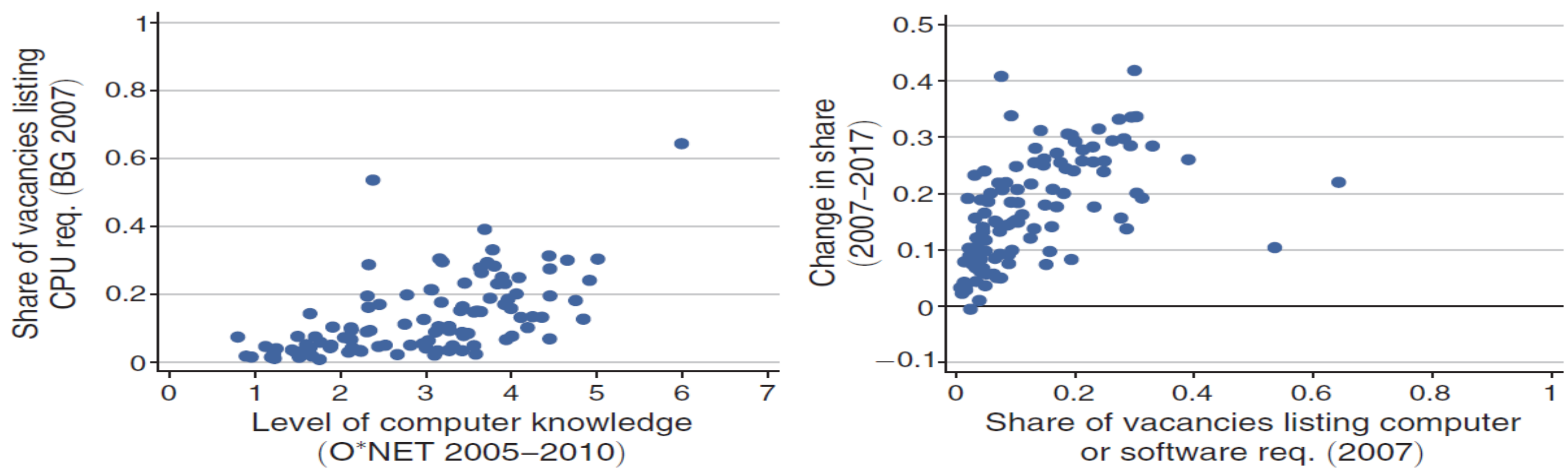
- **Explicit income dynamics with shocks**
- **Asset accumulation and wealth dynamics**
- **Mobility of firms and workers across regions**

N. Technology-induced job loss and earning dispersion: Braxton-Taska (2023)

- **Technology change requires workers to update skills to perform new tasks**
- **Those lacking the required updated skills get displaced, moving to occupations at which their current skills are still employable and receiving lower pay**
- **While those with updated skills gain, those switching to lesser jobs lose, leading to widened income inequality**

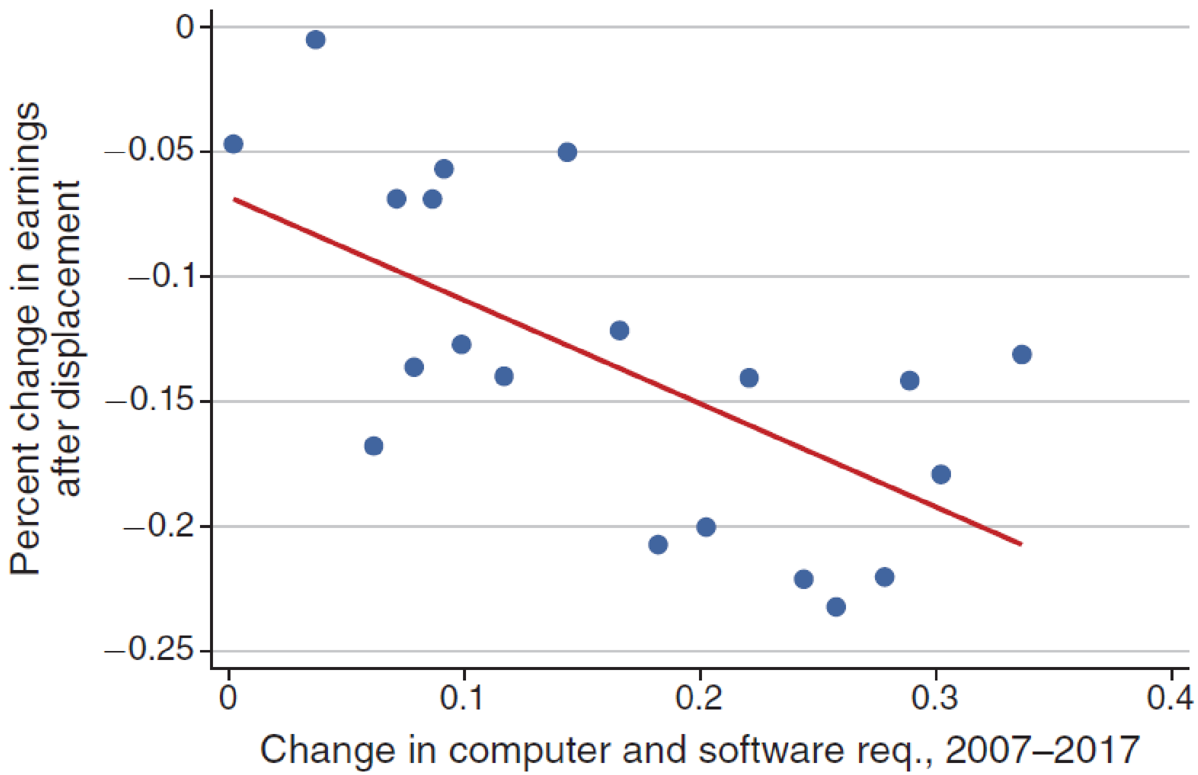
1. Facts

- **Technology changes by occupations**



Rank	SOC-4	Occupation	Chg. computer req. (2007–2017) (1)	Nonroutine cognitive (2)	Routine cognitive (3)	Nonroutine manual (4)	Routine manual (5)
Panel A. Occupations with largest increase in computer and software requirements							
1	1710	Architects	0.419	1.603	0.657	−0.187	−0.285
2	3310	Supervisors of protective service workers	0.408	1.036	0.160	1.845	−0.250
3	3390	Protective service workers	0.338	−0.656	1.160	−0.966	−1.036
4	1720	Engineers—aerospace/biomedical/computer	0.337	0.323	−0.468	−1.587	−0.868
5	4330	Financial clerks	0.336	−0.705	1.878	−0.633	−0.311
6	1721	Engineers—industrial/mechanical/nuclear	0.332	0.644	−0.084	−2.095	−0.558
7	1520	Mathematical science occupations	0.315	0.888	−0.789	−2.455	−1.355
8	4750	Oil, gas, and mining extraction workers	0.312	−0.290	0.340	0.635	2.158
9	1120	Advertising, marketing, and sales managers	0.306	1.815	−1.540	0.419	−1.506
10	2740	Media and communication equipment workers	0.304	0.097	0.605	−0.235	1.112
Panel B: Occupations with smallest increase in computer and software requirements							
1	3990	Personal care and service workers	0.044	−0.641	−2.490	0.894	−1.245
2	3730	Grounds maintenance workers	0.042	−1.010	−2.386	0.091	2.112
3	5130	Food processing workers	0.041	−0.832	0.150	−0.736	1.281
4	3720	Cleaners	0.039	−1.992	−1.330	−1.225	0.647
5	3920	Animal trainers and caretakers	0.036	−0.234	−1.760	1.154	−0.706
6	3520	Cooks and food preparation workers	0.033	−1.209	−0.585	−0.059	1.085
7	3590	Restaurant attendants, dishwashers, hosts	0.029	−1.758	−1.242	−0.041	0.762
8	4730	Helpers, construction trades	0.022	−0.624	−0.228	−0.214	1.099
9	3530	Food and drink servers	0.010	−1.040	−0.394	0.210	0.214
10	5330	Drivers—ambulance/bus/tractor trailer/taxi	−0.005	−1.207	0.476	2.487	1.323

