Endogenous Growth Theory

Ping Wang
Department of Economics
Washington University in St. Louis

March 2024

I. Introduction

A. Stylized Facts and Growth Empirics

(i) Basic Stylized Facts:

- on-going increasing K/L and Y/L ratios
- increasing wage rates with stationary interest rates
- stationary wage and capital income shares
- widened cross-country disparities

(ii) Sources of Data: Summers-Heston, Barro-Lee, Barro-Sala-i-Martin, others

(iii) Growth Accounting: Denison, Jorgenson, Tallman-Wang

(iv) Factor Accumulation and Growth

Country	I/Y (%)	Y_i/Y_{US}*100 (1990)	$\Delta Y_i/Y_i$ (%)	Country	H index	Y_i/Y_{US}*100 (1990)	$\Delta Y_i/Y_i$ (%)
U.S.	24.0	100	2.1	U.S.	11.8	100	2.1
Algeria	23.3	14	2.2	Argentina	6.7	19	0.7
Zambia	27.9	4	-0.8	Philippines	6.7	14	1.3
Guyana	25.1	7	-0.9	Korea	9.2	45	6.3
Japan	36.6	80	5.6	New Zealand	12.3	63	1.4
Singap.	32.6	60	6.4	Norway	10.6	81	3.7

(v) TFP Growth and Long-Run Development

Country	TFP Growth (%)	Contribution to Output Growth (%)
U.S.	0.4	13
Germany	1.6	49
U.K.	1.3	52
Chile	1.5	40
Mexico	2.3	37
Hong Kong	2.2	30
Singapore	-0.4	-5
Korea	1.2	12
Taiwan	1.8	20

(vi) Convergence:

Convergence in per capita real GDP (Baumol, Barro, Barro-Sala-i-Martin):

- **β-convergence -** $\frac{\dot{y}_i}{y_i} = \theta_i = \beta_0 + \beta y_i(0) + \ldots$

- **σ-convergence -** $\frac{d}{dt}[Var(y_i)] < 0$

Problems:

- **Galton Fallacy (Quah 1993)**
- **Twin-peak hypothesis (Quah 1996)**
- **Endogeneity problems; measurement errors**

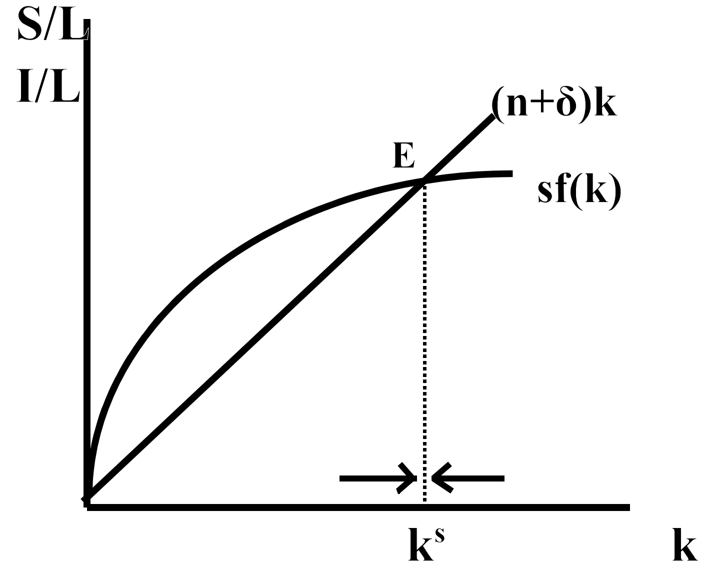
(vii) Sources of Growth:

- **Theory ahead of empirics**
- **Kitchen sink regressions**

(viii)Open Issues:

- **Robustness (Levine-Renelt 1992, de la Fuente 1997)**
- **TFP matters (Young 1995/Hsieh 1999, 2002)**
- **Test of endogenous vs. exogenous growth (Quah 1993, Jones 1995)**
- **Cross-country vs. country studies (Tallman-Wang 1994)**
- **Identification of endogenous threshold**

B. Neoclassical Exogenous Growth Theory



(i) Solow-Swan: use variable k/y to resolve the Harrod knife-edge problem

(ii) Allais-Phelps golden rule: S-S per capita consumption maximization

- $\max_k \; C/L = f(k) - (n+\delta)k$

$$\Longrightarrow \; f_k = n + \delta \; \Longrightarrow \; k^g$$

(iii) Ramsey-Cass-Koopman optimal exogenous growth:

a. Continuous-time (optimal control):

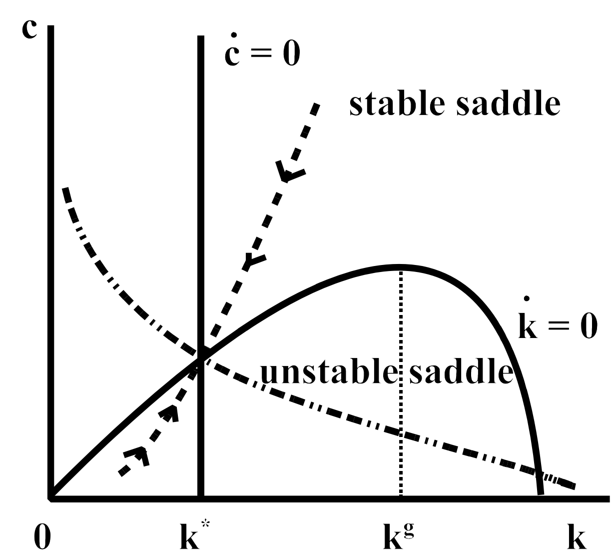
$$\max \quad U = \int_0^\infty u(c)\, e^{-\rho t}\, dt$$

$$s.t. \quad \dot{k} = f(k) - (n+\delta)k - c, \qquad k(0) = k_0 > 0.$$

- **Transversality condition TVC (λ = co-state):** $\lim_{t\to\infty} \lambda k e^{-\rho t} = 0$
- **Keynes-Ramsey equation:**

$$\dot{c} = \sigma(c)c[f_k - \rho - (n+\delta)], \quad \sigma(c) = -\frac{u_c}{u_{cc}c}$$

