Income Distribution

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A. Introduction

- **Stylized facts (U.S. over the past 4 or 5 decades):**
  - **wage inequality increased sharply: 90%-10% ratio rose by over 40%, documented by Katz-Autor (1999)**
  - **despite an increase in skill premium/between-group inequality, the majority of the increase in wage inequality is *residual*, due to unobserved characteristics of workers in the same education and demographic group**
- **While the literature provides adequate explanation on the between-group inequality, it is largely failed in explaining the within-the-skilled-group inequality, with only a few attempts including, Aghion (2000), Violante (2002), Jovanovic (2009) and Tang and Wang (2014)**
- **Most of the existing studies focus on ex ante fixed innate ability, such as *Glomm-Ravikumar (1992)*, Acemoglu (1999), Caselli (1999), *Aghion (2000)*, Galor-Moav (2000), *Violante (2002)* which results in counterfactually high persistency in inequality (cf. Gottschalk-Moffitt 1994)**
- **Inequality is also associated with geographic stratification, particularly within municipals and to some degree across different regions**
  - ***Banabou (1996)* offers a simple framework for human capital stratification**
  - ***Acemoglu-Dell (2009)* provide useful decomposition of wage inequalities**

- **Between-firm wage inequality may be driven by firm productivity, firm-worker match quality (Bils-Kudlyak-Lins 2023), trade (Helpman-Itskhoki-Redding 2010), different labor supply *(Erosa-Fuster-Kambourov-Rogerson 2024*), and occupation spillover (*Gottlieb-Hémous-Hicks-Olsen 2023*), but within-job (industry-occupation pair) wage inequality due to job match quality, performance pay and endogenous sorting (Tang-Tang-Wang 2023)**
- **Piketty (2014) emphasizes a sharp rise in top inequality**
  - **historical data: Piketty (2014)**
  - **new data: tax administrative data (no top coding), wealth data**
  - **methodological issues: Krusell-Smith (2015), Weil (2015)**
- **Wealth inequality:**
  - **super stars: Jones-Kim (2014), Aghion-Akcigit-Bergeaud-Blundell-Hemous (2015), Gabaix-Lasry-Lions-Moll (2015)**
  - **asset risk and nonliear taxation: Benhabib-Bisin (2016), Kaymak-Poschke (2016), Lusardi-Michaud-Mitchell (2017)**
  - **financial knowledge: *Lusardiy-Michaudz-Mitchell (2017)***
  - **automation: *Moll-Rachel-Restrepo (2019)***
  - **health shocks: Wang-Wong-Yao (2020)**
  - **survey: *De Nardi (2015)***
- **Inequality and growth: Matsuyama (2002), Jovanovic (2009), *Oberfield (2023)***

B. Education Provision, Growth and Inequality: Glomm-Ravikumar (1992)

- **Different from the representative-agent framework developed by Lucas (1988), this paper allows for human capital heterogeneity, which enables a clean study of the issues of growth vs. distribution as well as private vs. public education**

1. The Model

- **2-period lived agents, who work when young and consume when old (endogenous labor-leisure trade-off, with altruism)**
- **Preferences:** $V_t = \ln n_t + \ln c_{t+1} + \ln e_{t+1}$, **that is , and agent of generation-t cares leisure, consumption and the offspring's quality of education**
- **Human Capital:**
  - **distribution:** $G_t(h) \sim \log \text{normal } (\mu_t, \sigma_t^2)$
  - **evolution:** $h_{t+1} = \theta h_t^{\delta}(1-n_t)^{\beta} e_t^{\gamma}$, $\delta, \beta, \gamma \in (0,1)$ **(Lucas:** $\gamma = 0, \delta = \beta = 1$**)**
- **CRS production: output** $= h_{t+1}$

- **Two educational system:**
  - **public education:** $E_{t+1} = \tau_{t+1} H_{t+1},\ \ H_{t+1} = \int h_{t+1} dG_{t+1}(h_{t+1})$
    (income tax) (mean income)
  - **private education:** $e_{t+1} = h_{t+1} - c_{t+1}$

2. Optimization and Equilibrium

a. Public Education:

- **Individual optimization:**

$$\max_{n,c} \ln n_t + \ln c_{t+1} + \ln E_{t+1}$$

s.t. $c_{t+1} = (1-\tau_{t+1}) h_{t+1}$

$$h_{t+1} = \theta (1 - n_t)^{\beta} E_t^{\gamma} h_t^{\delta}$$

$$\Rightarrow \max_{n_t} \ln n_t + \ln[(1-\tau_{t+1})\theta E_t^{\gamma} h_t^{\delta}] + \beta \ln(1-n_t) + \ln E_{t+1}$$

- **FOC:** $1 - n_t = \dfrac{\beta}{1+\beta}$

- **Government optimization:**

$$\max_{\tau} \ln[(1-\tau_{t+1})h_{t+1} + \ln \tau_{t+1} H_{t+1} \qquad (\because n_t = \frac{1}{1+\beta}\text{ fixed})$$

$$\Rightarrow \max_{\tau} \ln(1-\tau) + \ln \tau$$

- **FOC:** $\tau = 1/2$
- **Equilibrium:**
  - **human capital evolution:** $h_{t+1} = \theta(\frac{\beta}{1+\beta})^{\beta}(\frac{1}{2})^{\gamma} H_t^{\gamma} h_t^{\delta} \equiv \mathrm{A} H_t^{\gamma} h_t^{\delta}$
  - **aggregate human capital:** $H_t = \exp[\mu_t + \frac{\sigma_t^2}{2}]$
    - **mean:** $\mu_{t+1} = \ln A + \gamma \ln H_t + \delta \mu_t$, **or,** $\mu_{t+1} = \ln \mathrm{A} + (\gamma + \delta)\mu_t + \frac{\gamma \sigma_t^2}{2}$
    - **variance (inequality measure):** $\sigma_{t+1}^2 = \delta^2 \sigma_t^2$

b. Private Education

- **Individual optimization**

$$\max_{n_t, c_{t+1}, e_{t+1}} \ln n_t + \ln c_{t+1} + \ln e_{t+1}$$

$$\text{s.t.} \quad h_{t+1} = \theta(1-n_t)^{\beta} e_t^{\gamma} h_t^{\delta}$$

$$c_{t+1} = h_{t+1} - e_{t+1}$$

$$\Rightarrow \max_{n_t, e_{t+1}} \ln n_t + \ln[\theta(1-n_t)^{\beta} e_t^{\gamma} h_t^{\delta} - e_{t+1}] + \ln e_{t+1}$$

- **FOCs:** $c_{t+1} \equiv e_{t+1} = \frac{1}{2} h_{t+1}$; $1 - n_t = \frac{\beta}{\frac{1}{2} + \beta} > \frac{\beta}{1+\beta}$ **(free-rider in public education)**

- **Equilibrium:**
  - $h_{t+1} = \theta\left(\frac{\beta}{\frac{1}{2}+\beta}\right)^{\beta}\left(\frac{1}{2}\right)^{\gamma} h_t^{\gamma+\delta} \equiv \mathrm{B}\, h_t^{\gamma+\delta}$ **(B > A)**
  - $\mu_{t+1} = \ln \mathrm{B} + (\gamma + \delta)\mu_t$
  - $\sigma_{t+1} = (\gamma + \delta)^2 \sigma_t^{\ 2}$

3. Growth vs. Inequality

- **Inequality:**
  - **Public education: inequality$\downarrow$ over time**
  - **Private education: inequality may decline (or rise) over time if** $\delta + \gamma < (\text{or} >) 1$
- **Is inequality harmful for growth?**
  - **public education:** $H_{t+1} = A H_t^{\gamma+\delta} \exp[-\frac{1}{2}\delta(1-\delta)\sigma_t^2] \Rightarrow d(\frac{H_{t+1}}{H_t}) / d\sigma_t^2 < 0$
  - **private education:** $H_{t+1} = B H_t^{\gamma+\delta} \exp[\frac{1}{2}(\gamma+\delta)(\gamma+\delta-1)\sigma_t^2]$

    $\Rightarrow d(\frac{H_{t+1}}{H_t}) / d\sigma_t^2 < (\text{or} >) 0 \text{ if } \delta + \gamma < (\text{or} >) 1$
- **Kuznets curve: the correlation between growth and inequality is consistent with the Kuznets curve under private education**

4. Political Economy and Institutional Choice: Public vs. Private Education

- **Mechanism: majority voting by the old (political economy) – ignore n_t (decision by the young)**
- **Value functions:**
  - **Public education:** $V^{old}(\text{public}) = 2\ln(\frac{1}{2}) + \ln h + \mu + \frac{\sigma^2}{2}$
  - **Private education:** $V^{old}(\text{private}) = 2\ln(\frac{1}{2}) + 2\ln h$
- **Median voter's decision:**
  - $V^{old}(\text{public}) - V^{old}(\text{private}) = [\mu - \ln h(\text{median})] + \frac{\sigma^2}{2} \quad = \frac{\sigma^2}{2} > 0$

    (ex ante mean μ = median < ex post mean $= \mu + \frac{\sigma^2}{2}$, because log normal distribution has a long tail)
  - **outcome: select public education system (U.S. : 86%- public education)**
- **Problem: under public education, the declined income inequality is inconsistent with the real world observation**

C. General Purpose Technology and Between/Within-Group Inequality: Aghion (2000)

- **Stylized facts in U.S. & U.K: within-group inequality started before between-group inequality**
- **Equipment price and skill premium – Krusell et al. (2000 Econometrica):**

$$y_t = A_t\{K_S^{\alpha}[\mu u^{\sigma} + (1-\mu)(\lambda k_e^{\rho} + (1-\lambda)S_t^{\rho})^{\frac{\sigma}{\rho}}]^{\frac{1-\alpha}{\sigma}}$$

under $\frac{1}{1-\sigma} > \frac{1}{1-\rho}$ **(stronger complementarity between** k_e **and** S **),**

equipment price$\downarrow \Rightarrow \frac{W_s}{W_u}\uparrow$

1. Between-Group Inequality

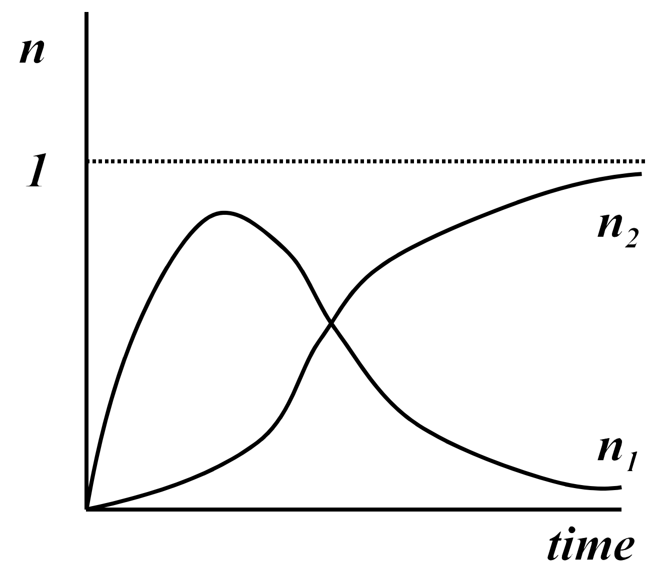
- **General purpose technology (GPT) experimentation and adoption require skilled labor**