- **Technology change and earning loss**



- **Earning gain among occupation stayers**

Dependent variable: change in log earnings over 12 months	(1)	(2)	(3)
Change in computer requirements	0.00270	0.00272	0.00205
	(0.000705)	(0.000680)	(0.000804)
Change in employment share		8.76e-05	0.000145
		(0.000595)	(0.000598)
Observations	150,330	150,330	150,330
R^2	0.005	0.005	0.005
Controls	Yes	Yes	Yes
Occupation definition	SOC-4	SOC-4	AD

2. The Model

- **z_j = technology at time j**
- **c_k = technology intensity of occupation k $\in \mathbb{K}$, with measure K**
- **vacancy posted with technology** $z_{k,j} = c_k z_j$

- **T generations of workers, each lives for T period, either employed (W) or unemployed (U), with human capita (skill)** $h \in \mathcal{H} \equiv [\underline{h}, \bar{h}]$
- **Matching function between vacancies and workers: M(v,s)**
- **Market tightness $\theta^x_{j,t}$ pins down**
  - **job finding rate:** $p\left(\theta^x_{j,t}(h,k,\omega)\right) = \dfrac{M\left(s^x_{j,t}(h,k,\omega), v^x_{j,t}(h,k,\omega)\right)}{s^x_{j,t}(h,k,\omega)}$
  - **firm hiring rate:** $p_f\left(\theta^x_{j,t}(h,k,\omega)\right) = \dfrac{M\left(s^x_{j,t}(h,k,\omega), v^x_{j,t}(h,k,\omega)\right)}{v^x_{j,t}(h,k,\omega)}$
- **Value of inexperienced unemployed workers:**

$$U^N_t(h,0) = b + \beta E\left[\hat{U}^N_{t+1}(h',0)\right], \ \forall t \leq T$$

$$U^N_{T+1}(h,0) = 0,$$

where $h' = H(h)$ **and**

$$\hat{U}^N_{t+1}(h',0) = \max_{(k,\omega)\in\mathcal{K}\times[0,1]} p\left(\theta^N_{t+1}(h',k,\omega)\right)W^N_{t+1}(h',\bar{z},k,\omega) + \left[1 - p\left(\theta^N_{t+1}(h',k,\omega)\right)\right]U^N_{t+1}(h',0),$$

and experienced unemployed workers have similar form

- **Continuation value of inexperienced employed workers:**

$$W_t^N(h,z,k,\omega) = \omega f(c_k z,h,N) + \beta E\Big\{\delta \hat{U}_{t+1}^N(h',k) + (1-\delta)\Big[\lambda_E \hat{W}_{k,t+1}^E(h',z',k,\omega) + (1-\lambda_E) \times \hat{W}_{t+1}^N(h',z',k,\omega)\Big]\Big\}, \forall t \leq T,$$

$$W_{T+1}^N(h,z,k,\omega) = 0,$$

where

- $h' = H(h), \quad z' = Z(z)$
- **value of on-the-job-search for an inexperienced worker is:**

$$\hat{W}_{t+1}^N(h',z',k,\omega) = \max_{(\tilde{k},\tilde{\omega})\in\mathcal{K}\times[0,1]} p\big(\theta_{t+1}^N(h',\tilde{k},\tilde{\omega})\big) W_{t+1}^N(h',\bar{z},\tilde{k},\tilde{\omega}) + \big[1 - p\big(\theta_{t+1}^N(h',\tilde{k},\tilde{\omega})\big)\big] W_{t+1}^N(h',z',k,\omega),$$

- value of on-the-job-search for an experienced worker is:

$$\hat{W}^E_{t+1}(h', z', k, \omega) = \max\Big\{ \max_{\tilde{\omega}\in[0,1]} p\big(\theta^E_{t+1}(h', k, \tilde{\omega})\big) W^E_{t+1}(h', \bar{z}, k, \tilde{\omega}) + \big[1 - p\big(\theta^E_{t+1}(h', k, \tilde{\omega})\big)\big] W^E_{t+1}(h', z', k, \omega),$$
$$\max_{(\tilde{k},\tilde{\omega})\in\mathcal{K}/\{k\}\times[0,1]} p\big(\theta^N_{t+1}(h', \tilde{k}, \tilde{\omega})\big) W^N_{t+1}(h', \bar{z}, \tilde{k}, \tilde{\omega}) \times \big[1 - p\big(\theta^N_{t+1}(h', \tilde{k}, \tilde{\omega})\big)\big] W^E_{t+1}(h', z', k, \omega) \Big\},$$

- All matches start with a frontier technology $\bar{z}$

3. Calibration

- Technology evolution: $Z(z) = z' = \begin{cases} z\mu, & \text{with pr. } \iota; \\ z, & \text{with pr. } 1 - \iota; \end{cases}$

  with decay at μ due to technology growth g: μ = 1/(1+g)

- **Matching function:** $M(s,v) = \frac{sv}{\left(s^{\xi} + v^{\xi}\right)^{1/\xi}}$ **with elasticity ξ = 1.6**
- **With quarterly timing, separation rate δ = 0.1 (Shimer 2005) and β = 0.99**
- **T = 120 (30 years), K = 10**
- **Entry cost for posting a vacancy κ is calibrated to target u = 6.8%**
- **Up-to-the task production function for a worker with experience x (Albrecht-Vroman 2002):** $f(c_k z, h, x) = \begin{cases} A_x c_k z, & \text{if } A_x h \geq c_k z \\ 0, & \text{otherwise;} \end{cases}$
  - **relative productivity of experienced A_E = 1.12 (Kambourov-Manovskii (2009)**
  - **probability to become experienced λ_E = 0.05**
- **Human capital evolution:** $H(h) = h' = \begin{cases} h\mu, & \text{with pr. } \iota; \\ h, & \text{with pr. } 1 - \iota \end{cases}$

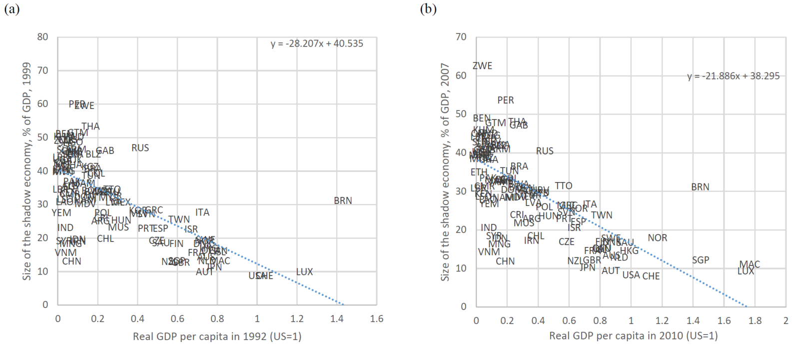
- **Main finding: on average, technology change accounts for 45% of earning declines from job loss, which subsequently results in widened wage inequality**

O. Informality and Wage Disparity: Liao-Wang-Wang (2024)

- **Models based on formal sector firms and workers may lead to underestimation of wage income disparity, which is particularly true in countries with a sizable shadow economy – the informal sector where firms may not be officially registered whereas workers (even in a formal firm) may not be recorded by tax authorities**
- **Empirical evidence indicates that in the early stage of economic development:**
  - **there are vast rural-urban migrations**
  - **there exist more sizable informality, particularly in Africa and Latin America**
  - **it features high dispersion in the intensity of migration and the extent of informality**
- **Research questions: as informal urban sector would be a potential outlet for migrant workers,**