- **Steady-state: golden rule is dynamically inefficient (over-accumulation)**
- **Dynamic equilibrium path: unique stable saddle**



b. Discrete-time (dynamic programming)

$$v(k) = \max_c \quad u(c) + \beta V(k')$$

$$s.t. \quad k' = f(k) + (1-\delta)k - c, \quad k(0) = k_0 > 0, n = 0.$$

- FOC: $u_c(c) = \beta V_k(k')$
- Benveniste-Scheinkman: $V_k(k) = \beta V_k(k')[f_k(k) + (1-\delta)]$
- Euler equation: $u_c(c) = \beta u_c(c')[f_k(f(k)+(1-\delta)k-c) + (1-\delta)]$

(iv) Problems of the exogenous growth models:

a. long-run growth exogenously determined by exogenous technical progress
b. lack of strong evidence in global convergence
c. failure to explain widened growth disparities
d. policy does not matter unless it can affect the rate of technical progress

C. Development of Endogenous Growth Theory

- Is the endogenous growth theory new?
- Seminal work before 1980s:
  - von Neumann (1937, translated 1945/46 RES): linear production & BGP
  - Solow (1956, at the end of his seminal paper): IRS & sustained growth
  - Pitchford (1960 ER): DRS with sustained growth
  - Shell (1966 AERP&P): inventive activity & growth
  - Wan (1970 REStud) - learning by doing & growth

- Creators of the new waves:
  - Romer (1986 JPE): general/knowledge capital
  - Lucas (1988 JME): human capital
  - Stokey (1988 JPE): learning-by-doing
  - Rebelo (1991 JPE): basic one & two sector models
  - Grossman-Helpman (1991 book) & Aghion-Howitt (1992 Econometrica): innovation & technical progress with imperfect competition

(i) New stylized facts:

(S1) club-convergence: bi-mode (Quah 1996)

(S2) divergent paths for physical and human capital:

US	high K	high H	
Macao	low K	high H	(Taiwan - H-intensive)
Congo	high K	low H	(Korea - K-intensive)
Zaire	low K	low H	

(S3) increasing rates of growth for the leading economy:

1700-1785	Netherlands	$\theta = -0.07\%$
1785-1820	UK	$\theta = 0.5\%$
1820-1890	UK	$\theta = 1.4\%$
1890-1970	US	$\theta = 2.3\%$

(S4) both skilled/unskilled migrate to rich countries:
why do the unskilled want to do so if paid by marginal products?

(S5) over-taking/lagging behind development experiences:
why did Korea/Taiwan by-pass the Phillipines?
why did Argentina dropped from the top?

(S6) capital deepening, product broadening, financial deepening, and world productivity and income distribution widening

(ii) How to design plausible models to endogenously determine the rate of economic growth and to match with these stylized facts?

	One-Sector	Two-Sector
CRS	(S1) Rebelo (1991) I Romer (1986) I Jones-Manuelli (1990) Barro (1990)	(S2) Lucas (1988) I Rebelo (1991) II Bond-Wang-Yip (1996)
IRS	(S3) Romer (1986) II Xie (1991)	(S4) Lucas (1988) II (S5) Xie (1994); Benhabib-Perli (1994) Laing-Palivos-Wang (1995) Bond-Wang-Yip (1996) (S6) Aghion-Howitt (1992) Grossman-Helpman (1989); Romer (1990) Greenwood-Jovanovic (1990)

Remarks: Other areas recently developed include,

a. **fiscal policy (government spending and finance)**
b. **money (inflation and growth)**
c. **trade (goods flows, factor movements, technology transfer)**
d. **demography (fertility, aging, gender gap, social security)**
e. **labor market (education, learning, training, unemployment, sectoral shift)**
f. **political economics (government infrastructure, size of nation, voting, institutional/organizational design)**

II. One-Sector Models

A. General Methodology

(i) Key:

Marginal products of reproducible factors are bounded below by a constant, thus requiring the production function be CRS or IRS in reproducible factors

(ii) Trick:

In perfectly competitive equilibrium, the model must be consistent with zero profit conditions, thus requiring CRS in privately provided factors

(iii) Decentralized problem:

a. consumer optimization:

$$\max\ U = \int_0^\infty u(c)e^{-\rho t}dt, \quad u(c) = \frac{c^{1-\alpha}-1}{1-\alpha}, \alpha = \sigma^{-1}$$

$$s.t.\quad \dot{\Omega} = r\Omega - c, \quad \Omega(0) = \Omega_0 > 0.$$

Transversality condition (λ = co-state): $\lim_{t\to\infty} \lambda\Omega e^{-\rho t} = 0.$

b. producer optimization:

$$\max \quad V = \int_0^\infty [f(k) - qx]\, e^{-\int_0^t r(s)\,ds}\, dt$$

$$s.t. \quad \dot{k} = x - \delta k, \qquad k(0) = k_0 > 0.$$

Transversality condition (μ = co-state): $\lim_{t\to\infty} \mu k e^{-\int_0^t r(s)\,ds} = 0.$

c. Key relationships:

- **Keynes-Ramsey equation:**

$$\theta = \frac{\dot{c}}{c} = \frac{r - \rho}{\alpha} \implies r = \rho + \alpha\theta \quad \text{(UU)}$$