- **Production:** $y = [\int_0^1 A(i)^\alpha x(i)^\alpha di]^{1/\alpha}$, $A(i) = \begin{cases} 1 & \text{if sector } i \text{ uses old GPT} \\ \gamma > 1 & \text{if sector } i \text{ uses new GPT} \end{cases}$
- **Skilled Labor:** $L_s(t) = L[1-(1-s)e^{-\beta t}]$
  - β **= speed of exogenous skill acquisition**
  - **1=** n_0 **(old GPT) +** n_1 **(experimenting new) +** n_2 **(new)**
- **Arrival of new GPT:**

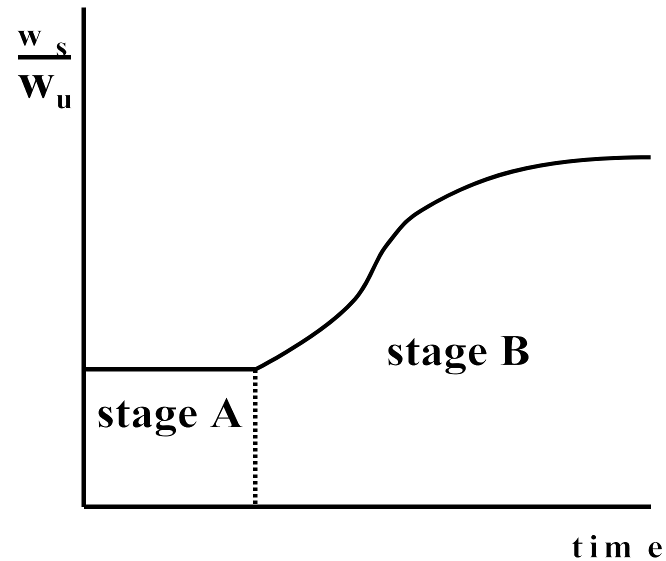
$$\lambda(n_2) = \begin{cases} \lambda_0 & \text{if } n_2 \le \bar{n} \\ \lambda_0 + \Delta & \text{if } n_2 \ge \bar{n} \end{cases}$$

where λ_0 **is small,** Δ **is large and** λ_1 **is the arrival of successful experimentation**

- **Population dynamics:**
  - $\dot{n}_1 = \lambda(n_2)n_0 - \lambda_1 n_1$
  - $\dot{n}_2 = \lambda_1 n_1$



- **Early stage (A):** $n_1 + n_2$ **is too small to absorb** $L_s \Rightarrow$ **integrated labor market with wage equalization, i.e.,**
  $(1-n_2)x_0 + n_1 L_1 + n_2 x_2 = L$
- **Later stage (B):** L_s **is fully absorbed by** n_1 **and** $n_2 \Rightarrow$ **segmented labor market with** $n_1 L_1 + n_2 x_2 = L_s$ **and** $(1-n_2)x_0 = L_u$



2. Within-Group Inequality

- **Machine lasts exactly two periods (with no depreciation within the two periods)**
- **Only a random fraction** (σ) **of workers get chance to adopt new GPT (crucial to create with-group heterogeneity)**
- **Continual adoption of new GPT yields higher productivity due to learning (at rate** τ**)**
- **By experience, learning of old GPT is more efficient (at rate** $\eta > \tau$**)**

- **Production**
  - **new GPT:** $y_t = A_t x_{ot}^{1-\alpha}$
  - **old GPT:** $z_t = A_{t-1}[(1+\eta)x_{1t}]^{1-\alpha}$
- **Technology evolution:** $A_t = (1+\gamma)A_{t-1}$
- **Labor and Population Identity:**
  - n_{ij} **(transition from** i **to** j **) with** i, j **= 0 (new) or 1 (old)**
  - $x_0 = (1+\tau)n_{00} + n_{10}$
  - $x_1 = n_{01} + n_{11}$
  - $n_{00} + n_{10} + n_{01} + n_{11} = 1$
- **Adaptability Constraints:** $\dot{n}_{00} \leq \sigma(n_{00} + n_{10})$ **and** $\dot{n}_{10} \leq \sigma(n_{01} + n_{11})$
- **Steady-State Transition:** $n_{10} = n_{01}$
- **Consumption Efficiency:** $u(c) = \sum \beta^t \ln c \Rightarrow 1 + r = \frac{1}{\beta}\frac{c_{t+1}}{c_t} = \frac{1}{\beta}(1+\gamma)$

- **Labor Demand:**
  - $$\frac{w_0}{w_1} = \frac{1+\gamma}{(1+\eta)^{1-\alpha}}\left(\frac{x_0}{x_1}\right)^{-\alpha}$$
  - $w_{00} = (1+\tau)w_0$; $w_{10} = w_0$; $w_{01} = w_{11} = w_1$

- **Labor Supply:**
  - **value functions:**
    - $v_{i0} = w_{i0} + \beta\{\sigma \max(v_{00}, v_1) + (1-\sigma)v_1\}$
    - $v_1 = w_1 + \beta\{\sigma \max(v_{10}, v_1) + (1-\sigma)v_1\}$
  - **cases:**
    - **when** $v_{10} < v_1$**, labor supply decision** $\Rightarrow x_0/x_1 = 0$
    - **when** $v_{10} > v_1$**, labor supply decision** $\Rightarrow x_0/x_1 = \chi$
    - **when** $v_{10} = v_1$ $(\frac{w_0}{w_1} = \Omega)$ $v_1 = w_1 + \beta\sigma v_1 + (1+\sigma)v_1$ $w_1 = \sigma(1-\beta)v_1$,

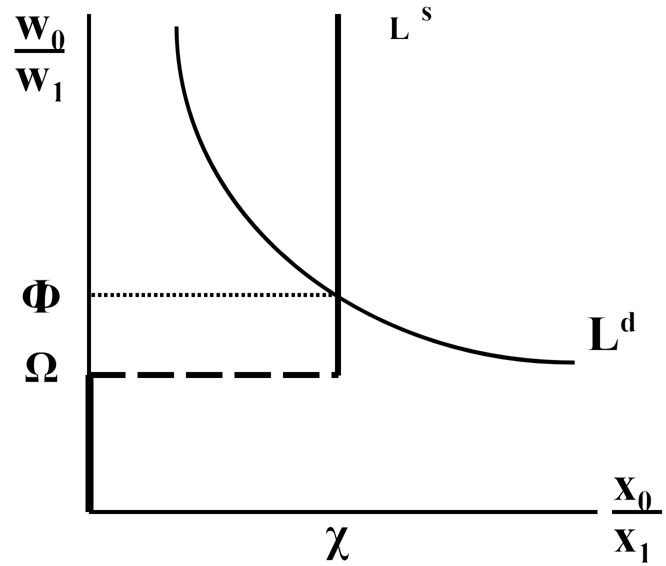
      $w_0 = \sigma[v_1 - \beta v_{00}]$, $w_{00} = (1-\beta\sigma)v_{00} - (1-\sigma)v_1$

- **Labor Market Equilibrium**

  $L^d = L^s \Rightarrow \quad \frac{w_0}{w_1} = \frac{1+\gamma}{(1+\eta)^{1-\alpha}}\left[\frac{1-\sigma}{\sigma(1+\sigma\tau)}\right]^{\alpha} \equiv \Phi(\underset{+}{\gamma}, \underset{-}{\sigma}, \underset{-}{\eta}, \underset{+}{\tau})$

- **Wage inequality within the skilled group:**

  - $= \max\ \{\frac{w_{00}}{w_0}, \frac{w_{00}}{w_1}\}$
  - $= \max\ \{\frac{w_{00}}{w_0}, \frac{w_{00}}{w_0}\frac{w_0}{w_1}\}$
  - $= (1+\tau)\quad \max\ \{1, \Phi\}$
  - **in general, within- group inequality rises when GPT size $(\gamma)\uparrow$, GPT learning $(\tau)\uparrow$, and monopoly rent $\uparrow(\sigma\downarrow$ or $\eta\downarrow)$**



- **Timing: evan at the early stage (A) when skill premium is zero, within-group inequality can arise already**
- **Problem: the underlying force driving within-group inequality is rather ad hoc**

D. Skill Transferability and Residual Wage Inequality: Violante (2002)

- **Stylized facts (US over the past 4 or 5 decades):**
  - **wage inequality increased sharply: 90%-10% ratio rose by over 40%, documented by Katz-Autor (1999)**
  - **despite an increase in skill premium/between-group inequality, the majority of the increase in wage inequality is *residual*, due to unobserved characteristics of workers in the same education and demographic group**
- **Previous studies on wage inequality focus on ex ante fixed innate ability**
  - **such as Acemoglu (1999), Caselli (1999), Aghion (2000), and Galor-Moav (2000)**
  - **counterfactually high persistency in inequality: Gottschalk-Moffitt (1994) find temporary components are as large as permanent ones**
- **Violante (2002) takes a deeper look at the data, finding that increased earning variability is due to:**
  - **more frequent job separation for a given turnover rate**
  - **more volatile dynamics of wages on the job and between jobs**
- **The above observations motivate the construction of a theory of inequality focusing on the accumulation and the transferability of specific human capital**
- **Key driving force: technology differences across machines of different vintages**