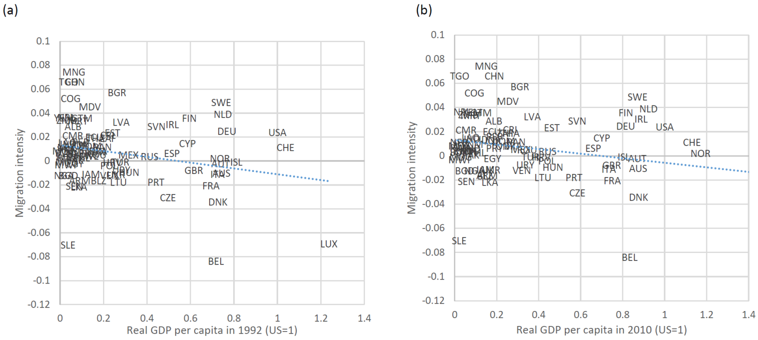
1. **What are the interplays between rural migration and informality?**
2. **What are their macroeconomic consequences?**
3. **What are the implications of migration and industry policies?**

Observations: Shadow economy and real GDP



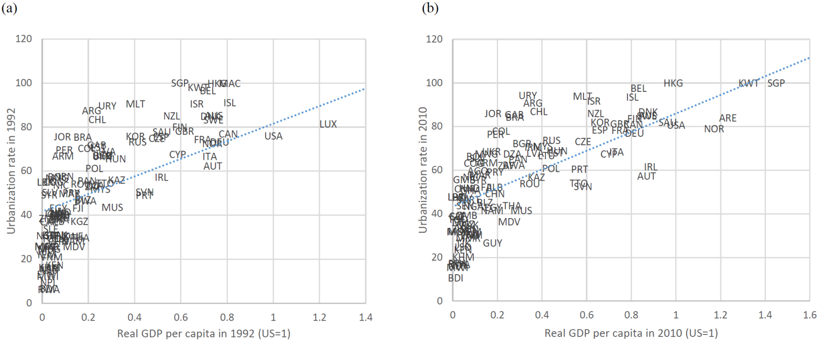
Data source: Size of shadow economy is from Medina and Schineider (2018). Real GDP per capita is from the PWT 10.0.

Observations: Migration intensity and real GDP



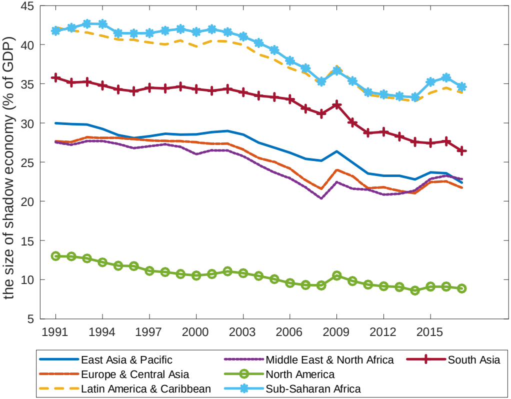
Data source: Migration intensity is computed using rural-urban employment ratio, which is taken from Global Jobs Indicator Data Base (JOIN), World Bank. See Liao, Wang, Wang and Yip (2022) for details of computation for migration intensity. Real GDP per capita data is taken from the PWT 10.0.

Observations: Urbanization and real GDP



Data source: Urbanization rates are taken from WDI. Real GDP per capita is from the PWT 10.0.

Size of shadow economy by region



Data source: The size of shadow economy is taken from Medina and Schneider (2019).

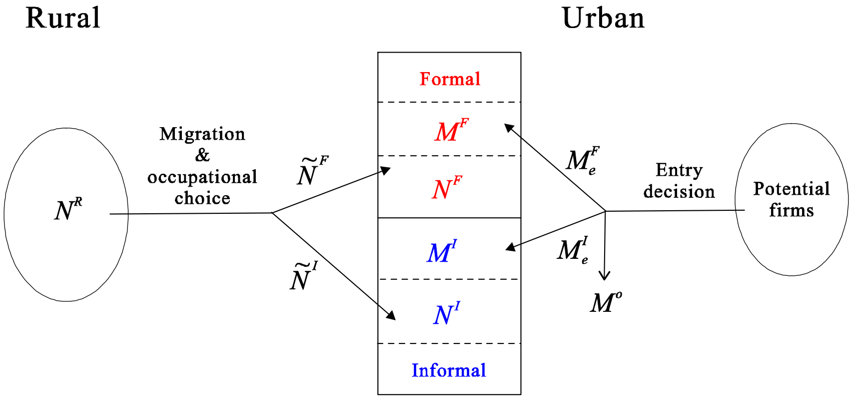
Related literature

- Rural-urban migration
  - Todaro (1969) and Harris and Todaro (1970): pioneers
  - **Lucas (2004)** studies human capital externality in urban areas
  - Liao, Wang, Wang and Yip (2022) consider tertiary education as a rural-urban migration channel
- Informal economy and development:
  - **Ulyssea (2018)** considers extensive and intensive margins of informality and finds that lower informality can be, but not necessarily, associated with higher output, TFP, or welfare.
  - Yuki (2007) emphasizes the important role of human capital accumulation in explaining the expansion of the urban formal/informal sector in the process of urbanization and development.

Environment

- Two geographical regions: Urban and rural
- Forms of production
  - Urban : { Formal sector: Melitz (2003) framework; Informal sector: DRTS technology, no fixed cost; Hand-to-mouth self-employed entrepreneurs }
  - Rural: Backyard farming
- Two groups
  - Urban firms: organizational choice of formal vs. informal; exit with a probability δ in every period
  - Workers:
    - Rural workers (our focus): migration and occupational choice decisions
    - Urban workers: passive, "inheriting" parents' occupations

Model overview



Rural households – production

- Rural working-age agents are hand-to-mouth farmers, relying on farming to make a living.
- Denote N^R as the number of rural agents cultivating rural land during a period, and Ω is total rural land. Total output in rural area is:

$$Q = z\left(N^R\right)^{\vartheta}\Omega^{1-\vartheta}, \quad z > 0,$$

where $z > 0$ is the farming technology and $\vartheta \in (0, 1)$ is the rural labor income share.

- Normalize total rural land to one. A rural farmer's output is

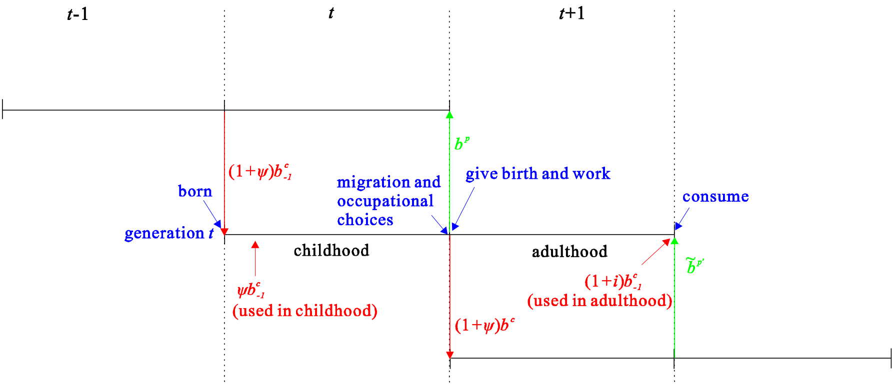
$$q = z\left(N^R\right)^{\vartheta-1}$$

and thereby a rural farmer's income, in value, is the total value of her output and equals $P^R q$.