- **Firm's Euler equation:**

$$\frac{\dot{q}}{q} = (r + \delta) - \frac{f_k}{q}$$

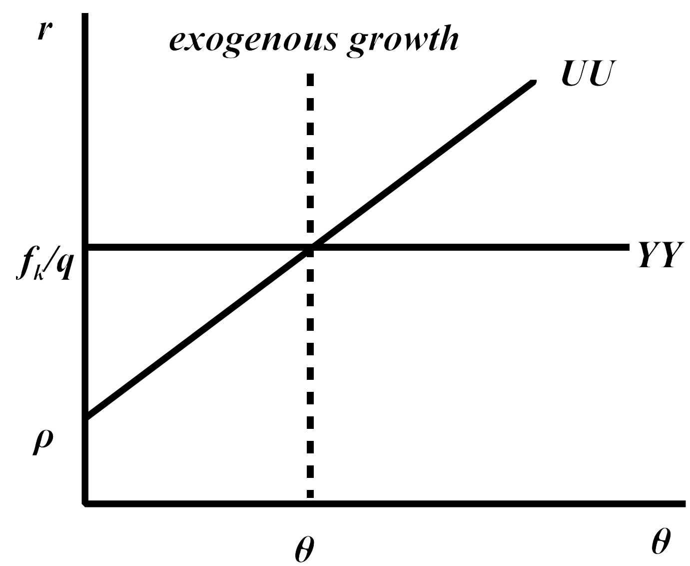
- **Balanced Growth Path (BGP):**

$$\frac{\dot{c}}{c} = const., \quad \frac{\dot{k}}{k} = const. \quad \Longrightarrow \quad \frac{\dot{q}}{q} = 0 \quad \Longrightarrow \quad r = \frac{f_k}{q} - \delta \quad \text{(YY)}$$

d. Determining the BGP endogenous growth rate:

$$\theta = \frac{(f_k/q) - (\rho + \delta)}{\alpha}$$

- **diagram: f_k = constant**
- **exogenous growth: given θ, UU and YY pin down r**
- **endogenous growth: given f_k/q, UU and YY pin down θ**



e. Key conditions:

Condition U: (Bounded Utility) $\rho > (1-\alpha)\theta^{max}$ **(Brock-Gale)**

Condition G: (Positive Growth) $\min_k \{f_k/q\} > \rho + \delta$ **(Jones-Manuelli)**

(iv) Social planner problem:

- **This is the same as the decentralized one if there are no externalities nor distortionary taxes**

- **In this case, the optimization can be simplified as:**

$$\max\ U = \int_0^\infty \frac{c^{1-\alpha}-1}{1-\alpha} e^{-\rho t}\, dt \quad s.t.\ \dot{k} = \frac{f(k)}{q} - \delta k - \frac{c}{q},\ k(0) = k_0,\ q(0) = 1$$

- **Remarks: $f_k(k)$ must be bounded below by a positive constant in order to support sustained growth, which includes three possible cases: (1) CRS, (2) asymptotic CRS, and (3) IRS. The second case includes the short-run diminishing return cases in Jones-Manuelli (1990) and in Pitchford (1960), where MPK = $A+\gamma Bk^{\gamma-1}$ and $[B_1^{\varepsilon}+B_2(L/k)^{1-1/\varepsilon}]^{1/(\varepsilon-1)}$, respectively. The third case can be found in Solow (1956).**

B. Ak Model: Rebelo (1988) I

(i) Setup: $y = f(k) = Ak$; $q = 1$

(ii) Endogenous growth rate: $\theta = \dfrac{A - (\rho + \delta)}{\alpha} > 0$

(iii) Main features:

- **a. common growth (for c, k, y): θ is increasing in (A, σ), decreasing in (ρ, δ)**
- **b. non-convergence (S1)**
- **c. any policy affecting the *level* of A has a growth effect**
- **d. no transitional dynamics**
- **e. CE <==> PO**

C. CRS Model with Uncompensated Positive Spillover: Romer (1986) I

(i) Setup: $y = f(k) = Ak^{1-\beta} K^{\beta}$; $q = 1$ (aggregate capital K = k in equilibrium)

$$\Longrightarrow \quad r = A(1-\beta)\left(\frac{\bar{K}}{k}\right)^{\beta} - \delta = A(1-\beta) - \delta \text{ in equilibrium}$$

(ii) Competitive common growth rate:

$$\theta = \frac{r - (\rho + \delta)}{\alpha} = \frac{A(1-\beta) - (\rho + \delta)}{\alpha} > 0$$

(iii) Main features:

a. θ is increasing in A and σ; decreasing in ρ, δ *and* β

b. non-convergence (S1)

c. any policy affecting A has a growth effect

d. no transitional dynamics

e. CE is not the same as PO

(iv) Pareto-optimal endogenous growth rate:

- **Substitute in K = k before differentiation and y = Ak**

  ==> $\theta^* = \dfrac{A - (\rho + \delta)}{\alpha} > 0$

  ==> $r < r^*; \quad \theta < \theta^*$

- **Under-investment in equilibrium is due to the free-rider problem, which is absent iff β = 0.**

(v) Remedy inefficiency using a Pigovian policy: Barro and Sala-i-Martin (1992)

a. **production subsidy:** $\dfrac{\beta}{1-\beta}(r+\delta)k \qquad \left(\tau^* = \dfrac{\beta}{1-\beta} \right)$

b. **factor price subsidy:** $\dfrac{f_k}{1-\beta} - \delta \qquad (q^* = 1-\beta)$

D. IRS Model with Uncompensated Positive Spillover: Romer (1986) II

(i) Setup: $y = f(k) = Ak^{1-\beta} K^{\beta+\gamma}$; $q = 1$; $\gamma \geq 0$ **(aggregate K=k in equilibrium)**

(ii) Equilibrium paths:

no longer common growth ==> need to distinguish different rates of growth

$$\theta_c = \frac{\dot{c}}{c} = \left(\frac{1-\beta}{\alpha}\right) A k^{\gamma} - \frac{\rho}{\alpha}$$

$$\theta_k = \frac{\dot{k}}{k} = A k^{\gamma} - \frac{c}{k}$$

(iii) Transformation - Xie (1991): $z = k/c$ **with** $1-\beta = \alpha$

$$\theta_z = \frac{\dot{z}}{z} = -\frac{1}{z} + \frac{\rho}{\alpha}$$ **(a Bernoulli equation for z)**

(iv) **Solution:** $z(t) = \frac{\alpha}{\rho} + (a - \frac{\alpha}{\rho})\, e^{(\rho/\alpha)t}$ **where *a* is an integration constant**

- **TVC:** $lim_{t\to\infty}\, \lambda k(t) e^{-\rho t} = \lim_{t\to\infty} c^{-\alpha} c z e^{-\rho t} = 0$

$$\Longrightarrow \quad \lim_{t\to\infty} c^{\beta}\left[\frac{\alpha}{\rho} e^{-\rho t} + (a - \frac{\alpha}{\rho})\, e^{(\frac{1}{1-\beta} - 1)\rho t}\right] = 0$$