1. The Basic Structure and Results

- **Technology frontier advances at rate $\gamma > 0$**
- **Each machine has two periods of productive life and does not depreciate after the first period (as in Aghion 2000)**
- **A machine M_j of age j matched with worker of skill z produces output:**
  $y_j = (1+\gamma)^{-\theta_j} z$
- **Matching surplus sharing rule: ξ to worker and 1-ξ to firm**
- **Value functions:**
  - **value of employed:**
    - **with machine M_0:** $V_0 = w_0 + \beta \max\{V_1, U\}$
    - **with machine M_j:** $V_1 = w_1 + \beta U$
  - **value of unemployed:** $U = \alpha V_0 + (1-\alpha) V_1$

    where β = productivity-adjusted discount factor

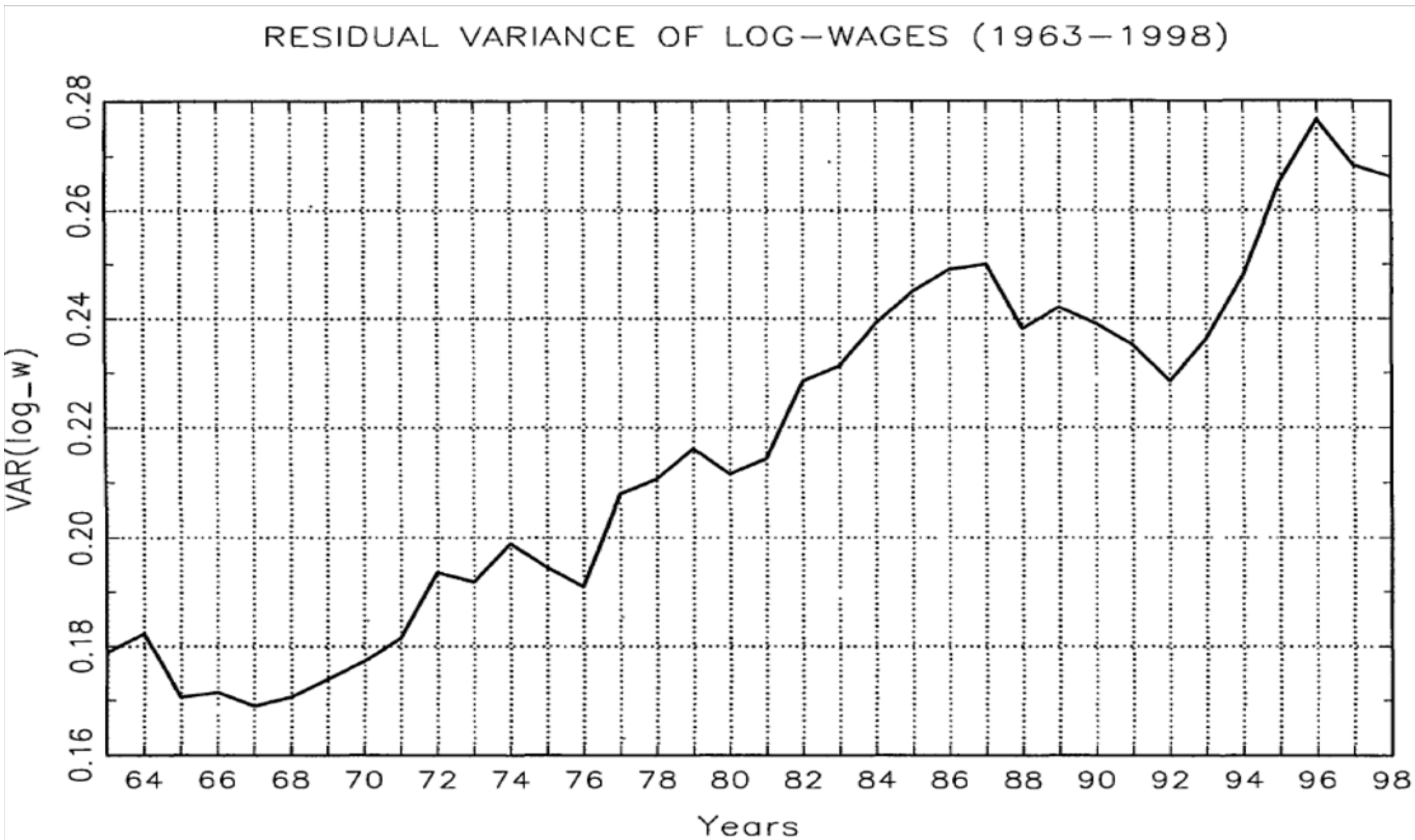
    α = probability of meeting a new machine
- **Separation decision for workers on new technologies: $\chi = \{0,1\}$**
  - **by construction, $w_0 > w_1$; thus, $U > V_1$**
  - **so if $\chi = 1$, we must have equal fractions of idle M_0 and M_1, i.e., $\alpha = 1/2$**
- **Wage inequality var(ln(w)) = $[(\theta \ln(1+\gamma)/2]^2 \simeq [(\theta\gamma)/2]^2$, depending exclusively on the technology differences across machines of different vintages (γ)**

2. Generalization: Vintage Human Capital

- **A worker on M_j may move on $M_{j'}$ with cumulated skills determined by the transferability process: $z_{jj'} = (1+\gamma)^{\tau[j'-(j+1)]}$ (following the adaptation structure in Aghion 2000)**
  - **the transferability of specific human capital is measured by τ**
  - **equilibrium skill levels:**
    - $z_{01} = 1$
    - $z_{00} = z_{11} = (1+\gamma)^{-\tau}$
    - $z_{10} = (1+\gamma)^{-2\tau}$
- **Productivity-adjusted wage: $w_{ij} = (1+\gamma)^{-\theta j}$**
- **Value functions: change to V_{ij} based on w_{ij}**
- **Worker's separation decision:**
  - $\tau \leq \theta \Rightarrow \chi = 1$ **for all** γ
  - $\tau > \theta \Rightarrow \chi = 1$ **for** $\gamma > \gamma_c$
- **Wage inequality: $var(\ln(w)) \simeq (\theta\gamma)^2\, var(j) + var(\ln(z)) - 2\theta\gamma\, cov(\ln(z),j)$**
  - **higher γ increases $var(\ln(z))$ and $cov(\ln(z),j)$, raising $var(\ln(w))$ if $\chi = 0$**
  - **the effect of γ on $var(\ln(w))$ is ambiguous if $\chi = 1$**

3. Calibration

- **Observation: residual wage inequality**



- **Parameterization**

Parameters	Moment to match (yearly average)	Source
$\gamma_L = .036$	growth of rel. price of equipment (< 1974)	Krusell et al. [2000]
$\gamma_H = .048$	growth of rel. price of equipment (> 1974)	Krusell et al. [2000]
$\theta = .7$	growth of real average wage = .024	Murphy and Welch [1992]
$\beta = .964$	rate of return on capital = .05	Cooley [1995]
$\kappa = 5$	labor share = .68	Cooley [1995]
$J = 28$	average age of equipment = 7.7	Bureau of Economic Analysis [1994]
$\lambda = .345$	wage growth within job = .03	Topel [1991]
$\tau = 1.90$	wage loss upon layoff = .23	Jacobson et al. [1993], Topel [1991]
$Z = 20$	transitory residual wage variance = .053	CPS data, Gottschalk and Moffitt [1994]
$\delta = .05$	separation rate from employment = .166	Blanchard and Diamond [1990]

- **Fitness of the Model**

	Variance of log wages		Variance of technologies	Variance of skills	Covariance component
	DATA	MODEL			
$\gamma_L = .035$	.053	.053	.008	.085	−.038
$\gamma_H = .048$	.089	.085	.014	.145	−.074
	Average age of capital	Average skill level	Wage growth within-job	Wage loss upon layoff	Separation rate
$\gamma_L = .035$	7.700	11.086	.030	−.230	.166
$\gamma_H = .048$	7.448	8.595	.044	−.305	.171

4. Open Issues

- **firm-specific technologies**
- **occupational mobility**
- **general vs. specific human capital**

E. Human Capital Stratification

- **In reality, households are stratified in various degrees by race, income, education and other socioeconomic indicators**
- **The Dissimilarity index (Duncan-Duncan 1955): using the 2000 Census data, Peng and Wang (2005) show highly stratified top 30 MSAs in the US:**

Metropolitan Statistical Area (MSA)	Dissimilarity Index
DC-Baltimore, Detroit	0.70 or higher
Milwaukee, Cleveland, St. Louis, New York	0.60 - 0.69
Philadelphia, Cincinnati, Chicago, Indianapolis	
Pittsburgh, Atlanta, Kansas City	0.50 - 0.59
Houston, Boston, Los Angeles	
Tampa, San Antonio, Phoenix, Minneapolis	0.40 - 0.49
San Diego, Norfolk, San Francisco	
Miami, Denver, Sacramento, Orlando	
Dallas, Seattle, Portland	0.39 or lower

- **It has been shown that since 1980, racial segregation in the U.S. has declined while economic segregation has risen**
- **Human capital and housing are believed the two primary sources of economic segregation**

1. The Model: Benobou (1996)

- **Interactions**
  - **Local positive spillovers – in human capital evolution**
  - **Global positive spillovers – in goods production**
- **Human Capital and Education**
  - **human capital evolution:** $h_{t+1}^{i} = \phi^{i}((1-u_{t}^{i})h_{t}^{i})^{\delta}(E_{t}^{i})^{1-\delta}$
  - **public education:** $E_{t}^{i} = \tau_{t}^{i}\int y_{t}^{i} dG_{t}^{i}(y_{t}^{i})$
- **Output:** $y_{t+1}^{i} = \mathrm{A}(H_{t})^{\alpha}(h_{t}^{i})^{1-\alpha}$
- **Combining the above relationships** $\Rightarrow h_{t+1}^{i} = \mathrm{B}^{i}(h_{t}^{i})^{\delta}(H_{t})^{\alpha(1-\delta)}(L_{t}^{i})^{(1-\alpha)(1-\delta)}$**, where** L^{i} **is a "local" human capital aggregator**

2. Segregated vs. Integrated Equilibrium

- **Segregated equilibrium features locational clustering by human capital/income**
- **Integrated equilibrium features mixture of groups with different human capital/income**
- **Two fundamental forces:**
  - **complementarity between L^i and h^i => segregation (assortative matching)**
  - **complementarity between H and h^i => integration (homogenizing)**

3. Results

- **Co-existence of segregated and integrated equilibria**
- **Integration lowers inequality as compared to segregation**
- **Integration lowers growth in SR but raises it in LR, because H has a larger scale effect in the long run**

F. Income Inequality Across Space and Time: Acemoglu-Dell (2009)

- **Stylized fact: large cross-country and within-country differences in per capita income**
- **Potential causes of such disparities:**
  - **differences in *human capital***
  - **differences in technological know-how**
  - **differences in production efficiency due to various institutions and organizations**

