Foundations of Dynamic Macroeconomic Analysis

Ping Wang, Washington University in St. Louis/NBER

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Introduction

This set of lecture notes provides foundations of macro analysis, focusing particularly on the following issues:

1. Fundamentals of Dynamic General Equilibrium
2. Optimal Growth in Discrete Time
3. Optimal Growth in Continuous Time

Basic references:

1. Acemoglu (2009): chs. 5-7
2. Aghion and Howitt (1998): chs. 1-2
3. Barro and Sala-i-Martin: ch. 2, secs. 4.1-4.3
4. Ljungqvist and Sargent (2000): chs. 3 and 11
5. Stokey and Lucas with Prescott (1989): chs. 3-5
6. Wang (2012), "Endogenous Growth Theory," Lecture Notes, Washington University-St. Louis.

Representative Household I

- Representative household is valid when the optimization of individual households can be represented as if there were a single household making the aggregate decisions using a representative preference subject to aggregate constraints.
- Consider a particular preferences representation:

$$\max \sum_{t=0}^{\infty} \beta^t u\left(c\left(t\right)\right)$$

- Let the excess demand of the economy be $\mathbf{x}(p)$.
- Key: whether this excess demand function $\mathbf{x}(p)$ can be obtained as a solution to the ***single*** household optimization problem.

Representative Household II

- Answer at the first glance: it cannot be so obtained in general as the weak axiom of revealed preferences for individuals need not hold for the aggregate.

Theorem H1 (Debreu-Mantel-Sonnenschein Theorem) Let $\varepsilon > 0$ and $N \in \mathbb{N}$. Consider a set of prices $\mathbf{P}_\varepsilon = \{p \in \mathbb{R}_+^N : p_j / p_{j'} \geq \varepsilon \text{ for all } j \text{ and } j'\}$ and any ***continuous*** function $\mathbf{x} : \mathbf{P}_\varepsilon \to \mathbb{R}_+^N$ that satisfies ***Walras's Law*** and is ***homogeneous of degree*** $\mathbf{0}$. Then there exists an exchange economy with N commodities and $H < \infty$ households, where the aggregate excess demand is given by $\mathbf{x}(p)$ over the set $\mathbf{P}_\varepsilon$.

- To yield a positive answer ensuring the excess demand function $\mathbf{x}(p)$ to be obtained as a solution to the single household optimization problem, we need to impose further restrictions, in particular, to ***remove strong income effects***.

Representative Household III

- A special but useful case for such a representation is to have ***linear*** value (indirect utility) function:

Theorem H2 (Gorman's Aggregation Theorem) Consider and economy with $N < \infty$ commodities and a set $\mathcal{H}$ of households. Suppose that the preferences of each household $h \in \mathcal{H}$ can be represented by an indirect utility function of the form

$$v^h\left(p, w^h\right) = a^h(p) + b(p)\, w^h$$

and that each household $h \in \mathcal{H}$ has a positive demand for each commodity. Then these preferences can be aggregated and represented by those of a representative household, with indirect utility

$$v(p, w) = a(p) + b(p)\, w,$$

where $a(p) \equiv \int_{h\in\mathcal{H}} a^h(p)\, dh$, and $w \equiv \int_{h\in\mathcal{H}} w^h dh$ is aggregate income.

Representative Household IV

- The class of preferences described in Theorem H-2 is referred to as "Gorman preferences" (1959 Econometrica).
- In this class, the Engel curve of each household for each commodity is linear and its slope is identical to all individuals for the same commodity.
- By Roy's Identity,

$$x_j^h\left(p, w^h\right) = -\frac{1}{b\left(p\right)}\frac{\partial a^h\left(p\right)}{\partial p_j} - \frac{1}{b\left(p\right)}\frac{\partial b\left(p\right)}{\partial p_j}w^h$$

Therefore, for each household, a linear relationship exists between demand and income and the slope, $-\frac{1}{b(p)}\frac{\partial b(p)}{\partial p_j}$, is ***independent*** of the household's identity (h).

Representative Household V

- Even under class of "Gorman preferences" a representative household exists, typical macro models require further restrictions on:
  - the abstract of distribution effects from the representative household's concern (***strong representation***);
  - the use of the representative household's preference as the welfare function of the aggregate economy (***normative representation***).

Representative Household VI

- Normative representation requires convexity, interiority and ***price-invariant basic value*** (o.w., one can transfer ε from low to high-valuation households for different p):

Theorem H3 (Normative Representation) Consider an economy with a finite number $N < \infty$ of commodities, a set $\mathcal{H}$ of households, and a ***convex*** aggregate production possibilities set Y. Suppose that the preferences of each household $h \in \mathcal{H}$ is represented by $v^h(p, w^h) = a^h(p) + b(p) w^h$ with $p = (p_1, ..., p_N)$ and that each household $h \in \mathcal{H}$ has a ***positive*** demand for each commodity.
1. Then any feasible allocation that maximizes the utility of the representative household $v(p, w) = \sum_{h \in \mathcal{H}} a^h(p) + b(p) w$, with $w \equiv \sum_{h \in \mathcal{H}} w^h$, is Pareto optimal.
2. If $a^h(p) = a^h$ for all p and all $h \in \mathcal{H}$ *(price-invariant basic value)*, then any Pareto optimal allocation maximizes the utility of the representative household.

Representative Firm I

- Representative firm is valid when the optimization of individual firms can be represented as if there were a single firm making the aggregate decisions using an aggregate production function subject to aggregate constraints.
- Consider price and output vectors: $p = (p_1, ..., p_N)$ and $y = (y_1, ..., y_N)$, with $p \cdot y = \sum_{j=1}^{N} p_j y_j$.
- Let $\mathcal{F}$ be the ***countable*** set of firms in the economy and

$$Y \equiv \left\{ \sum_{f \in \mathcal{F}} y^f : y^f \in Y^f \text{ for each } f \in \mathcal{F} \right\}$$

be the aggregate production possibility set (PPS).