- **Thus,** $a = \frac{\alpha}{\rho} \Longrightarrow z(t) = \frac{\alpha}{\rho}$ (so, $\theta_c = \theta_k = \theta$)

(v) **Competitive growth rates: (θ_y is strictly increasing and strictly concave)**

$$\theta = A k^{\gamma} - \frac{\rho}{\alpha}\ ; \quad \theta_y = (1+\gamma)\left[A^{\frac{1}{1+\gamma}} y^{\frac{\gamma}{1+\gamma}} - \frac{\rho}{\alpha}\right]$$

(vi) **Main features:**

a. **increasing growth rate (S2), at a diminishing speed if** $\gamma < 1$

b. **well-defined transitional dynamics**

c. **more sever free-rider problem.**

E. Public Capital: Barro (1990)

(i) Setup: $y = f(k) = Ak^{1-\beta} G^{\beta}$; $q = 1$ (public capital $G = \tau y$ in equilibrium)

(ii) Endogenous growth rate: $\theta = \frac{1}{\alpha}\left[(1-\beta)A^{1/(1-\beta)}\tau^{\beta/(1-\beta)}(1-\tau) - (\rho + \delta)\right]$

which is maximized at $\tau^* = \beta$ (optimal government size, max U too)

(iii) Discussion:
a. modified Laffer curve
b. growth-maximizing vs. welfare maximizing (generally different beyond this simple setting)

F. Long-Run CRS with Short-Run DRS: Jones-Manuelli (1990)

(i) Setup:

$y = Ak + Bk^{\gamma}$; $q = 1$ ($0<\gamma<1$); $\delta = 0$

$\Longrightarrow r = A + B\gamma k^{\gamma-1} \rightarrow A \quad as\ k \rightarrow \infty$ (thus, there is an asymptotic BGP)

(ii) Transformation: $z_1 = c/k;\ z_2 = y/k$

$$\theta_c = \frac{1}{\alpha}[A + \gamma B k^{\gamma-1} - \rho] = \frac{1}{\alpha}[A + \gamma(z_2 - A) - \rho]$$

$$\theta_k = \frac{y}{k} - \frac{c}{k} = z_2 - z_1$$

$$\theta_y = \frac{1}{z_2}[A + \gamma(z_2 - A)]\theta_k$$

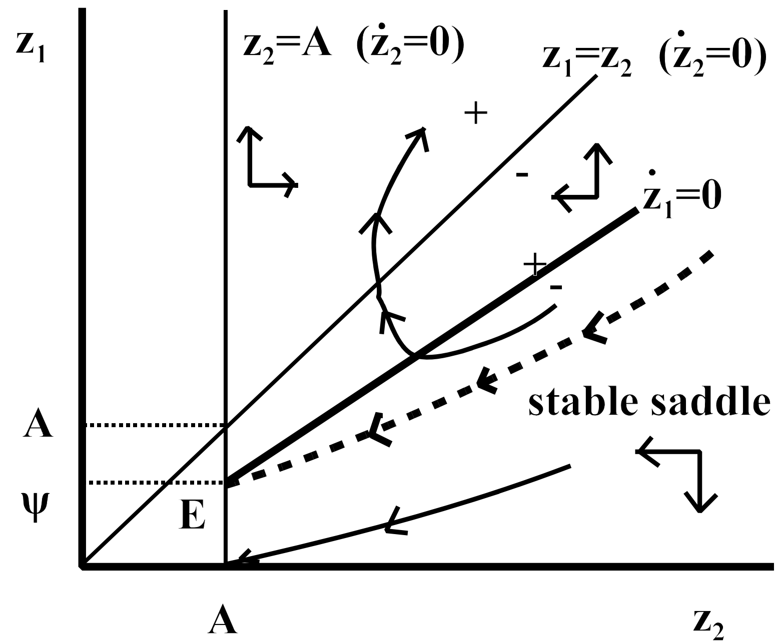
$$\theta_{z_1} = \frac{\dot{z}_1}{z_1} = (\frac{\gamma}{\alpha} - 1)(z_2 - A) + z_1 - \psi$$

$$\theta_{z_2} = \frac{\dot{z}_2}{z_2} = -\frac{1}{z_2}(1-\gamma)(z_2 - A)(z_2 - z_1)$$

where $\psi = \frac{1}{\alpha}[\rho - (1-\alpha)A] > 0$

(iii) The dynamical system: The case of $\alpha > \gamma$

- Well-defined transitional dynamics with a unique half saddle arm
- Along the transition, z_1 and z_2 move in the same direction toward the BGP
- Remark: for $\alpha < \gamma$, z_1 and z_2 move in the opposite direction, which is inconsistent with empirical observations



(v) Remarks: asymptotically linear production is generally useful for its clean long-run results and manageable short-run analytics.

III. Two-Sector Models

A. Physical and Human Capital with CRS: Lucas (1988) I

(i) Setup:

$$\max \qquad U = \int_0^\infty \frac{c^{1-\alpha} - 1}{1-\alpha} e^{-\rho t}\, dt$$

$$\text{s.t.} \quad \dot{k} = y - \delta k - c, \quad y = F(k, uh); \quad k(0) = k_0 > 0$$

$$\dot{h} = \varphi(1-u)h, \quad h(0) = h_0 > 0$$

(ii) Key: linear education evolution to simplify the analysis (simplified Uzawa)

(iii) Solution with Cobb-Douglas Production: (CE = PO)

$$\theta = \frac{\varphi - \rho}{\alpha} > 0$$

Only education matters (goods production technology does not matter)!