1. The Model

- **Measure of inequality (municipal m in country j) by the Theil index:**

$$T = \sum_{j=1}^{J} \frac{L_j}{L} \frac{y_j}{y} \left(\frac{\ln y_j}{y} \right) + \sum_{j=1}^{J} \frac{L_j}{L} \frac{y_j}{y} \left[\sum_{m=1}^{M_j} \frac{L_{jm}}{L_j} \frac{y_{jm}}{y_j} T_{jm} + \sum_{m=1}^{M_j} \frac{L_{jm}}{L_j} \frac{y_{jm}}{y_j} \ln \left(\frac{y_{jm}}{y_j} \right) \right]$$

where $T_{jm} = \sum_{i=1}^{L_{jm}} \frac{y_{jmi}}{L_{jm} y_{jm}} \ln \left(\frac{y_{jmi}}{y_{jm}} \right)$ **is the within-municipal m Theil index in country j**

- **Alternative measures: mean log deviation, variance/coefficient of variation, gini coefficient**

- **Wage inequality**

	90/10	Theil index	
		Between Country	Within Country
Municipals			
actual pop weights	34.2	0.250	*0.544*
equal pop weights	28.6	0.285	*0.622*
Regions			
actual pop weights	36.7	0.203	0.529
equal pop weights	32.7	0.139	0.615

  - **more *within* than between country inequalities**
  - **more inequality using *municipal* than region data**

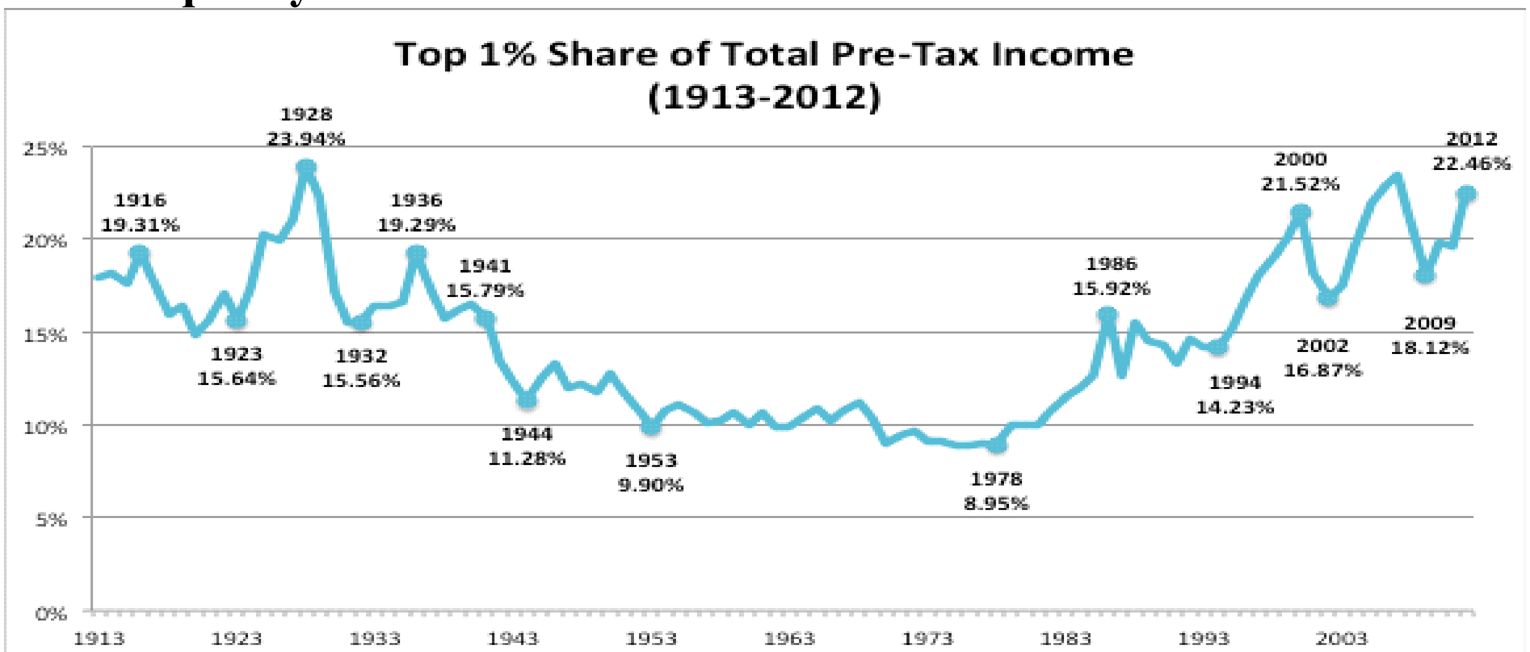
- **Decomposition of wage inequality measured by Theil index**

	Overall Inequality			**Residual Inequality**		
	Between Country	**Between Munic.**	**Within Munic.**	**Between Country**	**Between Munic.**	**Within Munic.**
Municipals						
actual pop weights	***0.265***	**0.067**	**0.424**	**0.033**	**0.040**	***0.389***
equal pop weights	***0.301***	**0.105**	**0.474**	**0.041**	**0.053**	***0.404***
U.S.		**0.050**	**0.365**		**0.020**	***0.291***

- **"residual" *within-the-skilled-group* inequalities account for a large portion of overall inequalities**
- ***within-municipal* disparities are most important for wage inequalities**
- **between-country disparities are important only for "non-residual" *between-skilled-and-unskilled-group* inequalities**
- **between-municipal disparities are never important**

G. The Battle between the Top 1% and the Remaining 99%: Pikety (2014)

- **Income inequality**



- **Wealth inequality**
  - **U.S. Wealth Inequality:** https://www.youtube.com/watch?v=QPKKQnijnsM
- **Capital In The 21st Century:**
  - **BBC:** **https://www.youtube.com/watch?v=HL-YUTFqtuI**
  - **ABC:** **https://www.youtube.com/watch?v=I05wLUuvQGM**

- **Methodological issues:**

  - **Piketty: r measures return to capital, g measures return to labor, so r > g implies widened inequality**
  - **Krusell-Smith (2015): Piketty's r > g theory works only with the unconventional definition of capital-output in terms of net capital (net of depreciation) and NNP**
  - **Weil (2015): market value of tradeable assets are incomplete measures for productive capital and wealth, missing**
    - **value of human capital**
    - **transfer wealth**
    - **these omitted types of wealth are distributed more equally than tradeable assets**

H. Wealth Inequality: De Nardi (2015)

- **Cagetti-De Nardi (2006): over the past 3 decades in the U.S., top 1% own 1/3 of national wealth, top 5% more than 1/2 (see also an older literature led by Wolff 1992, 1998)**
- **Can typical models predict such a high concentration of wealth?**

1. The Bewley (1977) Model of Permanent Income

- **Infinitely lived agents with time-additive preferences:** $E\left\{\sum_{t=1}^{\infty} \beta^t u(c_t)\right\}$
  - **u takes a CRRA form**
- **Labor endowment subject to an idiosyncratic labor productivity shock z, taking finite number of values and following a first-order Markov process with transition matrix Γ(z)**
- **A single asset *a* that may be used to insure against labor income risk**
- **Production of a single good Y using K and L under a CRS technology**

- **Household's problem:**

$$V(x) = \max_{(c,a')} \left\{ u(c) + \beta E\left[V(a', z') | x \right] \right\}$$

s.t.

$$c + a' = (1+r)a + zw$$

$$c \geq 0, \quad a' \geq \underline{a},$$

  - $\underline{a}$ **= net borrowing limit**
  - **state x = (*a*, z)**
- **In a stationary equilibrium, the distribution of people with (*a*, z) is constant**
- **Quantitative analysis by Aiyagari (1994): log(labor earning) follows AR(1) with autocorrelation = 0.6 and std dev of the innovations = 0.2**

	% wealth in top		
Gini	1%	5%	20%
U.S. data, 1989 SCF			
.78	29	53	80
Aiyagari Baseline			
.38	3.2	12.2	41.0

  - **wealth inequality largely underestimated compared to the 1989 Survey of Consumer Finance (not much improved even doubling std dev)**

2. A Overlapping-Generations Bewley Model with Survival Risk: Huggett (1996)

- **Agents live for at most N periods, subject to survival probability s_t of surviving up to t conditional on surviving at t-1**
- **Lifetime utility:** $E\left\{\sum_{t=1}^{N} \beta^t \left(\Pi_{j=1}^{t} s_t\right) u(c_t)\right\}$
- **Labor endowment is now age-specific: e(z, t)**
  - **again, z is Markov with transition Γ(z)**
- **No annuity, so people self-insure against earning risk and long life**
- **Those die prematurely leave accidental bequests**
- **Same production technology as in Bewley**
- **Household's problem:**

$$V(a,z,t) = \max_{(c,a')}\left\{u(c) + \beta s_{t+1} E\left[v(a',z',t+1)|z\right]\right\}$$

s.t.

$$c + a' = (1+r)a + e(z,t)w + T + b_t$$

$$c \geq 0, \quad a' \geq \underline{a} \quad and \quad a' \geq 0 \quad if \quad t = N$$

  - **T = lump-sum redistributed accidental bequests**
  - **b = social security payments to the retired**

- **Stationary equilibrium: similar to Bewley, with periodically balanced bequest transfers and government budget**
- **Quantitative results:**

Transfer wealth ratio	Wealth Gini	Percentage wealth in the top 1%	5%	20%	40%	60%	Percentage with negative or zero wealth
1989 U.S. data							
.60	.78	29	53	80	93	98	5.8–15.0
A basic overlapping-generations Bewley model							
.67	.67	7	27	69	90	98	17

  - **improved, but still far off for the top 1 or 5% wealth distribution**

3. Wealth Distribution in Variations of the Bewley Model

- **Benhabib-Bisin (2015): with intergenerational transmission and redistributive fiscal policy, the stationary wealth distribution is Pareto, driven critically by capital income and estate taxes**
- **Benhabib-Bisin-Zhu (2016): capital income shocks more important than labor income shocks**

4. Human Capital Transmission and Voluntary Bequests: De Nardi (2004)

- **Household's value:**

$$V(a,t) = \max_{c,a'}\left\{u(c) + s_t \beta E_t V(a', t+1) + (1-s_t)\phi(b(a'))\right\}$$