Rural households

- Each agent lives for two periods: childhood and adulthood.
- Consider an agent born in period t (a generation-t agent):
  - Childhood: attached to parent, receive $(1+\psi)\, b^c_{-1}$ transfer from parents and use up ψb^c_{-1} in childhood.
  - Adulthood: own one unit of labor and work in period $t+1$.
- At the end of childhood, make migration and occupational choice decisions:
  - If staying in rural: produce rural goods of q and earn an income of $P^R q$.
  - If migrating to cities: $\begin{cases} \text{earn } w^F \text{ if choosing the formal sector.} \\ \text{earn } w^I \text{ if choosing the informal sector.} \end{cases}$
- After the migration and the occupational choice decisions, they become adults:
  - Upon becoming adults: give remittance of b^P to parents.
  - Given birth to a child and transfer $(1+\psi)\, b^c$ to child, with ψb^c being paid as child-rearing cost.
  - Right before the end of the adulthood, receive transfer of $\tilde{b}^{p\prime}$ from children, consume c, and exit the market.

Timeline


t-1
t
t+1
b^p
$(1+\psi)b^c_{-1}$
born
generation t
migration and occupational choices
give birth and work
consume
childhood
adulthood
ψb^c_{-1}
(used in childhood)
$(1+i)b^c_{-1}$
(used in adulthood)
$\tilde{b}^{p'}$
$(1+\psi)b^c$

Rural households - staying in rural

- Denote V^R as the value function of staying in rural area:

$$V^R \equiv \max_{c^R, b^c, b^p} c^R + \beta^c u(b^c) + \beta^p u(b^p)$$

  - c^R : consumption
  - β^c (β^p): the altruistic factor towards child (parent)
  - $u' > 0, u'' < 0$

- A generation-t rural agent's lifetime budget constraint:

$$P^R c^R + (1+\psi) b^c + b^p = P^R q + (1+i) b^c_{-1} + \tilde{b}^p$$

  - ψ : child-rearing cost markup
  - P^R : price of rural-produced goods
  - b^c : transfer to child; $(1+\psi) b^c_{-1}$: total transfer from parents, with b^c_{-1} being carried over from childhood to adulthood
  - b^p: transfer to parent measured in generation-t's value unit
  - $\tilde{b}^{p'}$: amount of transfer from children received by generation-t agents

Rural households - migrating to cities

- The value of being a worker in urban formal sector V^F is:

$$V^F = \max_{c^U, b^c, b^p} c^U + \beta^c u(b^c) + \beta^p u(b^p)$$

$$\text{where } c^U \equiv c^F + \lambda c^I, \quad \lambda \in (0,1)$$

$$s.t. \;\; P^F c^F + P^I c^I + (1+\psi) b^c + b^p = (1-\tau^w) w^F + (1+i) b^c_{-1} + \tilde{b}^{p'}$$

  - c^U : consumption on urban traded goods
  - c^F : consumption on formal goods
  - c^I : consumption on informal goods
  - λ : quality of informal goods relative to formal goods perceived by urban agents
  - w^F : urban formal wage
  - τ^w : labor income tax rate

- The value of being a worker in urban informal sector V^I takes the same form as that for V^F except the income $(1-\tau^w) w^F$ in the budget constraint is replaced by w^I.

Migration decision and occupational choice

- Rural workers are heterogeneous in migration disutility $(1/\mu)$ and work-effort disutility $(1/\epsilon)$ in the formal sector and take the draws of μ and ϵ from Pareto distribution $G_\mu(\mu)$ and $G_\epsilon(\epsilon)$ at birth.

- Besides incurring disutility when working, formal workers need to pay income taxes τ^w, while informal workers do not have to pay taxes. Hence, w^F must be higher than w^I so that, at least some rural migrants are willing to work for the formal sector.

- A rural worker makes migration decision, plus occupational choice if needed, before entering adulthood:

  The 1st stage decision $(\mathbb{I}^M)$ Whether to migrate to cities?
  The 2nd stage decision $(\mathbb{I}^W)$ If migrating to cities, which sector to devote to?

- We solve rural workers' problem backwardly by solving the 2nd stage problem first.

The 2nd stage problem: Occupational choice 1

- The value function V^M of a migrant worker is given by

$$V^M(\epsilon) = \max_{\mathbb{I}^W \in \{0,1\}} \mathbb{I}^W \left(V^F - \frac{\chi_\epsilon}{\epsilon} \right) + \left(1 - \mathbb{I}^W\right) V^I$$

where $\chi_\epsilon > 0$ is the relative disutility of being a formal worker and $\mathbb{I}^W$ is an indicator function such that

$$\mathbb{I}^W = \begin{cases} 1 & \text{if the rural migrant works in the formal sector,} \\ 0 & \text{if the rural migrant works in the informal sector.} \end{cases}$$

That is

$$\mathbb{I}^{W^*} = \arg \max_{\mathbb{I}^W \in \{0,1\}} \mathbb{I}^W \left(V^F - \frac{\chi_\epsilon}{\epsilon} \right) + \left(1 - \mathbb{I}^W\right) V^I$$

- To focus on the nondegenerate equilibrium, we impose the following condition:

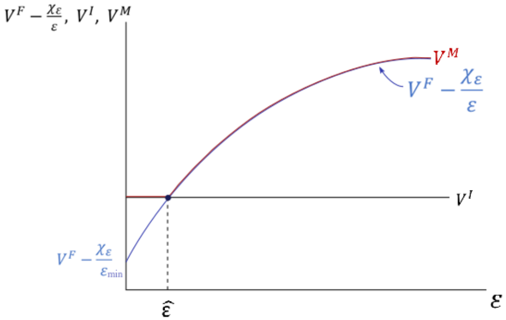
Condition F $\epsilon_{\min} < \frac{\chi_\epsilon}{V^F - V^I}$.

The 2nd stage problem: Occupational choice 2

- Under **Condition F**, since $V^F - \frac{\chi_\epsilon}{\epsilon}$ is strictly increasing in ϵ and V^I is constant in ϵ, $\exists$ a single cutoff $\hat{\epsilon}$ such that

$$\mathbb{I}^{W^*} = \begin{cases} 1 & \text{if } \epsilon \geq \hat{\epsilon}, \\ 0 & \text{if } \epsilon < \hat{\epsilon}, \end{cases}$$

with $\hat{\epsilon} = \frac{\chi_\epsilon}{V^F - V^I}$.



The 1st stage problem: Migration decision

- Denote V as the value function for a rural agent with disutility (μ, ϵ):

$$V(\mu,\epsilon) = \max_{\mathbb{I}^M \in \{0,1\}} \mathbb{I}^M \cdot \left[V^M(\epsilon) - \frac{\chi_\mu}{\mu} \right] + \left(1 - \mathbb{I}^M\right) V^R$$

where $\chi_\mu > 0$ is the relative magnitude of migration disutility and $\mathbb{I}^M$ is an indicator function such that

$$\mathbb{I}^M = \begin{cases} 1 & \text{if the rural worker decides to migrate,} \\ 0 & \text{if the rural worker decides to stay.} \end{cases}$$