B. Physical and Human Capital with IRS: Lucas (1988) II

(i) Setup: $y = F(k, uh, H) = Ak^{1-\beta}(uh)^{\beta}H^{\gamma}$ $\quad (H = h \text{ in equilibrium};\ \gamma > 0)$

H is due to uncompensated positive spillover from peers (γ = 0 => Lucas I)

(ii) Key:

linear human capital evolution accepts BGP
CRS in private factors ensures consistency with CE

(iii) Competitive BGP equilibrium:

common growth for c, k, F ==> $\theta_c = \theta_k = \frac{\dot{F}}{F} = \theta$

Cobb-Douglas PF ==> $\theta_h = (\frac{\beta}{\beta+\gamma})\theta_k = (\frac{\beta}{\beta+\gamma})\theta$

a. FOCs:

$$c^{-\alpha} = \mu \quad ==> \quad \theta = -\frac{1}{\alpha}\frac{\dot{\mu}}{\mu} \quad \text{(CC)}$$

$$\mu\frac{\beta F}{u} = \lambda\varphi h \quad ==> \quad \frac{\dot{\mu}}{\mu} + \frac{\dot{F}}{F} - \frac{\dot{u}}{u} = \frac{\dot{\lambda}}{\lambda} + \theta_h \quad \text{(FS)}$$

b. Euler equations:

$$\frac{\dot{\mu}}{\mu} = (\rho+\delta) - (1-\beta)\frac{F}{k} \quad ==> \quad \theta = \frac{1}{\alpha}\left[(1-\beta)\frac{F}{k} - (\rho+\delta)\right] \quad \text{(UU)}$$

$$\frac{\dot{\lambda}}{\lambda} = \rho - \varphi \quad \text{(LL)}$$

c. Transversality conditions: $\lim_{t\to\infty} \mu k(t) e^{-\rho t} = 0; \quad \lim_{t\to\infty} \lambda h(t) e^{-\rho t} = 0$

d. Transformation: $z_1 = c/k;\ z_2 = y/k = A(uh)^{\beta+\gamma}/k^{\beta}$

production function $==>$ $$\frac{\dot{F}}{F} = \beta\frac{\dot{u}}{u} + \theta \quad \text{(PP)}$$

e. BGP growth without common growth:

CC,FS,UU,LL,PP ==> $$\theta = \frac{(\beta+\gamma)(\varphi-\rho)}{\alpha\beta+(\alpha-1)\gamma} > \frac{\beta(\varphi-\rho)}{\alpha\beta+(\alpha-1)\gamma} = \theta_h$$

- **θ is more responsive to φ than in the case of $\gamma = 0$**
- **h does not growth at the same rate as c, k or y => S4**

(iv) Pareto Optimum

Similar to Romer II, people under-invest in CE due to the free-rider problem; one can remedy the inefficiency by an education subsidy.

(v) Transitional dynamics:

Lucas believes that k and h should move together along the saddle path toward the BGP, which has been shown incorrect by Xie (1994) and Benhabib-Perli (1994). Specifically, there may be dynamic indeterminacy in the sense that there is a continuum of transition paths converging to the unique BGP, depending crucially on how strong the positive externality and intertemporal substitution are.

C. Learning-by-doing and Growth: Lucas (1993)

- **Liberty ship (same blue print): Searle (1945) and Rapping (1965) identify 12-24% and 11-29% learning-by-doing effect in production**

(i) Setup:

Final good output: $y = F(n, z) = Anz^{\xi}$

Experience accumulation: $\dot{z} = G(n, z) = nz^{\xi}$

(ii) Key:

- **experiences grow over time**
- **more employment is better for both production and accumulating experiences**

(iii) Results:

$$z(t) = [z_0^{1-\xi} + (1-\xi)\int_0^t n(\tau)\,d\tau]^{1/(1-\xi)}$$

- **With** $n = \bar{n}$ **, we have:** $y(t) = A\bar{n}[z_0^{1-\xi} + (1-\xi)\bar{n}t]^{1/(1-\xi)}$
- **Rate of productivity:**

  $\mu(t) \equiv d\ln(z^{\xi})/dt = \xi\bar{n}z^{\xi-1} \rightarrow \xi/[(1-\xi)t]$ **if** $z_0 \rightarrow 0$

(iv) Main Findings:

a. **presence of scale effect:** $d\mu/d\bar{n} > 0$

b. **based on Rapping (1965),** $\xi = 0.2$**; so,** $d\mu/dt = 0.25$

c. **rapid decay in learning means that making a miracle requires continual emergence of new innovation**

d. **applying similar argument, process innovation (cost-reduction) cannot promote perpetual growth without developing new products**

(v) Problems:

a. **scale effect not empirically supported =>**
   (1) **Young (1991): bounded learning**
   (2) **Young (1998): removal of the scale effect**

b. **continual emergence of new products =>**
   (1) **product ladder models:**
   - **Romer (1990): intermediate goods broadening**
   - **Aghion-Howitt (1992): ex ante perfect competition & ex post monopoly in R&D**
   - **Grossman-Helpman (1991): monopolistic competition with zero profit ex post**
   - **Laing-Palivos-Wang (2002): continual development of new product in the presence of search frictions with zero profit ex ante**

   (2) **endogenous basket models:**
   - **Stokey (1988, 1995): LBD and new goods**

D. A General Two-Sector Endogenous Growth Model: Bond-Wang-Yip (1996)

- **Why should we treat H as the main driving force?**
- **Why should we believe in monotone transition?**
- **Why should we focus on the primal instead of the dual?**

(i) Setup:

Notation: C, K, H, X, Y all in levels; c = C/H, k = K/H, x = X/H, y = Y/H;
$k_x = (sK)/(uH)$, $k_y = [(1-s)K]/[(1-u)H]$;
$p = p_x/p_y = \lambda/\mu$ (goods are numéraire)

max $$U = \int_0^\infty \frac{c^{1-\alpha} - 1}{1-\alpha} e^{-\rho t}\, dt$$

s.t. $$\dot{K} = xH - \delta K - C, \quad x = uf(k_x); \quad K(0) = K_0 > 0$$

$$\dot{H} = yH - \eta H, \quad y = (1-u)g(k_y); \quad H(0) = H_0 > 0$$

(ii) Key:

a. **General CRS technologies**
b. **Intertemporal no-arbitrage**
c. **Polarization Theorem**

(iii) Competitive BGP equilibrium:

a. **FOCs:** $c^{-\alpha} = \mu \implies \theta = -\frac{1}{\alpha}\frac{\dot{\mu}}{\mu}$ **(CC)**