Representative Firm II

- When there is ***no price-dependent fixed cost*** (ensured under perfect competition in the absence of externalities), we have:

Theorem F (Representative Firm Theorem) Consider a competitive production economy with $N \in \mathbb{N} \cup \{+\infty\}$ commodities and a countable set $\mathcal{F}$ of firms, each with a production possibilities set $Y^f \subset \mathbb{R}^N$. Let $p \in \mathbb{R}_+^N$ be the price vector in this economy and denote the set of profit-maximizing net supplies of firm $f \in \mathcal{F}$ by $\hat{Y}^f(p) \subset Y^f$ so that for any $\hat{y}^f \in \hat{Y}^f(p)$, we have $p \cdot \hat{y}^f \geq p \cdot y^f$ for all $y^f \in Y^f$. Then there exists a representative firm with production possibilities set $Y \subset \mathbb{R}^N$ and a set of profit-maximizing net supplies $\hat{Y}(p)$ such that for any $p \in \mathbb{R}_+^N$, $\hat{y} \in \hat{Y}(p)$ if and only if $\hat{y} = \sum_{f \in \mathcal{F}} \hat{y}^f$ for some $\hat{y}^f \in \hat{Y}^f(p)$ for each $f \in \mathcal{F}$.

Equilibrium

- An economy $\mathcal{E}$ is described by preferences, endowments, production sets, consumption sets, and allocation of shares, that is, $\mathcal{E} \equiv (\mathcal{H}, \mathcal{F}, \mathbf{U}, \boldsymbol{\omega}, \mathbf{Y}, \mathbf{X}, \boldsymbol{\theta})$.
- An allocation $(\mathbf{x}, \mathbf{y})$ in $\mathcal{E}$, $\mathbf{x} \in \mathbf{X}$, $\mathbf{y} \in \mathbf{Y}$, is feasible if $\sum_{h \in \mathcal{H}} x_j^h \leq \sum_{h \in \mathcal{H}} \omega_j^h + \sum_{f \in \mathcal{F}} y_j^f$ for all $j \in \mathbb{N}$.

Definition E (Competitive Equilibrium) A competitive equilibrium for economy $\mathcal{E} \equiv (\mathcal{H}, \mathcal{F}, \mathbf{U}, \boldsymbol{\omega}, \mathbf{Y}, \mathbf{X}, \boldsymbol{\theta})$ is given by a feasible allocation $\left(\mathbf{x}^* = \{x^{h*}\}_{h \in \mathcal{H}}, \mathbf{y}^* = \{y^{f*}\}_{f \in \mathcal{F}}\right)$ and a price system p^* such that
1. (Firm optimization) For every firm $f \in \mathcal{F}$, y^{f*} maximizes profits: $p^* \cdot y^{f*} \geq p^* \cdot y^f \quad$ for all $y^f \in Y^f$.
2. (Household optimization) For every household $h \in \mathcal{H}$, x^{h*} maximizes utility: $U^h\left(x^{h*}\right) \geq U^h\left(x^h\right)$ for all x such that $x^h \in X^h$ and $p^* \cdot x^h \leq p^* \left(\omega^h + \sum_{f \in \mathcal{F}} \theta_f^h y^f\right)$.

Pareto Optimum

- The standard concept of optimality is Pareto optimum, though social optimum is often used in macroeconomics.

Definition O (Pareto Optimum) A feasible allocation $(\mathbf{x}, \mathbf{y})$ for economy $\mathcal{E} \equiv (\mathcal{H}, \mathcal{F}, \mathbf{U}, \omega, \mathbf{Y}, \mathbf{X}, \boldsymbol{\theta})$ is Pareto optimal if there exists no other feasible allocation $(\mathbf{x}', \mathbf{y}')$ such that $x'^h \in X^h$ for all $h \in \mathcal{H}$, $y'^f \in Y^f$ for all $f \in \mathcal{F}$, and $U^h\left(x'^h\right) \geq U^h\left(x^h\right)$ for all $h \in \mathcal{H}$ with $U^{h'}\left(x'^{h'}\right) > U^{h'}\left(x^{h'}\right)$ for at least one $h' \in \mathcal{H}$.

Welfare Theorems I

- Household $h \in \mathcal{H}$ is ***locally nonsatiated*** if, at each $x^h \in X^h$, $U^h(x^h)$ is strictly increasing in at least one of its arguments and $U^h(x^h) < \infty$.

Theorem W1 (First Welfare Theorem I) Suppose that $(\mathbf{x}^*, \mathbf{y}^*, p^*)$ is a competitive equilibrium of economy $\mathcal{E} \equiv (\mathcal{H}, \mathcal{F}, \mathbf{U}, \omega, \mathbf{Y}, \mathbf{X}, \boldsymbol{\theta})$ with $\mathcal{H}$ finite. Assume continuous preferences with all households locally nonsatiated. Then $(\mathbf{x}^*, \mathbf{y}^*)$ is Pareto optimal (PO).

Theorem W2 (First Welfare Theorem II) Suppose that $(\mathbf{x}^*, \mathbf{y}^*, p^*)$ is a competitive equilibrium of economy $\mathcal{E} \equiv (\mathcal{H}, \mathcal{F}, \mathbf{U}, \omega, \mathbf{Y}, \mathbf{X}, \boldsymbol{\theta})$ with $\mathcal{H}$ countably infinite. Assume continuous preferences with all households locally nonsatiated and

$$p^* \cdot \omega^* \equiv \sum_{h \in \mathcal{H}} \sum_{j=0}^{\infty} p_j^* \omega_j^h < \infty.$$

Then $(\mathbf{x}^*, \mathbf{y}^*, p^*)$ is Pareto optimal.