  - **value from leaving bequest by providing a worm glow (enjoyment of giving a la Andreoni (1989):**

$$\phi(b(a')) = \phi_1\left(1 + \frac{b(a')}{\phi_2}\right)^{1-\sigma}$$

  - **overall bequest motive: φ_1**
  - **bequest luxuriousness φ_2**
- **Two intergenerational linages:**
  - **human capital: inheritance in labor productivity**
  - **bequests**

- **Quantitative results**

Transfer wealth ratio	Wealth Gini	Percentage wealth in the top 1%	5%	20%	40%	60%	Percentage with negative or zero wealth
1989 U.S. data							
.60	.78	29	53	80	93	98	5.8–15.0
No intergenerational links, equal bequests to all							
.67	.67	7	27	69	90	98	17
No intergenerational links, unequal bequests to children							
.38	.68	7	27	69	91	99	17
One link: parent's bequest motive							
.55	.74	14	37	76	95	100	19
Both links: parent's bequest motive and productivity inheritance							
.60	.76	18	42	79	95	100	19

  - **unequal bequests do not matter**
  - **both intergenerational links matter to top group wealth distribution**

4. Entrepreneurship: Cagetti-De Nardi (2004)

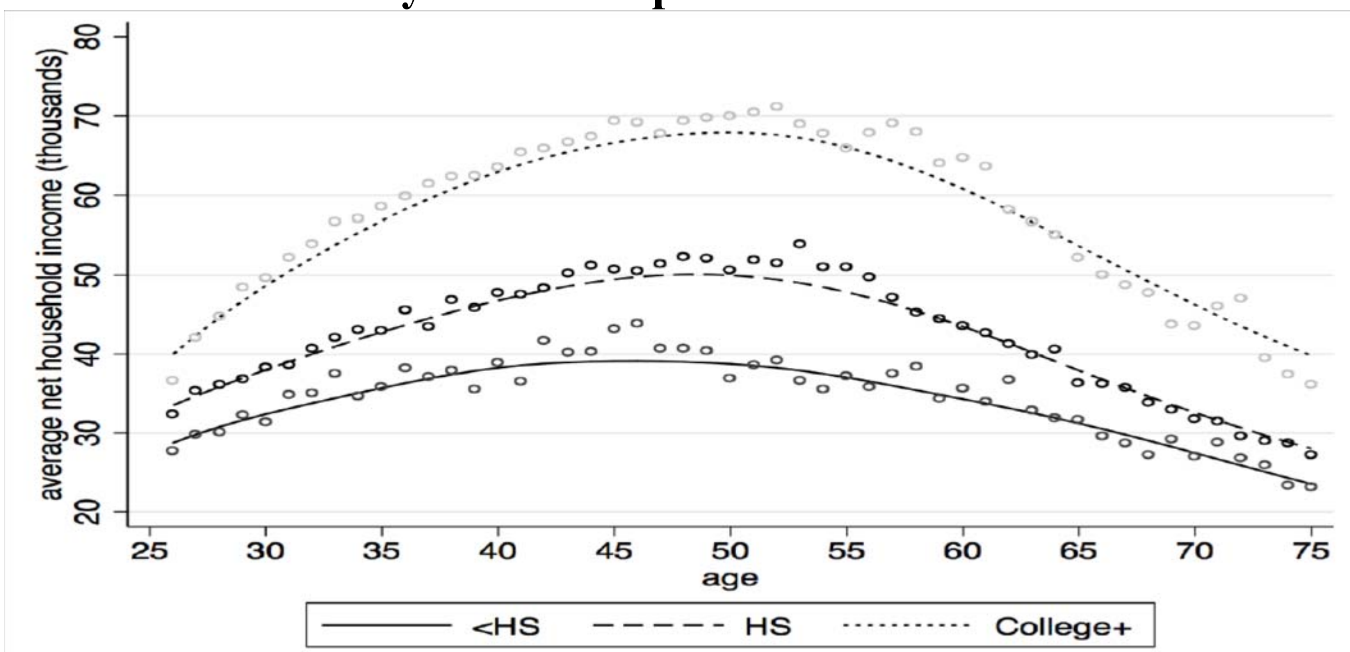
- **Agents are altruistic and face uncertainty about death time**
- **Occupational choice: workers vs. entrepreneurs**
  - **entrepreneurial production with working capital k and ability θ:** $f(k) = \theta k^{\nu} + (1-\delta)k$
  - **working capital subject to borrowing constraints, so** $k = a + b(a)$**, with borrowing b depending on asset collateral *a***
- **Quantitative findings:**

Wealth Gini	Fraction of entrepreneurs	Percentage wealth in the top 1%	5%	20%	40%
Data					
0.78	**10%**	**29**	**53**	**80**	**93**
Baseline model with entrepreneurs					
0.8	7.50%	31	60	83	94

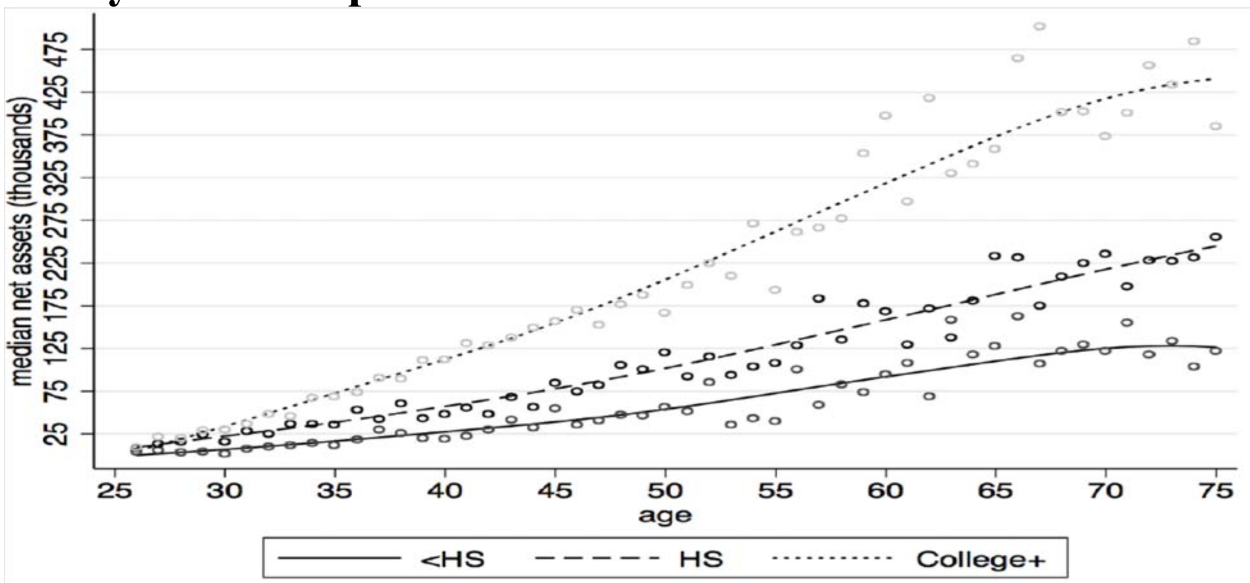
  - **over-estimation in top 5% wealth share especially under a smaller share of entrepreneurs**

I. Financial Knowledge and Wealth Inequalities: Lusardiy-Michaudz-Mitchell (2017)

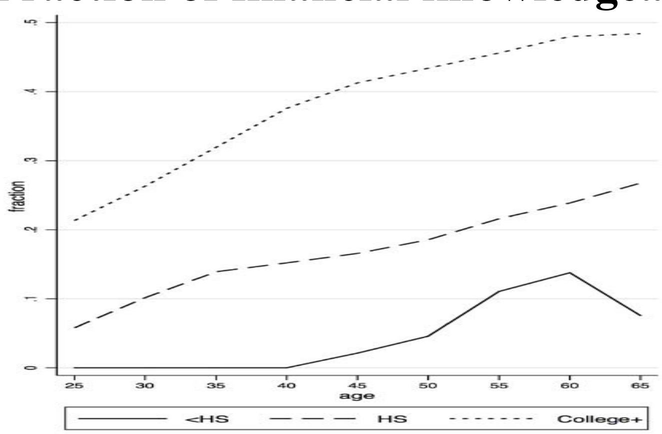
- Even the best fit model stated above is off, not to mention its ad hoc modeling strategy
- Can we fo better? A potential new avenue is to consider heterogeneous financial knowledge
- Education and lifecycle income profile:

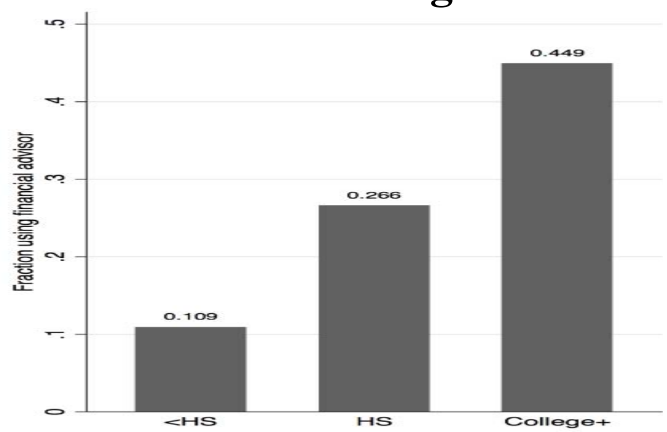


- **Lifecycle wealth profile:**



- **Fraction of financial knowledgeable and fraction of using financial advisors**





- **Financial knowledge => high return R, but with unit cost π**
- **With saving s, wealth *a* = Rs**
- **Household optimization:** $\max_{a,R} u(y - \pi R - a/R) + \beta u(a)$
  - **with log utility, wealth-income ratio is:** $\frac{a^*}{y} = \frac{y}{(2+\frac{1}{\beta})^2 \pi}$
    - **increasing in y**
    - **decreasing in π**
- **Model the evolution of financial knowledge:** $f_{t+1} = (1-\delta) f_t + i_t$
- **Cash on hand:** $x_t = a_t + y_t - oop_t$ **(oop = out of pocket expenditure)**
- **Wealth evolution:** $a_{t+1} = \widetilde{R}_\kappa(f_{t+1})(x_t + tr_t - c_t - \pi(i_t) - c_d I(\kappa_t > 0))$ **where κ = fraction of wealth in sophisticated financial asset and** $\widetilde{R}_\kappa(f_{t+1}) = (1-\kappa_t)\overline{R} + \kappa_t \widetilde{R}(f_t)$
- **Income process**

$$\log y_{e,t} = g_{y,e}(t) + \mu_{y,t} + \nu_{y,t}$$

$$\mu_{y,t} = \rho_{y,e}\mu_{y,t-1} + \varepsilon_{y,t}$$

$$\varepsilon_{y,t} \sim N(0, \sigma^2_{y,\varepsilon}),\ \nu_{y,t} \sim N(0, \sigma^2_{y,v})$$