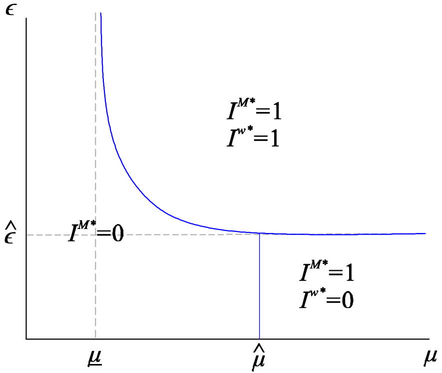
- **Condition IM** $\mu_{\min} < \frac{\chi_\mu}{V^I - V^R}$.
- Combining the two stages implies:

$$\left(\mathbb{I}^{M*}, \mathbb{I}^{W*}\right) = \begin{cases} (0,\cdot) & \text{for } \Gamma(\mu,\epsilon) < 0, \mu < \hat{\mu}, \\ (1,0) & \text{for } \Gamma(\mu,\epsilon) < 0, \mu \geq \hat{\mu}, \\ (1,1) & \text{for } \Gamma(\mu,\epsilon) \geq 0. \end{cases} \qquad \text{(IB)}$$

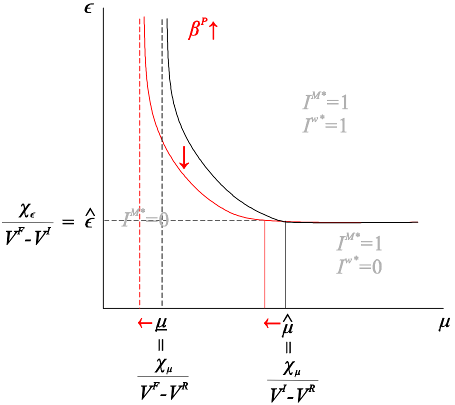
We are ready to write down the (IB) and the optimal migration and occupational choices for rural potential workers.

The IB for migration and occupational choice

- Let $\underline{\mu} = \frac{\chi_\mu}{V^F - V^R}$ be the smallest μ such that an agent with $\epsilon \to \infty$ is willing to migrate to cities.
- The figure shows the indifference boundary (IB) and the optimal decisions:



The IB with more remittance enjoyment or less migration disutility



Note: The figure shows the case where the price of goods in urban is relative expensive than goods in rural, $P^F > P^R$.

Workers' laws of motion

- The joint distribution of (μ, ϵ) is

$$\begin{aligned} \Lambda^R &= \{(\mu,\epsilon) \,|\, \Gamma(\mu,\epsilon) < 0, \mu < \hat{\mu}\}, \\ \Lambda^I &= \{(\mu,\epsilon) \,|\, \Gamma(\mu,\epsilon) < 0, \mu \geq \hat{\mu}\}, \\ \Lambda^F &= \{(\mu,\epsilon) \,|\, \Gamma(\mu,\epsilon) \geq 0\} = 1 - \Lambda^R - \Lambda^I. \end{aligned}$$

- Denote N_t^F, N_t^I and N_t^R the masses of workers in the formal sector, the informal sector, and rural agricultural sector at the beginning of period t.
- Migrant formal and informal workers and total migrant workers in period $t+1$ are:

$$\tilde{N}_{t+1}^F = N_t^R \Lambda^F, \; \tilde{N}_{t+1}^I = N_t^R \Lambda^I, \; \tilde{N}_{t+1} = \tilde{N}_{t+1}^F + \tilde{N}_{t+1}^I.$$

- Total workers in urban formal, urban informal and rural sectors evolve according to:

$$\begin{aligned} N_{t+1}^F &= N_t^F + N_t^R \Lambda^F, \\ N_{t+1}^I &= N_t^I + N_t^R \Lambda^I, \\ N_{t+1}^R &= N_t^R \left[1 - \Lambda^F - \Lambda^I\right] \end{aligned}$$

Urban production – overview

- Total mass of potential urban firms equals M (exogenously given).
- Three types of organization: Upon paying a fixed cost of $\bar{f}_e = w^F f_e$ to enter (where f_e is in terms of labor), urban potential firms make productivity draw and choose to be:
  - Formal firm: Output level depends on individual specific productivity φ.
  - Informal firm: Output level does not depend on individual specific productivity.
  - Urban hand-to-mouth self-employed entrepreneur.
- An one-time managerial cost for establishing and managing a firm (one owner per firm):

$$d(\varphi) = \frac{\xi \cdot 1}{\varphi}$$

with $\lim_{\varphi \to \varphi_{\min}} d(\varphi) = \frac{\xi \cdot 1}{\varphi_{\min}}$ and $\lim_{\varphi \to \infty} d(\varphi) = 0$.
- All potential urban firms exit with probability δ in every period.

Urban production – formal sector 1

- Following Melitz (2003), urban formal good Y^F is produced by:

$$Y^F = \left[\int_{\omega \in \Omega} y^F(\omega)^{\frac{\sigma-1}{\sigma}} d\omega \right]^{\frac{\sigma}{\sigma-1}}, \qquad \sigma > 1$$

where $y^F(\omega)$ is the quantity of good ω produced by an urban formal firm, with $\rho \equiv \sigma / (\sigma - 1)$.

- Output/consumption and revenues for each variety ω:

$$y^F(\omega) = Y^F \left[\frac{p^F(\omega)}{P^F} \right]^{-\sigma} \text{ and } r^F(\omega) = R^F \left[\frac{p^F(\omega)}{P^F} \right]^{1-\sigma}$$

where $P^F \equiv \left[\int_{\omega \in \Omega} p^F(\omega)^{1-\sigma} d\omega \right]^{\frac{1}{1-\sigma}}$ and $R^F \equiv \int_{\omega \in \Omega} r^F(\omega) d\omega$, $Y^F \equiv R^F / P^F$.

- Following Krugman (1980) with labor as the only factor of production, labor requirement for production of an urban formal firm with productivity φ is:

$$\ell^F = \bar{\ell}^F + x + \ell_v^F = e^{-S} f + x + \frac{y^F}{\varphi},$$

where $\bar{\ell}^F = e^{-S} f$ is the fixed overhead cost and x the government regulatory cost.

Urban production – formal sector 2

- Denote the wage rate paid by urban formal firms as w^F. Monopolistic pricing implies

$$p^F(\varphi) = \frac{w^F}{\rho\varphi}$$

implying $r^F(\varphi) = R^F\left(P^F\rho\varphi\right)^{\sigma-1}\left(w^F\right)^{1-\sigma}$ and $y^F(\varphi) = Y^F\left(P^F\rho\varphi\right)^{\sigma}\left(w^F\right)^{-\sigma}$.

- So more productive urban firms produce more and earn higher revenues:

$$\frac{y^F(\varphi_1)}{y^F(\varphi_2)} = \left(\frac{\varphi_1}{\varphi_2}\right)^{\sigma} \text{ and } \frac{r^F(\varphi_1)}{r^F(\varphi_2)} = \left(\frac{\varphi_1}{\varphi_2}\right)^{\sigma-1}.$$

- Subject to a corporate income tax rate of τ^C, an urban formal firm with productivity φ has the profit of:

$$\begin{aligned}\underset{+}{\pi^F(\varphi)} &\equiv \left(1-\tau^C\right)\left[r^F(\varphi) - w^F\ell^F(\varphi)\right] \\ &= \left(1-\tau^C\right)\left[\frac{R^F\left(P^F\rho\varphi\right)^{\sigma-1}\left(w^F\right)^{1-\sigma}}{\sigma} - w^F\left(e^{-S}f + x\right)\right].\end{aligned}$$

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Urban production – formal sector 3

- Since all firms with productivity φ charge the same price $p^F(\varphi)$, P^F can be rearranged as

$$P^F = \left[\int_0^{+\infty} p^F(\varphi)^{1-\sigma} M^F \mu(\varphi)\, d\varphi\right]^{\frac{1}{1-\sigma}}$$

  where

  - $M^F \equiv$ mass of operative formal firms in equilibrium.
  - $\mu(\varphi) \equiv$ (conditional) pdf of productivity levels of operative formal firms in equilibrium.