Welfare Theorems II

Theorem W3 (Second Welfare Theorem) Consider a Pareto optimal allocation $(\mathbf{x}^*, \mathbf{y}^*)$ in an economy $\mathcal{E} \equiv (\mathcal{H}, \mathcal{F}, \mathbf{U}, \boldsymbol{\omega}^*, \mathbf{Y}, \mathbf{X}, \boldsymbol{\theta}^*)$. Suppose that all consumption sets are ***convex***, all production sets are ***convex cones***, all utility functions are ***continuous*** and ***quasi-concave***, and interiority conditions are met such that (i) there exists $\chi < \infty$ such that $\sum_{h \in \mathcal{H}} x^h_{j,t} < \chi$ for all j and t; (ii) $\underline{0} \in X^h$ for each h; (iii) for any h and $x^h, \bar{x}^h \in X^h$ such that $U^h(x^h) > U^h(\bar{x}^h)$, there exists $\bar{T}(h, x^h, \bar{x}^h)$ such that $U^h(x^h[T]) > U^h(\bar{x}^h)$ for all $T \geq \bar{T}(h, x^h, \bar{x}^h)$; and (iv) for any f and $y^f \in Y^f$, there exists $\tilde{T}(f, y^f)$ such that $y^f[T] \in Y^f$ for all $T \geq \tilde{T}(f, y^f)$. Then there exists a price vector p^* and endowment and share allocations $(\boldsymbol{\omega}^*, \boldsymbol{\theta}^*)$ with $\omega^* = \sum_{h \in \mathcal{H}} \omega^{h*}$, $\boldsymbol{\theta}^* = \sum_{f \in \mathcal{F}} \theta^{f*}$, such that:

1. for all $f \in \mathcal{F}$, $p^* \cdot y^{f*} \geq p^* \cdot y^f$ for any $y^f \in Y^f$;
2. for all $h \in \mathcal{H}$, if $U^h(x^h) > U^h(x^{h*})$ for some $x^h \in X^h$ then $p^* \cdot x^h \geq p^* \cdot w^{h*}$, where $w^{h*} \equiv \omega^{h*} + \sum_{f \in \mathcal{F}} \theta^{f*} y^{f*}$.

Welfare Theorems III

- The Second Welfare Theorem can be readily used to prove the existence of a competitive equilibrium based on Brouwer/Kakutani.

Theorem W3 (Existence of Competitive Equilibrium)
Consider an economy $\mathcal{E}$ as described in Theorem W3, if $p^* \cdot w^{h*} > 0$ for each $h \in \mathcal{H}$, then there exists a competitive equilibrium $(\mathbf{x}^*, \mathbf{y}^*, p^*)$.

Dynamic General Equilibrium I

- To generalize the standard Arrow-Debreu-McKenzei general equilibrium (GE) analysis to an infinite-horizon dynamic framework (DGE), one needs to impose further restrictions, particularly in the following aspects:
  - finite value: bounded valuation of households/firms
  - infinite dimensional space: Banach Space, Hilbert Space, Polish Space with weak or weak* topology rather than standard product topology
  - infinite dimensional fixed point: Schauder and others
  - interiority in infinite dimension
  - information and Arrow-Debreu trade.

Dynamic General Equilibrium II

- In the context of optimal growth, such DGE frameworks are given by
  - discrete time:

$$\begin{array}{ll} & \max_{\{c_t,k_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u\left(c_t\right) \\ \text{s.t.} & k_{t+1} = f\left(k_t\right) + \left(1-\delta\right) k_t - c_t \\ & c_t, k_t \geq 0, k_0 > 0 \end{array}$$

  - continuous time:

$$\begin{array}{ll} & \max_{\{c(t),k(t)\}_{t=0}^{\infty}} \int_0^{\infty} \exp\left(-\rho t\right) u\left(c\left(t\right)\right) dt \\ \text{s.t.} & \dot{k}\left(t\right) = f\left(k\left(t\right)\right) - c\left(t\right) - \delta k\left(t\right) \\ & c\left(t\right), k\left(t\right) \geq 0, k\left(0\right) > 0. \end{array}$$

Dynamic Programming I

- Discrete-time infinite-horizon optimization problem:

$$\begin{aligned} & \sup_{\{x_t, y_t\}_{t=0}^{\infty}} \textstyle\sum_{t=0}^{\infty} \beta^t \tilde{U}_t(t, x_t, y_t) \\ \text{s.t.} \quad & y_t \in \tilde{G}(t, x_t) \quad \forall t \geq 0 \\ & x_{t+1} = \tilde{f}(t, x_t, y_t) \quad \forall t \geq 0 \\ & x_0 > 0 \quad \text{given} \end{aligned}$$

- Eliminate y_t and rewrite the optimization problem:

$$\begin{aligned} V_0^*(x_0) = \quad & \sup_{\{x_t\}_{t=0}^{\infty}} \textstyle\sum_{t=0}^{\infty} \beta^t U_t(x_t, x_{t+1}) \\ \text{s.t.} \quad & x_{t+1} \in G_t(t, x_t) \quad \forall t \geq 0 \\ & x_0 > 0 \quad \text{given.} \end{aligned}$$

Dynamic Programming II

- Under stationarity, $U_t = U$ and $G_t = G$, yielding the following recursive problem:

$$\begin{aligned} V^*(x_0) = \quad & \sup_{\{x_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(x_t, x_{t+1}) \\ \text{s.t.} \quad & x_{t+1} \in G(x_t) \quad \forall t \geq 0 \\ & x_0 \quad \text{given} \end{aligned}$$