$r = r_x(k_x) = p\,r_y(k_y);\ w = w_x(k_x) = p\,w_y(k_y)$ **(FPE)**

==> Stolper-Samuelson Theorem in endog. growth
==> $k_x(p)$, $k_y(p)$, $r(p)$, $w(p)$

Case 1: $k_x < k_y \implies \theta_w < 0 < \theta_p < \theta_r;\quad dk_i/dp < 0$

Case 2: $k_x > k_y \implies \theta_r < 0 < \theta_p < \theta_w;\quad dk_i/dp > 0$

b. Euler equations:

$$\theta_p = (r - \delta) - (\frac{w}{p} - \eta) \quad \text{(INA)}$$

Case 1: $k_x < k_y$

$$==> \quad dr/dp > 0; \; d(w/p)/dp < 0 \;\; ==> \; d\theta_p/dp > 0$$

Case 2: $k_x > k_y$

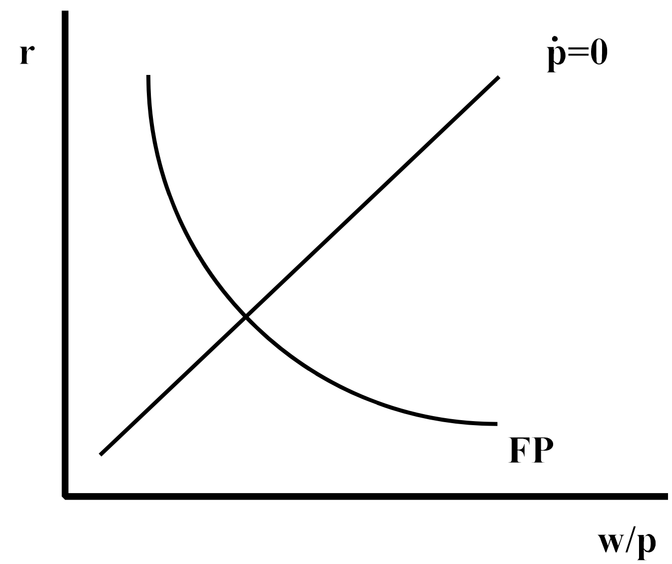
$$==> \quad dr/dp < 0; \; d(w/p)/dp > 0 \;\; ==> \; d\theta_p/dp < 0$$

c. Transversality conditions:

$$\lim_{t \to \infty} \mu K(t) e^{-\rho t} = 0$$

$$\lim_{t \to \infty} \lambda H(t) e^{-\rho t} = 0$$

(or the sum of the two limits goes to 0)



d. Factor evolution:

$$\frac{\dot{K}}{K} = \frac{u}{k}f(k_x(p)) - \delta - \frac{c}{k}$$

$$\frac{\dot{H}}{H} = (1-u)g(k_y(p)) - \eta$$

==> common growth ($\theta_C = \theta_K = \theta_H = \theta_X = \theta_Y = \theta$) and $\theta_p = 0$

e. Full employment:

$uk_x(p) + (1-u)k_y(p) = k$ **==>** $u = u(p,k)$, $du/dp < 0$; $du/dk < (>)\, 0$ **for 1 (2)**

Rybczynski Theorem in dynamic setting: ($dx/dp < 0$, $dy/dp > 0$)

Case 1: $k_x < k_y$

==> $\theta_x < 0 < \theta_k < \theta_y$; $dx/dk < 0$, $dy/dk > 0$

Case 2: $k_x > k_y$

==> $\theta_y < 0 < \theta_k < \theta_x$; $dx/dk > 0$, $dy/dk < 0$

f. Endogenous growth rate (Conditions U, G, FP):

$$\theta = \frac{1}{\alpha}[r(p) - (\rho + \delta)] = \theta(p) \quad \text{where } d\theta/dp > (<)\, 0 \text{ in Case 1 (2)}$$

(iv) Transitional Dynamics

$$\theta_p = (r(p) - \delta) - (\frac{w}{p}(p) - \eta)$$

$$\theta_c = \frac{1}{\alpha}[r(p) - (\rho + \delta)] - y(p,k) + \eta$$

$$\theta_k = \frac{1}{k}x(p,k) - y(p,k) - \frac{c}{k} + (\eta - \delta)$$

Key features:

a. **polarization between p and (c,k) ==> saddle-path stability**

b. **distortionary taxes such that price and physical factor intensity measures are reversed ==> instability or dynamic indeterminacy**

- 3x3 dynamical system ($c = \frac{C}{H}, k = \frac{K}{H}$):

$$\begin{aligned} \frac{\dot{p}}{p} &= (r-\delta) - (\frac{w}{p} - \eta) \\ \frac{\dot{c}}{c} &= \frac{1}{\alpha}(r(p) - (\rho+\delta)) - y(p,k)) + \eta \\ \frac{\dot{k}}{k} &= \frac{x(p,k)}{k} - y(p,k) - \frac{c}{k} - (\delta - \eta) \end{aligned}$$

- Jacobean:

$$\begin{bmatrix} \dot{p} \\ \dot{c} \\ \dot{k} \end{bmatrix} = \begin{bmatrix} p\theta_p & 0 & 0 \\ J_{21} & 0 & -cy_k \\ J_{31} & -1 & x_k - y_k - y - (\delta - \eta) \end{bmatrix} \begin{bmatrix} p - \bar{p} \\ c - \bar{c} \\ k - \bar{k} \end{bmatrix}$$

where $J_{11} = p\theta_p > (<)\ 0$ in case 1 (2), $J_{23} = -y_k < (>)\ 0$ in case 1 (2), $J_{33} = x_k - y_k - y - (\delta - \eta)$

- To sign J_{33}, note that:
  - in case 1, $x_k < 0$ and $y_k > 0$, so

$$J_{33} = x_k - y_k - \frac{x-c}{k} < 0$$

  - in case 2, $x_k > \frac{x}{k} > 0$ (Rybczynski) and $y_k < 0$, so

$$\begin{aligned} J_{33} &= x_k - y_k - \frac{x-c}{k} \\ &= \frac{c}{k} - y_k + \frac{x}{k}\left(\frac{k}{x}x_k - 1\right) > 0 \end{aligned}$$