- **Out of pocket expenditure process:**

$$\log oop_{e,t} = g_{o,e}(t) + \mu_{o,t} + \nu_{o,t}$$

$$\mu_{o,t} = \rho_{o,e}\mu_{o,t-1} + \varepsilon_{o,t}$$

$$\varepsilon_{o,t} \sim N(0, \sigma^2_{o,\varepsilon}),\ \nu_{o,t} \sim N(0, \sigma^2_{o,v})$$

- **Bellman equation:**

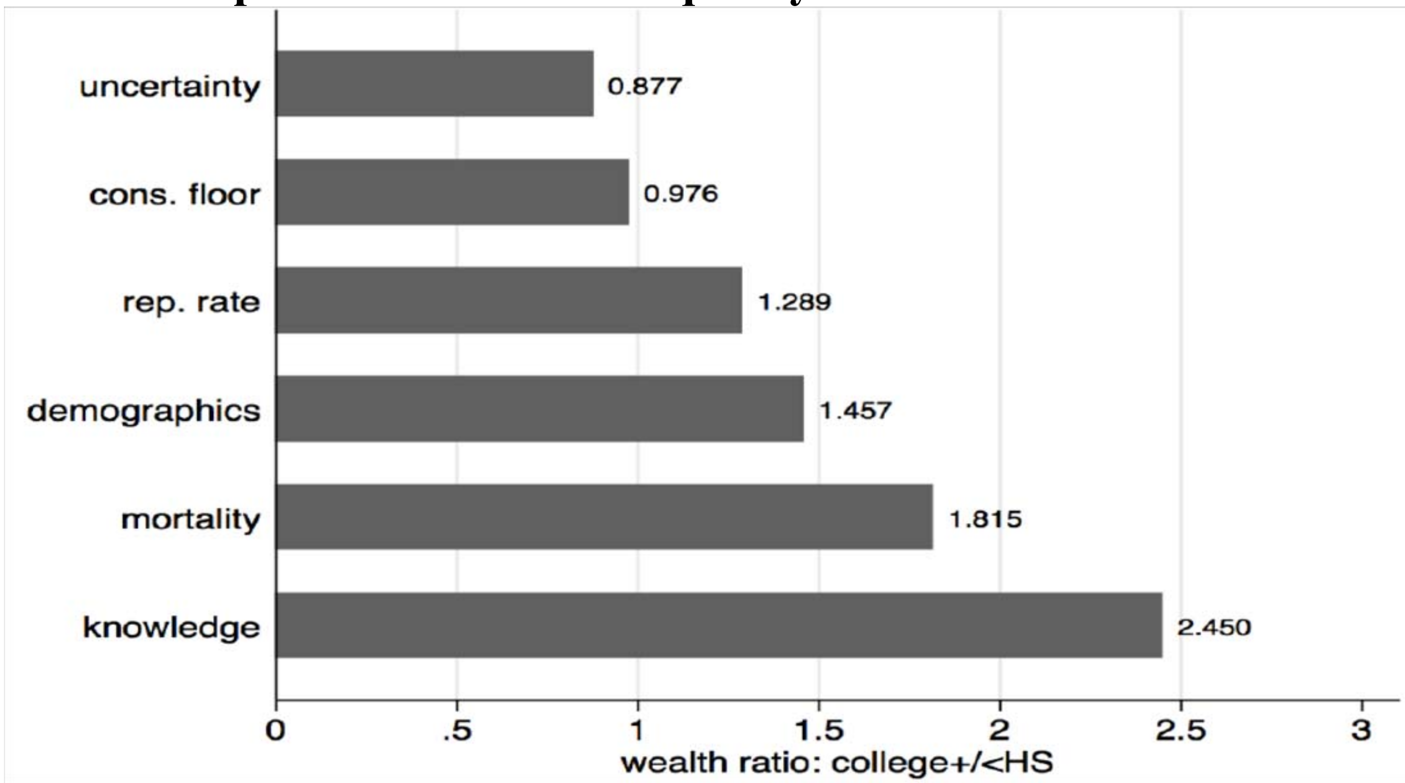
$$V_d(s_t) = \max_{c_t, i_t, \kappa_t} n_{e,t} u(c_t/n_{e,t}) + \beta p_{e,t} \int_{\varepsilon}\int_{\eta_y}\int_{\eta_o} V(s_{t+1}) dF_e(\eta_o) dF_e(\eta_y) dF(\varepsilon)$$

$$a_{t+1} = \widetilde{R}_\kappa(f_{t+1})(a_t + y_{e,t} + oop_{e,t} + tr_t - c_t - \pi(i_t) - c_d I(\kappa_t > 0))$$

$$f_{t+1} = (1-\delta) f_t + i_t$$

$$\widetilde{R}_\kappa(f_{t+1}) = (1-\kappa_t)\overline{R} + \kappa_t \widetilde{R}(f_t).$$

- **Calibration results (using Tauchen 1986 discretization of the two processes):**
  - **decomposition of wealth inequality**



  - **importance of financial knowledge: accounting for 30-40% of wealth inequality of the retired, even more important than replacement rate, demographics and health mortality factors**

J. Automation, Uneven Growth and Distribution: Moll-Rachel-Restrepo (2019)

- **Individuals differ in skill z with density ℓ_z, facing a Poisson death rate p and replaced by those of the same skill**
- **Individual optimization:**

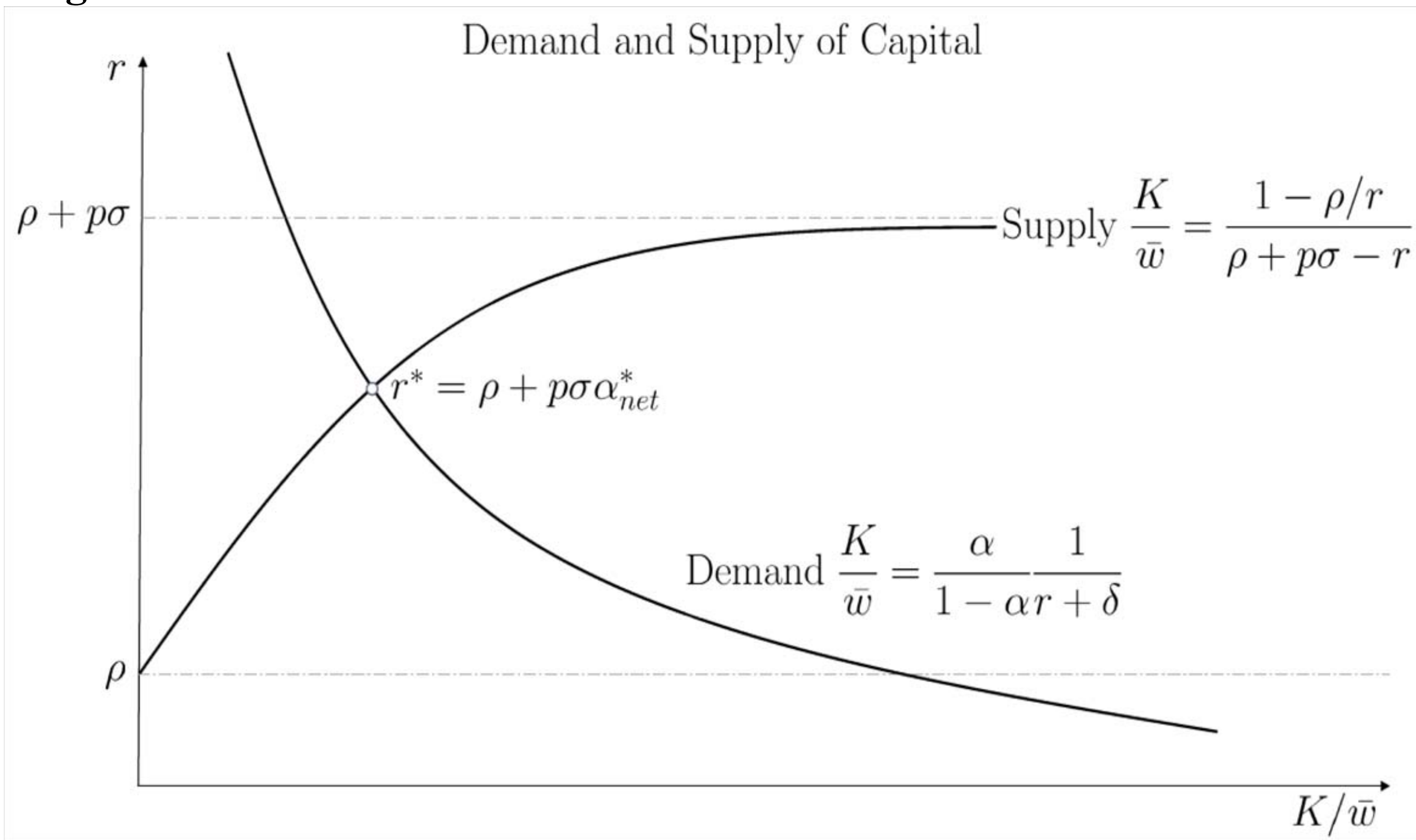
$$\max_{\{c_z(s),a_z(s)\}_{s\geq 0}} \int_0^\infty e^{-(\varrho+p)s}\frac{c_z(s)^{1-\sigma}}{1-\sigma}ds$$

$$\text{s.t. } \dot{a}_z(s) = w_z + ra_z(s) - c_z(s), \text{ and } a_z(s) \geq -w_z/r$$

  - **non-negative income**
  - **incidental bequest with new born having $a_z(0) = 0$**
- **Production:** $Y = A\prod_z Y_z^{\gamma_z} \quad \text{with} \quad \sum_z \gamma_z = 1$ **and** $\ln Y_z = \int_0^1 \ln \mathcal{Y}_z(u)du$
  - **each skill z works on a task $\mathcal{Y}_z(u)$ in sector z that produces output Y_z**
  - **task production:** $\mathcal{Y}_z(u) = \begin{cases} \psi_z\ell_z(u) + k_z(u) & \text{if } u \in [0, \alpha_z] \\ \psi_z\ell_z(u) & \text{if } u \in (\alpha_z, 1] \end{cases}$
  - **α_z measures the degree of automation**
  - **(A, γ_z, α_z) summarize technologies: TFP, sector-biased technical changes and automation**