- The functional problem is (Bellman equation):

$$V(x) = \sup_{y \in G(x)} \{U(x, y) + \beta V(y)\} \quad \forall x \in X.$$

Dynamic Programming III

- Feasible set:
  $\Phi(x_t) = \{\{x_s\}_{s=t}^{\infty} : x_{s+1} \in G(x_s), s = t, t+1, ...\}$.
- Let X be a compact subset of R^K and
  $X_G = \{(x,y) \in X \times X : y \in G(x)\}$.
- Assumption A1:
  - (a) $G(x) \neq \emptyset$ for all $x \in X$;
  - (b) G is compact-valued and continuous;
  - (c) G is an increasing set s.t. $x \leq x' \Rightarrow G(x) \subset G(x')$.
- Assumption A2:
  - (a) $\forall\ x_0 \in X$ and $x \in \Phi(x_0)$,
    $\lim_{n\to\infty} \sum_{t=0}^{n} \beta^t U(x_t, x_{t+1}) = V_\infty < \infty$;
  - (b) $U : X_G \rightarrow R$ is continuous;
  - (c) $U(x,y)$ is strictly increasing in the first K elements for each $y \in X$;
  - (d) $U(x,y)$ is strictly concave in (x,y);
  - (e) U is continuously differentiable over intX_G.

Dynamic Programming IV

- Applying sandwich theorem, we can show:

Theorem DP1 (Equivalence of Values) Under A1(a) and A2(a), the solution to the recursive problem $V^*(x_0)$ is equivalent to the solution to the functional problem $V(x)$, i.e., $V^*(x) = V(x)$ for all $x \in X$.

- By recursive substitution, it is straightforward to obtain:

Theorem DP2 (Principle of Optimality) Under A1(a) and A2(a), consider a feasible plan $x^* \in \Phi(x_0)$ that attains $V^*(x_0)$ in the recursive problem. Then

$$V^*(x_t^*) = U(x_t^*, x_{t+1}^*) + \beta V^*(x_{t+1}^*)$$

for $t = 0, 1, ...$ with $x_0^* = x_0$. Moreover, if any $x^* \in \Phi(x_0)$ attains $V(x)$ in the functional problem, then it attains the optimal value in the recursive problem.

Dynamic Programming V

- With continuity as well as compactness of the constraint set, we can apply Beige's Theorem of Maximum and Weierstrass Theorem to obtain:

Theorem DP3 (Existence of Solutions) Under A1(a,b) and A2(a,b), there exists a unique continuous and bounded function $V : X \longrightarrow R$ that satisfies the Bellman equation. Moreover, for any $x_0 \in X$, an optimal plan $x^* \in \Phi(x_0)$ exists.

- With concavity and differentiability, we have:

Theorem DP4 (Differentiability of the Value Function) Under A1(a,b) and A2(a,b,d,e), consider an optimal plan x^* and define π as the policy function satisfying $x^*_{t+1} = \pi(x^*_t)$. Further assume that $x \in \text{int}X$ and $\pi(x) \in \text{int}G(x)$. Then $V(\cdot)$ is differentiable at x, with gradient $DV(x) = D_x U(x, \pi(x))$.

Dynamic Programming VI

- Let (s, d) be a norm space and $T : S \to S$ be an operator mapping S into itself. If for some $\beta \in (0, 1)$,

$$d(Tz_1, Tz_2) \leq \beta d(z_1, z_2) \quad \forall z_1, z_2 \in S$$

Then T is a contraction mapping (with modulus β).

Theorem DP5 (Contraction Mapping Theorem) Let (s, d) be a complete norm space and suppose that $T : S \to S$ is a contraction. Then T has a unique fixed point, $\hat{z}$, i.e., there exists a unique $\hat{z} \in S$ such that $T\hat{z} = \hat{z}$.

Dynamic Programming VII

- In practice, the following theorem provide sufficient conditions for a contraction map:

Theorem DP6 (Blackwell's Sufficient Conditions for a Contraction) Let $X \subseteq R^k$, and $B(X)$ be the space of bounded functions $f : X \to R$ defined on X equipped with the sup norm $||\cdot||$. Suppose that $B'(X) \subset B(X)$, and let $T : B'(X) \to B'(X)$ be an operator satisfying the following two conditions:
1. (Monotonicity) For any $f, g \in B'(X)$, $f(x) \leq g(x)\ \forall x \in X$, $(Tf)(x) \leq (Tg)(x)\ \forall x \in X$;
2. (Discounting) $\exists\ \beta \in (0,1)$ s.t. $[T(f+c)](x) \leq (Tf)(x) + \beta c$ $\forall f \in B(X)$, $c \geq 0$, and $x \in X$.
Then T is a contraction (with modulus β) on $B'(X)$.

Dynamic Programming VIII

- Applying Contraction Mapping Theorem to the value function V, we can establish its properties:

Theorem DP7 (Value Function Properties) Under A1(a,b,c) and A2(a,b,c,d), $V(x)$ in the Bellman equation is strictly increasing in all of its arguments and strictly concave.
Theorem DP8 (Necessity and Sufficiency) Under A1(a,b,c) and A2(a,b,c,d,e), a sequence $\{x_t^*\}_{t=0}^{\infty}$ such that $x_{t+1}^* \in \text{int}G(x_t^*)$, $t = 0, 1...,$ is optimal for the recursive problem given x_0 if and only if it satisfies the following:
1. (Euler Equations) $D_y U(x, \pi(x)) + \beta D_x U(\pi(x), \pi(\pi(x))) = 0$
2. (Transversality Conditions) $\lim_{t\to\infty} \beta^t D_x U(x^*, x_{t+1}^*) x_t^* = 0.$

Dynamic Programming IX

- In standard optimal growth models, A1(a,b,c) and A2(a,b,c,d,e) are all met.
- Example: consider $U = \ln(c_t)$ and $G = k_t^{\alpha} - c_t$, so the Bellman equation is: $V(x) = \max_{y \geq 0}\{\ln(x^{\alpha} - y) + \beta V(y)\}$
  - (Euler) $\frac{1}{x^{\alpha}-y} = \beta V'(y)$
  - (Benveniste-Scheinkman) $V'(x) = \frac{\alpha x^{\alpha-1}}{x^{\alpha}-y}$
  - (Policy function) $k_{t+1} = \alpha\beta k_t^{\alpha}$, $c_t = (1-\alpha\beta)k_t^{\alpha}$.