- Define $\bar{\varphi}^F \equiv \left[\int_0^{+\infty} \varphi^{\sigma-1}\mu(\varphi)\, d\varphi\right]^{\frac{1}{\sigma-1}}$ as the average productivity of urban operative formal firms. Then,

$$P^F = \frac{w^F {M^F}^{\frac{1}{1-\sigma}}}{\rho\bar{\varphi}^F} = P^F\left(\bar{\varphi}^F\right).$$

$$R^F = M^F r^F\left(\bar{\varphi}^F\right), \;\; \Pi^F = M^F \pi^F\left(\bar{\varphi}^F\right), \;\; Y^F = {M^F}^{\frac{\sigma}{\sigma-1}} y^F\left(\bar{\varphi}^F\right).$$

Urban production – informal sector

- The technology of urban informal firms is

$$y^I = a^I (\ell^I)^\gamma, \quad \gamma \in (0,1)$$

  where $a^I > 0$ is the technology scaling factor for urban informal sector.
- The profit of an informal firm is:

$$\pi^I = (1-\zeta)\left(P^I y^I - w^I \ell^I\right)$$

  where $\zeta = \zeta_0\left(\bar{\zeta} + (1-\bar{\zeta})\right) \in (0,1)$ is the probability of being fined $(\zeta_0 \bar{\zeta})$ and asked for bribes $(\zeta_0\left(1-\bar{\zeta}\right))$, and $\bar{\zeta}$ is the share of firms being fined.
- Assume that informal firms pay their employees at a wage rate $w^I < VMPL = P^I \gamma a^I \left(\ell^I\right)^{\gamma-1}$:

$$w^I = \kappa P^I \gamma a^I \left(\ell^I\right)^{\gamma-1}$$

  with $\kappa \in (0,1]$ being the informal wage markdown.
- The profit of an informal firm can be rewritten as

$$\pi^I = (1-\zeta)\, P^I y^I (1-\kappa\gamma) > 0.$$

Government's technology

- The government provides public infrastructure that helps lowering formal firms' fixed costs of production in operation:

$$S = S_0 G_g,$$

where $S_0 > 0$ is the government's technology scaling factor, and G_g is government expenditure.

- Total taxes T collected by the government in period t is:

$$T = \tau^W w^F N^F + \tau^C M^F \bar{\pi}^F + \zeta_0 \bar{\zeta} M^I \pi^I,$$

where M^F and M^I are masses of formal and informal firms.

- Assume that the government runs a balanced budget in every period:

$$T = G_g.$$

Formal vs. informal cutoff profit 1

- In a stationary equilibrium, a firm either exits immediately, if it finds not worth running a business, or produces and earns the same profits in each period.
- The *expected value* of a firm with productivity φ is:

$$v(\varphi) = \max\{\sum_{t=0}^{\infty}(1-\delta)^t \pi^I, \sum_{t=0}^{\infty}(1-\delta)^t \pi^F(\varphi)\} = \max\{\frac{\pi^I}{\delta}, \frac{\pi^F(\varphi)}{\delta}\}$$

- Recall that $\pi^F(\varphi)$ is increasing in φ with $\lim_{\varphi\to\infty}\pi^F(\varphi) = \infty$, and $\pi^I > 0$. There exists a formal vs. informal cutoff productivity such that

$$\hat{\varphi} \equiv \inf\left\{\varphi \geq 0 : \pi^F(\varphi)/\delta \geq \pi^I/\delta\right\}.$$

- As π^I depends on w^I and P^I, an urban firm will choose to operate as an informal firm if

$$\frac{\pi^I(\hat{\varphi})}{\delta} - \frac{\xi}{\varphi} \geq v^o$$

where $v^o > 0$ is the outside option for an urban firm.

Formal vs. informal cutoff profit 2

- Denote $\tilde{\varphi} < \hat{\varphi}$ such that among non-formal firms with $\varphi < \hat{\varphi}$, $\frac{G(\tilde{\varphi})}{G(\hat{\varphi})}$ of them choose not to participate, and $\frac{G(\hat{\varphi})-G(\tilde{\varphi})}{G(\hat{\varphi})}$ of them choose to operate as informal firms.
- Under $\tilde{\varphi}$ and $\hat{\varphi}$, we can compute the conditional formal firms' productivity $\mu(\varphi)$ and the probability of being a formal firm, an informal firm, and a hand-to-mouth self-employed entrepreneur:

$$\begin{aligned}
\mu(\varphi) &= \begin{cases} \frac{g(\varphi)}{1-G(\hat{\varphi})} & \text{if } \varphi \geq \hat{\varphi}, \\ 0 & \text{if } \varphi < \hat{\varphi}. \end{cases} \\
P_r^{\text{formal}} &= 1 - G(\hat{\varphi}), \\
P_r^{\text{informal}} &= G(\hat{\varphi}) - G(\tilde{\varphi}). \\
P_r^{\text{o}} &= G(\tilde{\varphi}).
\end{aligned}$$

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Formal vs. informal cutoff profit 3

- The average productivity and average profit are:

$$\bar{\varphi}^F(\hat{\varphi}) = \left[\frac{1}{1-G(\hat{\varphi})}\int_{\hat{\varphi}}^{+\infty}\varphi^{\sigma-1}g(\varphi)\,d\varphi\right]^{\frac{1}{\sigma-1}}$$

$$\bar{\pi}^F = \left(1-\tau^C\right) r^F\left[\bar{\varphi}^F(\hat{\varphi})\right]/\sigma - \left(1-\tau^C\right) w^F\left(e^{-S}f+x\right)$$

- At the cutoff $\hat{\varphi}$, we have $\pi^F(\hat{\varphi}) = \pi^I$, so $r^F(\hat{\varphi}) = \sigma\left[w^F\left(e^{-S}f+x\right)+\frac{\pi^I}{(1-\tau^C)}\right]$.
- **Formal-informal Cutoff Profit (FICP)** By plugging in the derived $r^F(\hat{\varphi})$ into $\bar{\pi}^F$, we can rewrite $\bar{\pi}^F$ as

$$\bar{\pi}^F = \left\{\left[\left(1\text{-}\tau^C\right) w^F\left(e^{-S}f+x\right)+\pi^I\right]\left[\left(\frac{\bar{\varphi}^F(\hat{\varphi})}{\hat{\varphi}}\right)^{\sigma-1}\text{-}1\right]+\pi^I\right\} \qquad \text{(FICP)}$$

- Under a given $\tilde{\varphi}$ and π^I, the FICP condition
  - is downward sloping in $\hat{\varphi}$, with $\lim_{\hat{\varphi}\to 0}\bar{\pi}^F = \infty$ and $\lim_{\hat{\varphi}\to\infty}\bar{\pi}^F = \pi^I$.
  - behaves similar to the ZCP in Melitz (2003).

Establishment condition 1

- **Assumption** (Nondegenerate) $v^o > w^F f_e$.
- The expected value of establishing an urban firm satisfies:

$$P_r^{\text{informal}} \cdot \left(\frac{\pi^I}{\delta} - \frac{\xi}{\tilde{\varphi}}\right) + P_r^{\text{formal}} \cdot \left(\frac{\bar{\pi}^F}{\delta} - \frac{\xi}{\tilde{\varphi}}\right) = v^o$$

- **Establishment Condition (EC):**

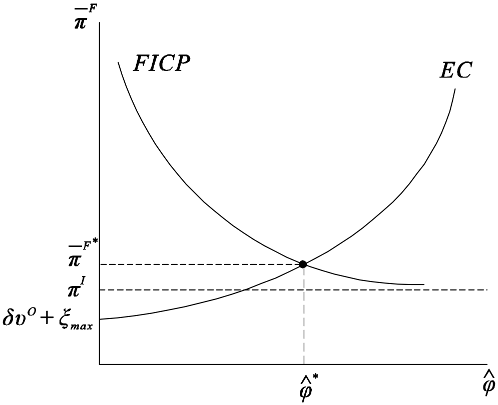
$$\bar{\pi}^F = \frac{\delta}{1 - G(\hat{\varphi})} \left\{ v^o - [1 - G(\tilde{\varphi})] \left(\frac{\pi^I(\hat{\varphi})}{\delta} - \frac{\xi_{\max}}{\tilde{\varphi}/\varphi_{\min}} \right) \right\} + \pi^I(\hat{\varphi}) \qquad \text{(EC)}$$

where $\xi_{\max} \equiv \frac{\xi}{\varphi_{\min}}$ is the maximum managerial cost for establishing and managing a firm.