- **Market-Clearing:**
  - **labor:** $\int_0^1 \ell_z(u)du = \ell_z$
  - **capital:** $K = \sum_z \int_0^{\alpha_z} k_z(u)du = \sum_z \ell_z \int_0^{\infty} a_z(s)pe^{-ps}ds$
- **Assumption I (immediate adoption of available automation technology)**
  $\frac{w_z}{\psi_z} > R \quad \text{for all } z$
- **Under A-I, equilibrium features**
  - **output:** $Y = \mathcal{A}K^{\sum_z \gamma_z \alpha_z} \prod_z (\psi_z \ell_z)^{\gamma_z(1-\alpha_z)}$
  - **factor prices:** $w_z = (1-\alpha_z)\frac{\gamma_z}{\ell_z}Y$ **and** $R = \alpha \frac{Y}{K}$
  - **TFP growth:** $d\ln \text{TFP}_\alpha = \sum_z \gamma_z \ln\left(\frac{w_z}{\psi_z R}\right) d\alpha_z > 0$**, rising in α_z under A-I**
- **Steady-state equilibrium:**
  - **equating capital demand and supply:** $\frac{1-\rho/r^*}{p\sigma + \rho - r^*} = \frac{\alpha}{1-\alpha}\frac{1}{r^* + \delta}$

- **diagrammatic illustration:**



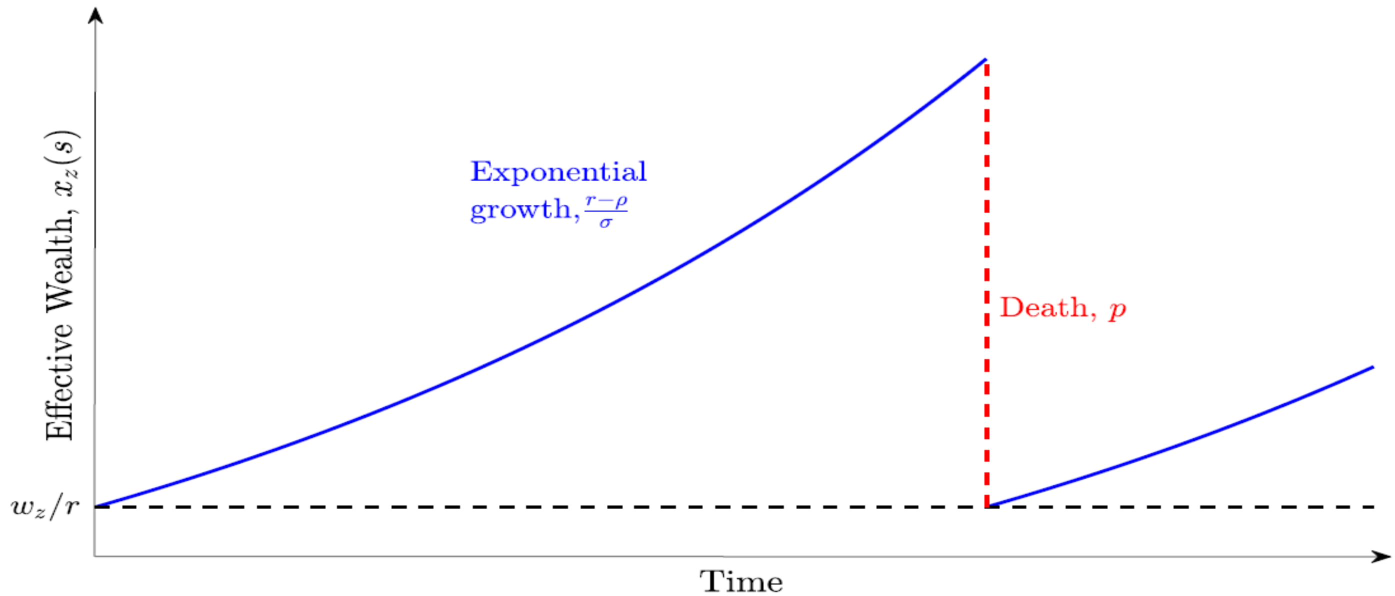
- **S-S return to wealth $r^* = \rho + p\sigma\alpha^*_{net}$, rising with the net capital share α^*_{net} that is increasing in the average degree of automation α (not the distribution of α_z)**

- **steady-state effect of automation on aggregate output:**

$$d\ln Y^* = \frac{1}{1-\alpha} d\ln \mathrm{TFP}_\alpha + \frac{\alpha}{1-\alpha} d\ln(K/Y)^* > 0$$

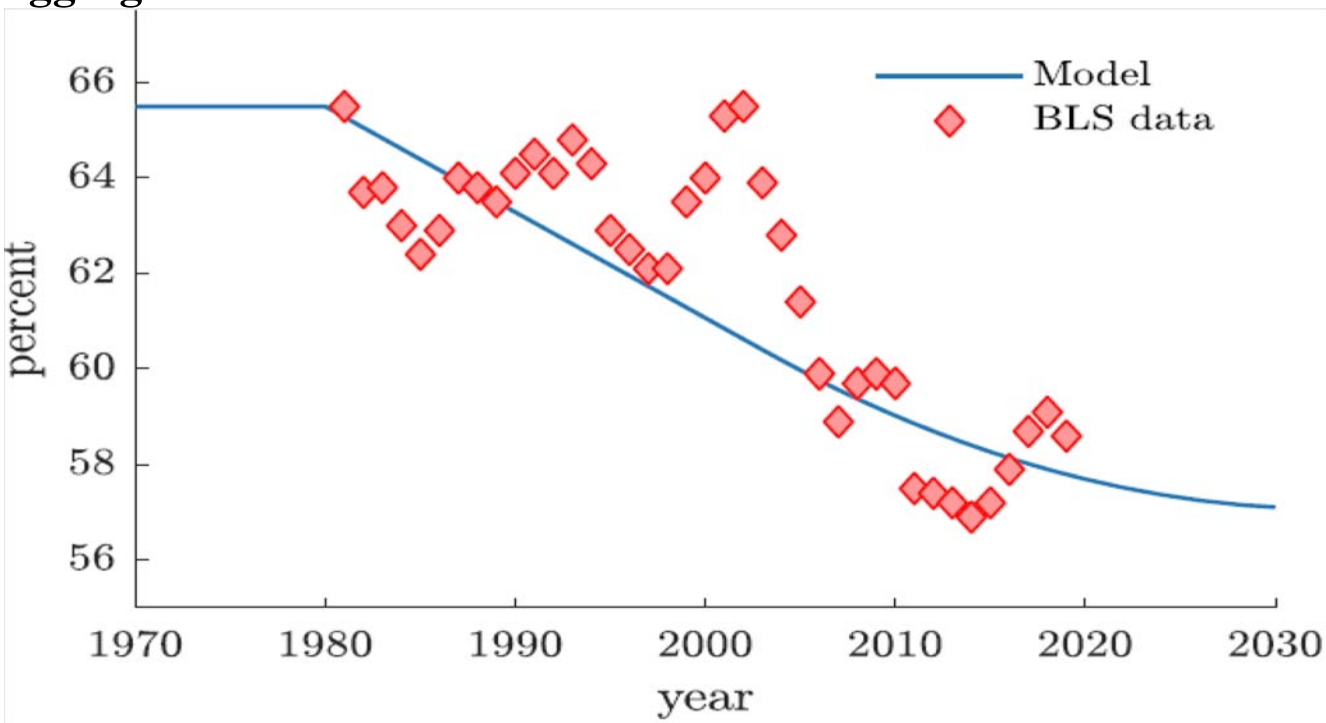
- **steady-state effect of automation on relative wage and average wage w*:**
  - **higher α_z => lower w_z^*/w^***
  - **$\exists\ \bar{p}$ s.t.**
    - **for $p < \bar{p}$, higher α_z => average wage w* rises**
    - **for $p > \bar{p}$, higher α_z => average wage w* falls**
  - **automation can lead to wage stagnation under higher death rate**
    - **higher p => capital supply more inelastic in the long run**
    - **less output expansion as a result of automation**
    - **so negative displacement effect can wipe out positive productive effect, leading to lower wage bill and lower average wage**

- **Distribution:**
  - **effective wealth $x_z(s) = a_z(s) + w_z^*/r^*$**
  - **effective wealth distribution: random exponential growth with Poisson death => Pareto wealth distribution**