- Define $J_{2\times 2} = \begin{bmatrix} 0 & -cy_k \\ -1 & x_k - y_k - y - (\delta - \eta) \end{bmatrix}$, so
  - $\mathrm{tr}(J_{2\times 2}) = J_{33} < (>) \ 0$ in case 1 (2)
  - $\det(J_{2\times 2}) = -cy_k < (>) \ 0$ in case 1 (2)
- So we have:
  - case 1: price dynamics unstable, quantity dynamics saddle
  - case 2: price dynamics stable, quantity dynamics unstable
- Thus, saddle-path (1 stable and 2 unstable roots) in both cases

- With distortionary taxes or sector-specific positive externalities, price dynamics may be reversed

	price	quantity	roots	prop
ben	1 unstable 2 stable	1 saddle 2 unstable	1(-),2(+)	saddle
gen 1	stable	saddle	2(-),1(+)	indet
gen 2	unstable	unstable	3(+)	source

- Thus distortionary taxes or sector-specific positive externalities can lead to the possibility of unstable source (3 unstable roots) and dynamic indeterminacy (2 stable roots).

(v) Extensions:

- a. **factor taxation**
- b. **dynamic Heckscher-Ohlin**
- c. **dynamic sector shifts & economic transition**
- d. **public vs. private sector**
- e. **formal vs. informal sector**
- f. **rural vs. urban**

IV. Development of the Literature

(i) Endogenous Technological Progress & Adoption

- **Rustichini-Schmitz (1991): imitation vs innovation**
- **Aghion-Howitt (1992): creative destruction**
- **Parente-Prescott (1994): adoption barriers**
- **Stokey (1995): R&D & endogenous basket of goods**
- **Jovanovic-Nyarko (1996): learning & technology choice**
- **Eicher-Turnovsky (1998): no-scale effect model**
- **Thesmar-Thoemig (2000): adoption & organization**
- **Laing-Palivos-Wang (2002): product diffusion**

- **Aghion (2002): general purpose technology (GPT)**
- **Chen-Mo-Wang (2002): matching & adoption**
- **Boucekkine-Licandro-Puch-del Rio (2004): vintage capital**
- **Boldrin-Levine (2004): competitive market for R&D**
- **Acemoglu-Aghion-Zilibotti (2006): distance to frontier technology**

(ii) Endogenous Human Capital Accumulation

- **Stokey (1991): human capital and product quality**
- **Tssidon (1992): moral hazard trap to growth**
- **Acemoglu (1996): microfoundation human capital externality**
- **Benobou (1996): stratification and growth**
- **Ferschtman-Murphy-Weiss (1996): social status and human capital**
- **Redding (1996): human capital and R&D**
- **Tamura (2001): teacher and growth**
- **LLoyd-Ellis-Bernhart (2002): managerial capital**
- **Laing-Palivos-Wang (2003): human capital vintage**
- **Manuelli-Seshadri (2006): human capital & productivity differences**
- **Ghatak-Morellib-Sjostromc (2007): occupational choice & trickle-up**
- **Jiang-Wang-Wu (2009): entrepreneurial capital**

(iii) Endogenous Fertility Choice

- **Becker-Murphy-Tamura (1990): human cap. & fertility**
- **Wang-Scotese-Yip (1994): fertility & growth cycles**
- **Palivos (1995): fertility & multiple growth paths**
- **Galor-Weil (1996): gender gap**
- **Galor-Weil (2000, 2005): demographic transition**
- **Greenwood-Seshadri (2005): technology & baby boom**
- **Mullin-Wang (2005): timing of childbearing**
- **Soares (2005): mortality & fertility**
- **Golosov-Jones-M. Tertilt (2006): efficiency in fertility choice**
- **Bara-Leukhina (2009): demographic transition & industrial evoluation**

(iv) Government Spending, Taxation and Growth

- **Lucas (1990): human capital accumulation & tax incidence**
- **Perotti (1993): distributive policy & growth**
- **Jones-Manuelli-Rossi (1993): dynamic tax incidence & endogenous growth**
- **Stokey-Rebelo (1994): flat-rate capital taxation & growth**
- **Bond-Wang-Yip (1996), Mino (1997), Milesi-Ferretti & Roubini (1998): factor taxation & growth**
- **Grossman-Helpman (1998): intergenerational redistribution**

- **Krusell & Rios-Rull (1999): government size & growth**
- **LLoyd-Ellis (2000): public education**
- **Werning (2007): nonlinear tax incidence**
- **Acemoglu (2009): political economy & tax incidence**
- **Chen-Chen-Wang (2009): labor-market frictions & dynamic tax incidence**

(v) Money, Inflation and Growth

- **Wang-Yip (1991): 3 approaches in a unified framework**
- **Wang-Yip (1992): money & endogenous growth**
- **Gomme (1993) & Ireland (1994): money & growth under cash-in-advance**
- **Van der Ploeg-Alogoskoufis (1994): money & growth under money-in-the-utility-function**
- **Jones-Manuelli (1995): growth effects of inflation**
- **Jha-Wang-Yip (2002), growth dynamics with money**
- **Chang-Chang-Lai-Wang (2007), growth dynamics with money & banking**
- **Wang-Xie (2009): labor-market frictions & welfare cost of inflation**

(vi) Financial Intermediation, Credit Creation and Growth

- **Greenwood-Jovanovic (1990): risk pooling and effective monitoring**
- **Bencivenga-Smith (1991): liquidity management**
- **Becsi-Wang-Wynne (1996): funds pooling and endogenous market structure**
- **Aghion-Bolton (1997): financial trickle down & growth**
- **Acemoglu-Zilibotti (1997): diversification & growth**
- **Levine-Zervos (1998): stock market & growth**
- **Aghion-Dewatripont-Rey (1999): financial discipline & growth**
- **Becsi-Wang-Wynne (1999): financial big push & big crash**
- **Chen-Chiang-Wang (2004): production approach to Schumpeterian growth & finance**
- **Hwang-Jiang-Wang (2007): financial collusion & equilibrium crash**
- **Jiang-Wang-Wu (2009): credit constraint & entrepreneur-led growth**