- In EC,
  - $\bar{\pi}^F$ is upward sloping in $\hat{\varphi}$ under a given $\tilde{\varphi}$ and π^I;
  - $\bar{\pi}^F$ is increasing in $\tilde{\varphi}$ if the markup of informal over self-employed is less than the shaped parameter, η_φ.

FICP and EC

- Under a given $\tilde{\varphi}$ and π^I, the upward-sloping EC and the downward-sloping FICP intersect and determine the unique equilibrium $\left(\hat{\varphi}^*, \bar{\pi}^{F*}\right)$.



Stationary equilibrium conditions 1

Firm side

- Total potential firms as $M \equiv M^F + M^I + M^O$ is exogenously given.
- Denote M_-^i $(i = F, I)$ as the mass of firms before the death of existing firms, the entrance of new firms and the migration of rural migrant workers; M_e^i the total mass of newly entered firms and M_{en}^i as the net increase in the mass of firms:

$$
\begin{aligned}
M^i &= (1-\delta) M_-^i + M_e^i \text{ and } M_e^i = M_{en}^i + \delta M_-^i. \\
M^F(\underset{-}{\hat{\varphi}}) &= [1 - G(\hat{\varphi})] M, \\
M^I(\tilde{\varphi}, \hat{\varphi}) &= [G(\hat{\varphi}) - G(\tilde{\varphi})] M.
\end{aligned}
$$

- Given M_-^F and M_-^I, the net increase in formal and informal firms are:

$$
M_{en}^F\left(\hat{\varphi}; M_-^F\right) = M^F(\hat{\varphi}) - M_-^F \text{ and } M_{en}^I(\underset{-}{\tilde{\varphi}}, \underset{+}{\hat{\varphi}}; \underset{-}{M_-^I}) = M^I(\tilde{\varphi}, \hat{\varphi}) - M_-^I
$$

- Stationary state for firms requires that the mass of firms grows at a constant rate in order to accommodate migrant workers.

Stationary equilibrium conditions 2

Informal wage rate

- In equilibrium, $P^I = \lambda P^F(\hat{\varphi})$. For an informal firm with $\tilde{\varphi}$, its profit must satisfy $\frac{\pi^I(\hat{\varphi})}{\delta} - \frac{\xi}{\tilde{\varphi}} = v^o$. Since $\frac{\xi}{\tilde{\varphi}} = \frac{\xi_{\max}}{\tilde{\varphi}/\varphi_{\min}}$, we obtain

$$w^I(\underset{-}{\hat{\varphi}}; \underset{+}{\tilde{\varphi}}; \underset{-}{\xi_{\max}}, \underset{-}{\zeta_0}, \underset{+}{\lambda}) = \gamma \left[\frac{\delta}{(1-\gamma)(1-\zeta)} \left(v^o + \frac{\xi_{\max}}{\tilde{\varphi}/\varphi_{\min}} \right) \right]^{-\frac{1-\gamma}{\gamma}} \left(\lambda a^I P^F(\underset{-}{\hat{\varphi}}) \right)^{\frac{1}{\gamma}},$$

  where w^I is decreasing in $\hat{\varphi}$.
- From the labor demand for individual informal firms, we have:

$$\ell^I(\underset{+}{\hat{\varphi}}; \underset{-}{\tilde{\varphi}}; \underset{+}{\xi_{\max}}, \underset{+}{\zeta_0}, \underset{-}{\lambda}, \underset{+}{\kappa}) = \left(\frac{\kappa}{\gamma} \right)^{\frac{1}{1-\gamma}} \left[\frac{\delta}{(1-\gamma)(1-\zeta)} \left(v^o + \frac{\xi_{\max}}{\tilde{\varphi}/\varphi_{\min}} \right) \frac{1}{\lambda a^I P^F(\underset{-}{\hat{\varphi}})} \right]^{\frac{1}{\gamma}}$$

  with ℓ^I increasing in $\hat{\varphi}$.

Stationary equilibrium conditions 3

Informal wage rate

- Define $W^I \equiv w^I \ell^I$ as the total labor cost of an urban informal firm. We have:

$$W^I(\underset{0}{\hat{\varphi}(x, \tau_c, S_0)}; \underset{-}{\tilde{\varphi}}; \underset{+}{\xi_{\max}}, \underset{+}{\zeta_0}, \underset{0}{\lambda}, \underset{+}{\kappa}) = \left(\frac{\kappa}{\gamma^\gamma}\right)^{\frac{1}{1-\gamma}} \frac{\delta}{(1-\gamma)(1-\zeta)} \left(v^o + \frac{\xi_{\max}}{\tilde{\varphi}/\varphi_{\min}}\right).$$

That is, for given $\tilde{\varphi}$, total labor cost of an informal firm is independent of $\hat{\varphi}$ and λ, and is increasing in $\xi_{\max}$, ζ and κ.

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Stationary equilibrium conditions 4

Labor market equilibrium

- Define ($i = F, I$):
  - L^i and L^i_- (N^i and N^i_-): firm side's (household side's) total workers in sector i after and before all decisions.
  - $\tilde{N}^i$: new migrants in sector i
- (Aggregating from firm side) Define $\bar{\ell}^F_p(\hat{\varphi}) = (\sigma - 1)\frac{\bar{\pi}^F(\hat{\varphi})}{w^F} + \sigma\left(e^{-S}f + x\right)$. Total workers in sector i is :

$$
\begin{aligned}
L^F &= \left[(1-\delta) M^F_- + M^F_e\right]\bar{\ell}^F_p(\hat{\varphi}) + M^F_e f_e, \\
L^I &= (1-\delta) M^I_- \ell^I(\hat{\varphi};\cdot) + M^I_e\left(\frac{w^F}{w^I(\hat{\varphi};\cdot)} f_e + \ell^I(\hat{\varphi};\cdot)\right).
\end{aligned}
$$

- (Aggregating from household side) Total workers in sector i is :

$$
L^F = \tilde{N}^F + L^F_- \quad \text{and} \quad L^I = \tilde{N}^I + L^I_-,
$$

where $L^F_- = M^F_- \ell^F\left(\bar{\varphi}^F\right)$ and $L^I_- = M^I_- \ell^I(\hat{\varphi};\cdot)$.