Optimal Control I

- Consider a canonical continuous-time optimization problem (P):

$$\begin{aligned} & \max_{\{\mathbf{x}(t),\mathbf{y}(t),t_1\}} W = \int_0^{t_1} f(t,\mathbf{x}(t),\mathbf{y}(t))dt \\ \text{s.t.} \quad & \dot{\mathbf{x}}(t) = G(t,\mathbf{x}(t),\mathbf{y}(t)) \ \ \forall t \\ & \mathbf{x}(t) \in \mathcal{X}(t), \ \mathbf{y}(t) \in \mathcal{Y}(t) \ \ \forall t; \quad \mathbf{x}(0) = \mathbf{x}_0 \end{aligned}$$

- Special case: finite time horizon $t_1 < \infty$ with one state x and one control y with $\dot{x}(t) = g(t,x(t),y(t))$ where $y(t)$ is continuous in t and hence $x(t)$ continuously differentiable so that variational arguments can be applied:
  - $f : [0,t_1] \times R \times R \to R$ and $g : [0,t_1] \times R \times R \to R$ are continuously differentiable;
  - $\mathcal{X}$ and $\mathcal{Y}$ are nonempty and convex;
  - $W(x(t),y(t)) < \infty$ for any admissible pair $(x(t),y(t))$;
  - $(\hat{x}(t),\hat{y}(t)) \in \text{int}\mathcal{X} \times \mathcal{Y}$ (never involving discontinuities) $\Rightarrow$ $W(\hat{x}(t),\hat{y}(t)) \geq W(x(t),y(t))$.

Optimal Control II

- Calculus of variation arguments:

$$y(t,\epsilon) = \hat{y}(t) + \epsilon\eta(t) \in \text{int}\mathcal{Y}, \forall t \in [0,t_1], \forall \epsilon \in [-\epsilon'_\eta, \epsilon'_\eta]$$

- Feasible path of state:

$$\dot{x}(t,\epsilon) = g(t,x(t,\epsilon),y(t,\epsilon)) \forall t \in [0,t_1] \text{ and } x(0,\epsilon) = x_0$$

- Optimization over ϵ for all variations:

$$\begin{aligned} W(\epsilon) &= \int_0^{t_1} [f(t,x(t,\epsilon),y(t,\epsilon)) + \lambda(t) g(t,x(t,\epsilon),y(t,\epsilon)) + \dot{\lambda}(t) x(t,\epsilon)] dt \\ &- \lambda(t_1) x(t_1,\epsilon) + \lambda(0) x_0 \end{aligned}$$

$$W'(0) = \int_0^{t_1} [f_x(t,\hat{x}(t),\hat{y}(t)) + \lambda(t) g_x(t,\hat{x}(t),\hat{y}(t)) + \dot{\lambda}(t)] x_\epsilon(t,0) dt$$

$$+ \int_0^{t_1} [f_y(t,\hat{x}(t),\hat{y}(t)) + \lambda(t) g_y(t,\hat{x}(t),\hat{y}(t))] \eta(t) dt - \lambda(t_1) x_\epsilon(t_1,0) = 0.$$

Optimal Control III

Theorem OC1 (Necessary Conditions) Consider the problem (P) with $t_1 < \infty$, one state-one control and f and g continuously differentiable. Suppose that (P) has an interior continuous solution $(\hat{x}(t), \hat{y}(t)) \in \text{int}\mathcal{X} \times \mathcal{Y}$. Then there exists a continuously differentiable costate function $\lambda(\cdot)$ defined on $t \in [0, t_1]$ such that:

(FOC)	$f_y(t, \hat{x}(t), \hat{y}(t)) + \lambda(t) g_y(t, \hat{x}(t), \hat{y}(t)) = 0$
(Euler)	$\dot{\lambda}(t) = -[f_x(t, x(t,\epsilon), y(t,\epsilon)) + \lambda(t) g_x(t, x(t,\epsilon), y(t,\epsilon))]$
(Terminal)	$\lambda(t_1) = 0$

- (Complementary slackness) $\lambda(t_1)(x(t_1) - x_1) = 0$ if $x(t_1) \geq x_1$, with $\lambda(t_1) > 0$ if $x(t_1) = x_1$.
- If t_1 is given (as is $x(t_1) = x_1$), then there is no need for $\lambda(t_1) = 0$.

Optimal Control IV

- Hamiltonian (which is continuously differentiable as well): $H(t, x(t), y(t), \lambda(t)) = f(t, x(t), y(t)) + \lambda(t) g(t, x(t), y(t))$.