(vii) Trade, Growth and Development

- **Rivera-Batiz and Romer (1991): economic integration**
- **Ventura (1997): international interdependence**
- **Bond-Trask-Wang (2003): dynamic Heckscher-Ohlin**
- **Nishimura-Shimomura (2003): trade and indeterminacy**

- **Yi (2003): vertical specialization and growth**
- **Grossman (2004): allocation of talent and trade**
- **Farmer-Lahiri (2005): trade and indeterminacy**
- **Wan (2002): learning, trade and industrialization**
- **Bond-Jones-Wang (2004): learning from export**
- **Ghironi-Melitz (2005): trade & firm dynamics**
- **Peng-Thisse-Wang (2006): vertical integration & agglomeration-led growth**
- **Alvareza-Lucas (2007): trade & firm growth**
- **Grossman & Rossi-Hansberg (2008): task outsourcing & growth**
- **Riezman-Wang (2009): outsourcing & demand-led growth**

(viii)Dynamic Indeterminacy

- **Benhabib-Farmer (1994): one-sector, IRS**
- **Boldrin-Rustichini (1994): two-sector, IRS**
- **Benhabib-Perli (1994): two-sector, IRS**
- **Xie (1994): two-sector, IRS**
- **Bond, Wang and Yip (1996): distortionary taxes**
- **Benhabib-Meng-Nishimura (2000): two-sector, CRS with sector-specific externalities**

- **Mino (2001): two-sector, CRS with sectoral externalities**
- **Nishimura-Venditti (2002): two-sector, CRS with intersectoral externalities**
- **Nishimura-Shimomura-Wang (2005): multi-sector, CRS with sector-specific externalities**
- **Mino-Shimomura-Wang (2005): multi-sector, OG with endogenous occupational choice**
- **Mino-Nishimura-Shimomura-Wang (2007): two-sector, CRS with sectoral externalities in discrete time**

(ix) Income Distribution and Growth

- **Galor-Zeira (1993): income distribution in OLG**
- **Glomm-Rivikumar (1992): public vs. private education and inequality**
- **Galor-Moav (2000): ability-biased technical progress & wage inequality**
- **Krusell, Ohanian, Rios-Rull & Violante (2000): techical progress, relative price change & inequality**
- **Ghiglino-Sorger (2002): indeterminacy & wealth distribution**
- **Aghion (2002): between and within-the-skilled-group inequality**
- **Matsuyama (2002): inequality and industrial transformation**
- **Eeckhout-Jovanovic (2002): production knowledge spillovers & inequality**

- **Fender-Wang (2003): educational choice & inequality**
- **Galor-Moav (2004), inequality & development**
- **Laing-Palivos-Wang (2003): vintage & inequality**
- **Foellmi-Zweimuller (2006): inequality & demand-led innovation**
- **Huggett-Ventura-Yaron (2006): human capital & earnings distribution dynamics**
- **Manuelli-Seshadri (2007): human capital & wealth distribution**
- **Kambourov-Manovskii (2009): occupational shifts & inequality**

(x) Search, Matching, Unemployment and Growth

- **Aghion and Howitt (1994): unemployment**
- **Laing, Palivos and Wang (1995): learning, matching and unemployment**
- **Acemoglu (1997): training and innovation**
- **Huang-Laing-Wang (2004): unemployment & crime**
- **Chen-Chen-Wang (2009): growth effects of human capital policy with labor-market frictions**
- **Chen-Mo-Wang (2009): micro-matching and turnover**
- **Jovanovic (2009): micro-matching and technology cycle**

V. Concluding Remarks: Perspectives of Growth Theory

(i) From Macro to Micro: Hopenhayn (1992), Acemoglu (1996), Berliant-Peng-Wang (2002), Eeckhout-Jovanovic (2002), Klette-Kortum (2004), Ghironi-Melitz (2005), Berliant-Reed-Wang (2006), Atkeson-Kehoe (2007), Jovanovic (2009), Peng-Riezman-Wang (2009)

(ii) From Representative-Agent to Heterogenous-Agents: Glomm-Ravikumar (1992), Benobou (1996), Aghion-Bolton (1997), Grossman (2004), Mino-Shimomura-Wang (2005), Ghatak-Morellib-Sjostromc (2007), Manuelli-Seshadri (2006), Kambourov-Manovskii (2009), Chen-Chen-Wang (2009)

(iii) From Balanced Growth to Transitional Dynamics and Nonbalanced Growth: Kongsamut-Rebelo-Xie (2002), Matsuyama (2000, 2002), Hansen-Prescott (2002), Bond-Trask-Wang (2003), Bond-Jones-Wang (2004), Acemoglu-Guerrieri (2008), Chang-Wang-Xie (2009)

(iv) From Market Perfections to Imperfections: Grossman-Helpman (1991), Aghion-Howitt (1992, 1994), Laing-Palivos-Wang (1995, 2001, 2003), Eaton-Kortum (1999, 2002), Bernard-Eaton-Jensen-Kortum (2003), Melitz (2003),

Peng-Thisse-Wang (2006), Alvareza-Lucas (2007)

(v) From Cross-Country to Country Studies: Young (1992), Tallman-Wang (1994), Lee-Liu-Wang (1994), Loayza (1996), Hsieh (1999), Thanpapura-Wang (2002), Hsieh-Klenow (2007)

(vi) From Individual Incentives to Economic Institutions and Organizations: Banerjee-Newman (1998), Jovanovic (2000), Thesmer-Thoenig (2000), Castro-Clementi-McDonald (2004), Atkeson-Kehoe (2005), Acemoglu-Johnson-Robinson (2001, 2005, 2008), Samaniego (2006), Buera-Shin (2009), Riezman-Wang (2009)

Remarks:

- **The literature has largely diminished, becoming inessential over the past 2 decades**
- **The major self-destroying factor are the focus on technicalities without much deep insights and the lack of innovative ideas to handle heterogeneous agents and stochasticities with unbalanced sectoral, regional or national growth**
- **Moreover, a unified framework to model growth and cycles that can be put to data remains to be challenged**