Stationary equilibrium conditions 5

Labor market equilibrium

- Population laws of motion:

$$
\begin{aligned}
N^F &= \tilde{N}^F + N^F_-, \\
N^I &= \tilde{N}^I + N^I_-.
\end{aligned}
$$

- Denote θ^i_j as the growth rate of j in sector i. In a stationary equilibrium,

$$\frac{\theta^F_M}{\theta^F_N} = \frac{N^F_- / M^F_-}{\bar{\ell}^F + f_e},$$

and

$$\frac{\theta^I_M}{\theta^I_N} = \frac{N^I_- / M^I_-}{\ell^I(\hat{\varphi};\cdot) + \frac{w^F}{w^I(\hat{\varphi};\cdot)} f_e}.$$

Stationary equilibrium conditions 6

Labor market equilibrium

- The labor market equilibrium conditions are:

$$\tilde{N}^F = \delta M_-^F f_e + M_{en}^F \left(\bar{\ell}_p^F(\hat{\varphi}) + f_e\right), \quad \text{(FLE)}$$

$$\tilde{N}^I(\hat{\varphi}) = \frac{w^F}{w^I}\left\{\delta M_-^I f_e + \left[\Theta + \frac{[G(\hat{\varphi}) - G(\tilde{\varphi})]}{[1 - G(\hat{\varphi})]} M_-^F - M_-^I\right]\left(\frac{W^I}{w^F} + f_e\right)\right\}, \quad \text{(ILE)}$$

where $w^I = w^I(\underset{-}{\hat{\varphi}}; \underset{+}{\tilde{\varphi}}; \underset{-}{\xi_{\max}}, \underset{-}{\zeta_0}, \underset{+}{\lambda})$, $W^I = W^I(\hat{\varphi}(x, \underset{0}{\tau_c}, S_0); \underset{-}{\tilde{\varphi}}; \underset{+}{\xi_{\max}}, \underset{+}{\zeta_0}, \underset{0}{\lambda}, \underset{+}{\kappa})$, and $\Theta = \Theta(\underset{+}{\hat{\varphi}}; \underset{-}{\xi_{\max}}, \underset{-}{x}, \underset{+}{S_0}, \underset{-}{f_e}) \equiv \frac{[G(\hat{\varphi}) - G(\tilde{\varphi})]}{[1-G(\hat{\varphi})]} \frac{\tilde{N}^F - \delta M_-^F f_e}{\left(\underset{+}{\bar{\ell}_p^F(\hat{\varphi})} + f_e\right)}$.

- From the **informal labor market clearing condition (ILE)** above, we can derive: $\tilde{\varphi} = \underset{+}{\tilde{\Phi}(\hat{\varphi})}$, which is positively sloped in plane-$(\hat{\varphi}, \tilde{\varphi})$.
  - The FICP, EC and ILE, together with households' optimal decisions determine the equilibrium in this economy.

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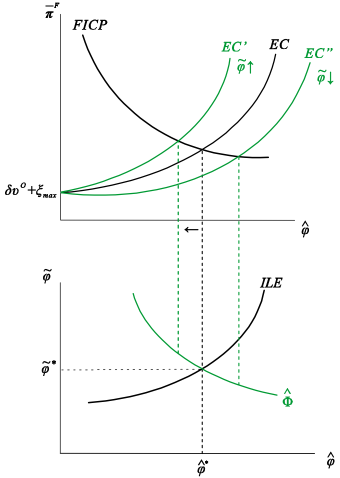
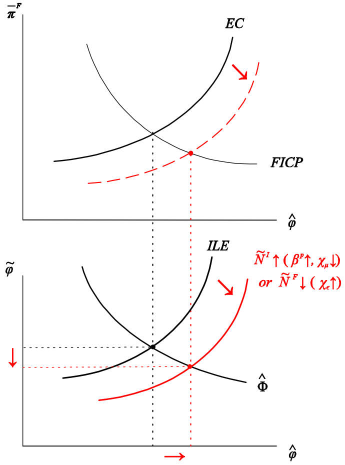
The equilibrium



- Given $\{N_-^i, M_-^i\}$, $i = \{F, I\}$, EC shifts up to the left when $\tilde{\varphi}$ is higher, intersecting with FICP at a lower $\hat{\varphi}$. EC shifts down to the right when $\tilde{\varphi}$ is lower, intersecting with FICP at a higher $\hat{\varphi}$.
- We thus combine EC and FICP to derive $\tilde{\varphi} = \hat{\Phi}(\underset{(-)}{\hat{\varphi}})$, which is negatively sloped in plane-$(\hat{\varphi}, \tilde{\varphi})$.
- Also note that in equilibrium, $\pi^I(\hat{\varphi}) = \delta(v^o + \frac{\xi}{\tilde{\varphi}})$.

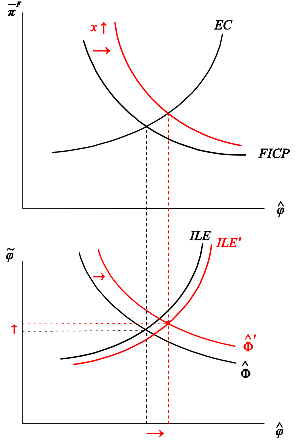
General equilibrium comparative statics 1



When households are more altruistic in remittance giving ($\beta^p \uparrow$) or migration disutility is lower ($\chi_\mu \downarrow$)

- $\tilde{N}^I$ increases, shifting ILE to the right.
- In response, EC must shift to the right.
- The result is an increase in $\hat{\varphi}$ and a decrease in $\tilde{\varphi}$ – an expansion of the informal sector.
- Positive correlation between remittance and informality, as well as migration and informality
- A higher aggregate informal output, but an ambiguous effect on aggregate formal output or aggregate output.
- Similar effect with a higher disutility from formal employment ($\chi_\epsilon \uparrow$): $\tilde{N}^F \downarrow$.

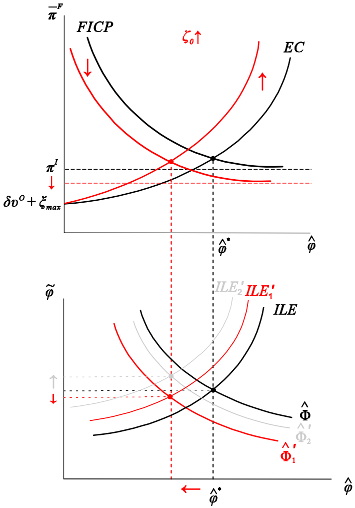
General equilibrium comparative statics 2



When the regulatory cost x increases,

- FICP rotates up to the right (keeping π^I constant), shifting $\hat{\Phi}$ up to the right.
- Less labor demand in the formal sector shifts ILE to the right.
- In equilibrium,
    - $\hat{\varphi}$ increases, meaning that running a formal firm is more "costly" and needs a higher productivity to compensate the increase in the regulatory cost.
    - $\tilde{\varphi}$ also increases, meaning that the overall threshold in running a business increases due to the increase in cost of running businesses.
- Depending on the relative increases in $\hat{\varphi}$ and $\tilde{\varphi}$, the size of the informal sector may shrink or expand.
- High dispersion in the size of informality.
- Other exercise: $S_0 \downarrow$.

General equilibrium comparative statics 3



When the probability of being fined or asked for bribes is higher ($\zeta_0 \uparrow$)

- π^I decreases, FICP shifts down, EC rotates up, and $\hat{\Phi}$ locus shifts down. Besides, ILE shifts up.
- In equilibrium,
  - $\hat{\varphi}$ decreases because the informal sector is relatively less profitable compared to the formal sector.
  - $\tilde{\varphi}$ could increase or decrease, depending on the shifts in ILE and Φ locus.
- This leads to an expansion of the formal sector but an ambiguous effect on the informal sector, though each informal firm is down-sized.
- Formal output share and formal employment share both rise.

Takeaway

- We model rural-urban migration and occupational choices and urban firm organizational choices:
  - A higher altruism on remittance or a lower migration disutility implies:
    - an expansion of the informal sector;
    - a positive correlation between remittance and informality, as well as between migration and informality;
    - a higher aggregate informal output, but an ambiguous effect on aggregate formal output or aggregate output.
  - When running a formal firm is more costly (e.g., a larger regulatory cost or a worse infrastructure provision):
    - the size of the informal sector may shrink or expand, depending on the relative changes of the two productivity cut-offs;
    - this implies a large variation in the size of informality.