Theorem OC2 (Maximum Principle) Consider (P) given in Theorem OC1. Suppose that (P) has an interior continuous solution $(\hat{x}(t), \hat{y}(t)) \in \text{int}\mathcal{X} \times \mathcal{Y}$. Then there exists a continuously differentiable function $\lambda(t)$ such that the optimal control $\hat{y}(t)$ and the corresponding path of the state variable $\hat{x}(t)$ satisfy the following necessary conditions (NC):

$$
\begin{array}{ll}
\text{(FOC)} & H_y(t, \hat{x}(t), \hat{y}(t), \lambda(t)) = 0, \forall t \in [0, t_1] \\
\text{(Euler)} & \dot{\lambda}(t) = -H_x(t, \hat{x}(t), \hat{y}(t), \lambda(t)), \forall t \in [0, t_1] \\
\text{(State Evolution)} & \dot{x}(t) = H_\lambda(t, \hat{x}(t), \hat{y}(t), \lambda(t)), \forall t \in [0, t_1]
\end{array}
$$

with $x(0) = x_0$ and $\lambda(t_1) = 0$. Moreover, $H(t, x, y, \lambda)$ also satisfies the Maximum Principle: $H(t, \hat{x}(t), \hat{y}(t), \lambda(t)) \geq H(t, \hat{x}(t), y, \lambda(t)), \forall y \in \mathcal{Y}, \forall t \in [0, t_1]$.

Optimal Control V

- For global (rather than local) maximum:

Theorem OC3 (Mangasarian Sufficiency Conditions) Suppose an interior continuous pair $(\hat{x}(t), \hat{y}(t)) \in \text{int}\mathcal{X} \times \mathcal{Y}$ exists and satisfies necessary conditions (NC). Suppose (i) $\mathcal{X} \times \mathcal{Y}$ is convex and (ii) given $\lambda(t)$, $H(t, x, y, \lambda)$ is jointly concave in $(x, y) \in \mathcal{X} \times \mathcal{Y}, \forall t \in [0, t_1]$. Then $(\hat{x}(t), \hat{y}(t))$ achieves the global maximum of (P) given in Theorem OC1. If H is strictly concave in $(x, y), \forall t \in [0, t_1]$, then such a solution is unique.
Theorem OC4 (Arrow-Kurz Sufficiency Conditions) Suppose $(\hat{x}(t), \hat{y}(t)) \in \text{int}\mathcal{X} \times \mathcal{Y}$ exists and satisfies (NC) and define $M(t, x(t), \lambda(t)) = \max_{y \in \mathcal{Y}} H(t, x(t), y, \lambda(t))$ as the Maximized Hamiltonian. Suppose (i) $\mathcal{X}$ is convex and (ii) $M(t, x, \lambda)$ is concave in $x \in \mathcal{X}, \forall t \in [0, t_1]$. Then $(\hat{x}(t), \hat{y}(t))$ achieves the global maximum of (P) given in Theorem OC1. If $M(t, x, \lambda)$ is strictly concave in $x, \forall t \in [0, t_1]$, then such a solution is unique.

Optimal Control VI

- We now generalize the theorems to the multivariate case:

Theorem OC5 (Maximum Principle) Consider (P) with $t_1 < \infty$ and f and g continuously differentiable. Suppose it has an interior continuous solution $(\widehat{\mathbf{x}}(t), \widehat{\mathbf{y}}(t)) \in \text{int}\mathcal{X} \times \mathcal{Y}$ and define $H(t, \mathbf{x}, \mathbf{y}, \boldsymbol{\lambda}) = f(t, \mathbf{x}(t), \mathbf{y}(t)) + \boldsymbol{\lambda}(t) \cdot G(t, x(t), y(t)), \lambda(t) \in R^{K_x}$.
Then $(\widehat{\mathbf{x}}(t), \widehat{\mathbf{y}}(t))$ satisfy the following necessary conditions:

(FOC)	$D_y H(t, \widehat{\mathbf{x}}(t), \widehat{\mathbf{y}}(t), \boldsymbol{\lambda}(t)) = 0, \forall t \in [0, t_1]$
(Euler)	$\dot{\boldsymbol{\lambda}}(t) = -D_X H(t, \widehat{\mathbf{x}}(t), \widehat{\mathbf{y}}(t), \boldsymbol{\lambda}(t)), \forall t \in [0, t_1]$
(State Evolution)	$\dot{\mathbf{x}}(t) = D_\lambda H(t, \widehat{\mathbf{x}}(t), \widehat{\mathbf{y}}(t), \boldsymbol{\lambda}(t)), \forall t \in [0, t_1]$
(Terminal)	$\boldsymbol{\lambda}(t_1) = 0$ ($\boldsymbol{\lambda}(t_1) > 0$ if t_1 given)

If $\mathcal{X}$ is convex and $M(t, \mathbf{x}, \boldsymbol{\lambda}) = \max_{\mathbf{y}(t) \in \mathcal{Y}} H(t, \mathbf{x}(t), \mathbf{y}, \boldsymbol{\lambda}(t))$ is concave in $\mathbf{x} \in \mathcal{X}, \forall t \in [0, t_1]$, then $(\widehat{\mathbf{x}}(t), \widehat{\mathbf{y}}(t))$ achieves the global maximum of (P) given above. If $M(t, \mathbf{x}, \boldsymbol{\lambda})$ is strictly concave in $\mathbf{x}, \forall t \in [0, t_1]$, then such a solution is unique.

Optimal Control VII

- We next consider the case of infinite horizon with one control and one state (P′):

$$\begin{aligned} & \max_{\{x(t),y(t)\}} W = \int_{t_0}^{\infty} f(t,x(t),y(t))dt \\ \text{s.t.} \quad & \dot{x(t)} = g(t,x(t),y(t)) \;\; \forall t \\ & x(t) \in \mathcal{X}(t),\; y(t) \in \mathcal{Y}(t) \;\; \forall t; \quad x(t_0) = x_0 \end{aligned}$$

- Problem (P′) can be rewritten in value function form:

$$\begin{aligned} V(t_0, x(t_0)) = \quad & \sup_{(x(t),y(t))\in\mathcal{X}(t)\times\mathcal{Y}(t)} \int_{t_0}^{\infty} f(t,x(t),y(t))dt \\ \text{s.t.} \quad & \dot{x(t)} = g(t,x(t),y(t)) \;\; \forall t \end{aligned}$$

- (Principal of Optimality): If $(\hat{x}(t), \hat{y}(t))$ solves (P′), then $\forall t \geq t_0$, (V'):

$$V(t_0, x(t_0)) = \int_{t_0}^{t} f(t', \hat{x}(t'), \hat{y}(t'))dt' + V(t, \hat{x}(t))$$

- This is analogous to the Bellman equation in dynamic programming.

Optimal Control VIII

Theorem OC6 (Maximum Principle) Consider (P′) with f and g continuously differentiable. Suppose it has an interior continuous solution $(\hat{x}(t), \hat{y}(t)) \in \text{int}\mathcal{X} \times \mathcal{Y}$ and define $H(t, x(t), y(t), \lambda(t)) = f(t, x(t), y(t)) + \lambda(t) g(t, x(t), y(t))$. Then:

$$H(t, \hat{x}(t), \hat{y}(t), \lambda(t)) \geq H(t, \hat{x}(t), y(t), \lambda(t)), \forall y(t) \in \mathcal{Y} \;\; \forall t$$

Moreover, the following necessary conditions (NC′) are met:

(FOC)	$H_y(t, \hat{x}(t), \hat{y}(t), \lambda(t)) = 0 \;\; \forall t$
(Euler)	$\dot{\lambda}(t) = -H_x(t, \hat{x}(t), \hat{y}(t), \lambda(t)) \;\; \forall t$
(State Evolution)	$\dot{x}(t) = H_\lambda(t, \hat{x}(t), \hat{y}(t), \lambda(t)) \;\; \forall t$

- (Hamilton-Jacobi-Bellman Equation) Differentiating the value function (V') using Leibniz's Rule yields:

$$f(t, \hat{x}(t), \hat{y}(t)) + \frac{\partial V(t, \hat{x}(t))}{\partial t} + \frac{\partial V(t, \hat{x}(t))}{\partial x} g(t, \hat{x}(t), \hat{y}(t)) = 0 \;\; \forall t.$$

Optimal Control IX

- Sufficiency conditions ensuring global maximum and uniqueness:
  - (i) $\mathcal{X}$ is a convex;
  - (ii) $M(t, x, \lambda)$ is strictly concave in $x \in \mathcal{X} \ \ \forall t$;
  - (iii) $\lim_{t\to\infty} \lambda(t)(\hat{x}(t) - \tilde{x}(t)) \leq 0 \ \ \forall \tilde{x}(t)$ associated with an admissible control path $\tilde{y}(t)$.
- (Transversality Condition, TVC) Consider (P′) given in Theorem OC5. Suppose $V(t, \hat{x}(t))$ is differentiable in x and t for t sufficiently large and that $\lim_{t\to\infty} \frac{\partial V(t,\hat{x}(t))}{\partial t} = 0$. Then the pair $(\hat{x}(t), \hat{y}(t))$ satisfies the necessary conditions (NC′) and the transversality condition $\lim_{t\to\infty} H(t, \hat{x}(t), \hat{y}(t), \lambda(t)) = 0$.
- The above TVC is weaker than $\lim_{t\to\infty} \lambda(t) = 0$, which is counterpart of $\lambda(t_1) = 0$ in finite horizon.

Optimal Control X

- With discounting in infinite horizon, we modify (P′) to (P″) with $W = \int_0^\infty f(t, x(t), y(t))e^{-\rho t}dt$.

Theorem OC7 (Maximum Principle) Consider (P″) with f and g continuously differentiable. Suppose it has an interior continuous solution $(\hat{x}(t), \hat{y}(t)) \in \text{int}\mathcal{X} \times \mathcal{Y}$ and define current-value Hamiltonian as $H(t, x(t), y(t), \lambda(t)) = f(t, x(t), y(t)) + \lambda(t)g(t, x(t), y(t))$. Then:

$$H(t, \hat{x}(t), \hat{y}(t), \lambda(t)) \geq H(t, \hat{x}(t), y(t), \lambda(t)), \forall y(t) \in \mathcal{Y} \;\; \forall t$$

Moreover, the following necessary and sufficiency conditions are met:

(FOC)	$H_y(t, \hat{x}(t), \hat{y}(t), \lambda(t)) = 0 \;\; \forall t$
(Euler)	$\dot{\lambda}(t) = \rho\lambda(t) - H_x(t, \hat{x}(t), \hat{y}(t), \lambda(t)) \;\; \forall t$
(State Evolution)	$\dot{x}(t) = H_\lambda(t, \hat{x}(t), \hat{y}(t), \lambda(t)) \;\; \forall t$
(TVC)	$\lim_{t \to \infty} H(t, \hat{x}(t), \hat{y}(t), \lambda(t))e^{-\rho t} = 0.$

Optimal Control XI

Corollary OC7 (Stronger form of TVC) Consider (P′′) given in Theorem OC7. Suppose in addition that
(i) f is weakly monotone in (x, y) and g is weakly monotone in (t, x, y),
(ii) $\exists m > 0$ such that $\left|g_y\left(t, x(t), y(t)\right)\right| \geq m \;\; \forall t$ and $\forall$ admissible $(x(t), y(t))$,
(iii) $\exists n < \infty$ such that $\left|f_y\left(x(t), y(t)\right)\right| \leq n \;\; \forall \left(x(t), y(t)\right)$.
Then, TVC can take a stronger form:

$$\lim_{t \to \infty} \lambda(t)\hat{x}(t)e^{-\rho t} = 0$$

- This stronger form can be readily generalized to the multivariate case (Kamihigashi 2001).

Optimal Control XII

$Q(t,x) = \{(p,z) \in R \times R^{K_x} | p \leq f(t,x,y), z = G(t,x,y), y \in \mathcal{Y}\}$.

Theorem OC8 (Existence of Solutions) Consider (P) with $t_1 \longrightarrow \infty$, satisfying:

1. $\mathcal{X}, \mathcal{Y} \neq \emptyset$, compact & closed-valued, and uhc;
2. f and G are continuous;
3. The sets of admissible $(\mathbf{x}(t), \mathbf{y}(t))$, $\Omega(0, x_0)$, and $Q(t,x)$ are nonempty for $x_0 \in \mathcal{X}$ and $\forall\ (t,x)$, and $Q(t,x)$ is closed & convex-valued, and uhc $\forall\ (t,x)$;
4. For any $[t_1, t_1{+}\delta]$ and any $\epsilon > 0$, $\exists$ continuous function $\Phi_{t_1\delta_\epsilon}(t)$ s.t. (i) for any $T \in [0, \infty]$, $\int_0^T \Phi_{t_1\delta_\epsilon}(t)dt \leq \Phi < \infty$ and (ii) $\|G(t,x,y)\| \leq \Phi_{t_1\delta_\epsilon}(t)$-$\epsilon f(t,x,y)$ $\forall t \in [t_1, t_1{+}\delta]$, $(x,y) \in \mathcal{X} \times \mathcal{Y}$;
5. $\exists$ a positive function $\phi(t)$ such that (i) $\int_0^\infty \phi(t)dt \leq \phi < \infty$ and (ii) $f(t,x,y) \leq \phi(t)$ $\forall t$ and $\forall (x,y) \in \mathcal{X} \times \mathcal{Y}$

Then, $\exists\ (\hat{x}(t), \hat{y}(t)) \in \Omega(0, x_0)$ that is a solution to (P) with $t_1 \longrightarrow \infty$ that $W(\hat{x}(t), \hat{y}(t)) = \bar{W} \leq W(x'(t), y'(t))$ for any $(x'(t), y'(t)) \in \Omega(0, x_0)$.

Optimal Control XIII

- In standard optimal growth models, the conditions specified in Theorem OC8 are all met.
- Example:

$$\max_{\{k(t),c(t)\}_{t=0}^{\infty}} \int_0^\infty u(c(t))e^{-\rho t}dt$$
$$\text{s.t.} \quad \dot{k}(t) = f(k(t)) - \delta k(t) - c(t)$$

  - (Current-value Hamiltonian)

$$H(k,c,\lambda) = u(c(t)) + \lambda(t)[f(k(t)) - \delta k(t) - c(t)]$$

  - (FOC)

$$u_c(c(t)) = \lambda(t)$$

  - (Euler)

$$\dot{\lambda}(t) = \rho\lambda(t) - \lambda(t)[f_k(k(t)) - \delta]$$

  - (TVC)

$$\lim_{t\to\infty} \lambda(t)k(t)e^{-\rho t} = 0.$$

Takeaways

- Aggregation:
  - Gorman + removal of strong income effects with Engle curve/iso-cost curve slopes uniform across households/firms
  - normative representation with price-invariant basic value/fixed cost
- Dynamic programming:
  - Equivalence between recursive and functional problems
  - Beige Theorem of Maximum
  - Blackwell sufficient conditions for Contraction Mapping Theorem
- Optimal control:
  - finite time calculus of variation + HJB $\Longrightarrow$ infinite time optimal control
  - Pontryagin Maximum Principle with current-value Hamiltonian and Arrow-Kurz Sufficiency + transversality.

Remarks I

- From deterministic (perfect foresight) to stochastic (rational expectations):
  - stochastic dynamic programming: straightforward extension in particular with finite dimensional state space for shocks
  - stochastic control: straightforward extension in particular with finite dimensional state space shocks
- From representative agent to heterogeneous agents:
  - heterogeneous households and inequalities
  - heterogeneous firms and firm distribution
  - aggregation validity and tractability (uniform, normal/log normal, gamma, Pareto and Fréhet).

Remarks II

- From infinite horizon-infinite lifetime to infinite horizon-finite lifetime overlapping generations (OLG)
- The prototypical setup is in discrete time (Allais 1947, Samuelson 1958), though there are continuous-time OLG with variational survival (Cass-Yarri 1966, Blanchard 1985) – generalized lifecycle models
- The simple 2-period lifetime consumption-loan structure with an initial old (born in time 0):

generation	1	2	3	...
born in 0	ω_o			
born in 1	ω_y	ω_o		
born in 2		ω_y	ω_o	
born in 3			ω_y	...

1. Classic economy ($\omega_o > \omega_y$): CE is PO
2. Samuelsonian economy ($\omega_y > \omega_o$): CE is suboptimal.

Remarks III

- While the OLG framework is clean, tractable and useful for modeling heterogeneous agents, there are several issues.
  1. Transfer from infinity problem (Gamow's Hotel): to remove Samuelsonian inefficiency, need discounting + Gaussian or minimum curvature (cf. Balasko-Shell 1980; Wang 1993)
  2. Lack of market clearing "at infinity" (i.e., Cantor infinity, requiring different dual as in Wang 1987)
  3. Generic market incompleteness in dated goods trading (no Arrow-Debreu securities; Bewley turnpike): 3-period lifetime may lead to market completeness (cf. Wang 1993)
  4. Large square economy with an infinity of agents and an infinite of dated goods (cf. Wilson 1981): problem of double infinity – single infinity is well defined, as in MasColell (Banach space) or Hildenbrandt (Hilbert space)
  5. Generic intergenerational externalities (via preferences or technologies)
  6. Concept of optimality: forward-looking PO and social-discounted social welfare maximum.