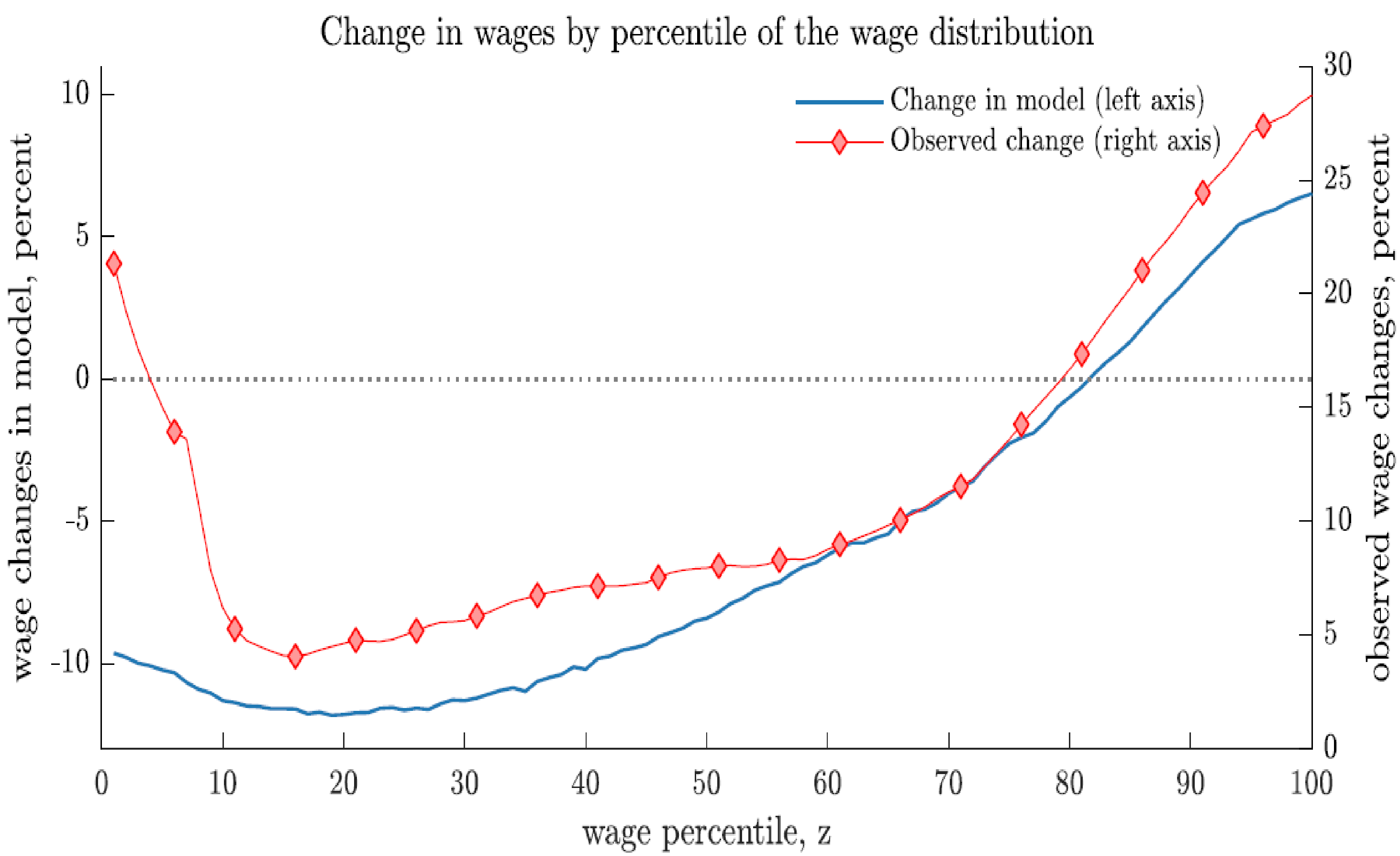


- **Pareto** $g_z(x) = \left(\frac{w_z^*}{r^*}\right)^{\zeta} \zeta x^{-\zeta-1}$ **with tail** $\frac{1}{\zeta} = \frac{1}{p}\frac{r^*-\rho}{\sigma} = \alpha_{net}^*$
- $\Pr(wealth \geq a) = \sum_z \ell_z \left(\frac{a + w_z^*/r^*}{w_z^*/r^*}\right)^{-1/\alpha_{net}^*}$, $\Pr(income \geq y) = \sum_z \ell_z \left(\frac{\max\{y, w_z^*\}}{w_z^*}\right)^{-1/\alpha_{net}^*}$
- **Pr of top-q wage earners:** $\Pr(skill = z | top\ q) = \frac{\ell_z w_z^{1/\alpha_{net}^*}}{\sum_v \ell_v w_v^{1/\alpha_{net}^*}}$
- **share of national income held by top-q:** $S(q) = \Lambda q^{1-\alpha_{net}^*}$

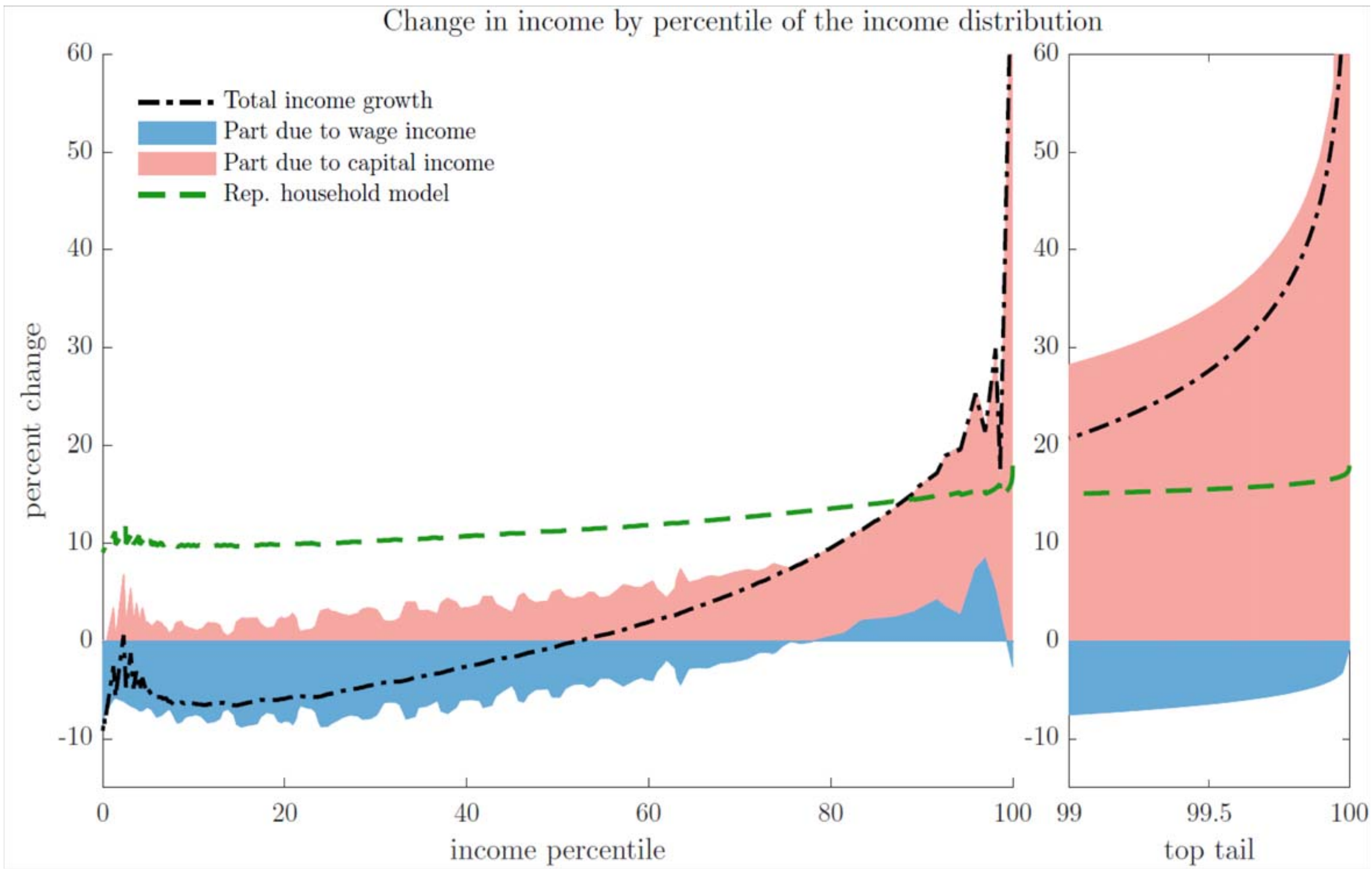
- **Calibration results:**
  - **p = 3.85%**
  - **α(1980) = 0.345, α(1980) = 0.428**
  - **aggregate labor share:**



- **predicted wage distribution:**

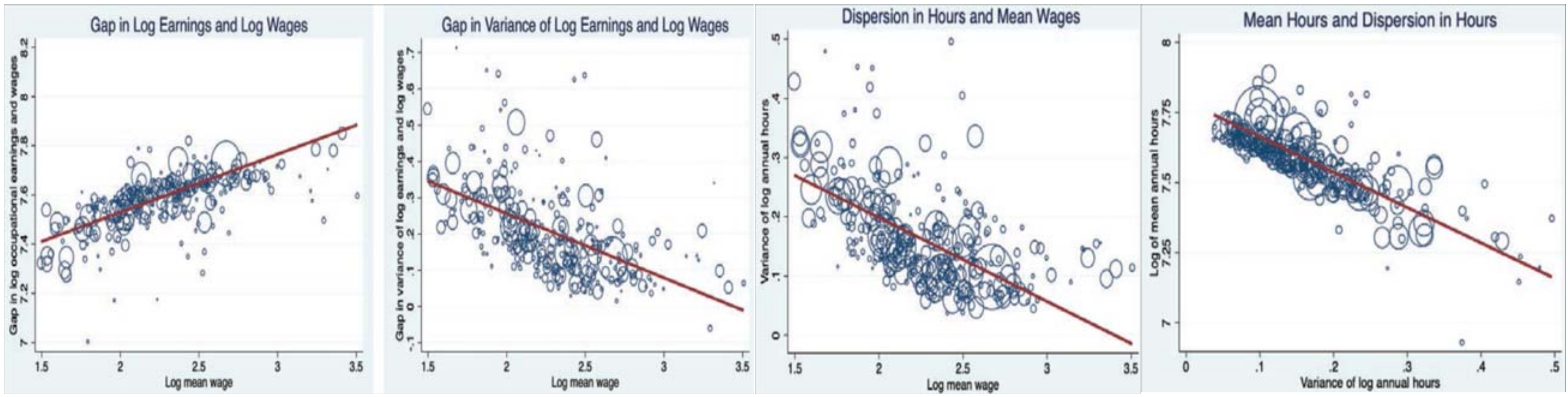


- **decompose changes in predicted income distribution**



K. Labor Supply and Inequality: Erosa-Fuster-Kambourov-Rogerson (2024)

- **IPUMS-CPS data over 1976-2015 indicate:**



  - **large quantitative differences in inequality in wages and earnings both across and within occupations**
  - **occupations with high mean wages exhibit larger gaps in mean log earnings/mean log wages**
  - **occupations with high mean wages exhibit smaller gaps between the within occupation variance of log earnings/variance of log wages**
  - **negative relationship between the within occupation variance of log hours and log mean wages**
  - **negative relationship between log mean hours and the within occupation variance of log hours**

- **Consider a Roy model with 3 occupations: (H, M, L), each with 1/3 employment share, ranked by mean hours**
- **Key average data moments:**
  - **log mean wage = 2.61, 2.28, 1.93 (earning: 10.32, 9.87, 9.41)**
  - **variance of log wage = 0.33, 0.28, 0.28 (earning: 0.46, 0.48, 0.60)**
- **3 dimensions of generalization of standard Roy:**
  - **endogenous work hour decision**
  - **heterogeneous tastes for leisure (and hence labor supply elasticities)**
  - **nonlinearity of efficiency units of labor as a function of labor hours, varying across occupations**
- **Preference: a continuum of individuals of mass one, with type i individual's utility given by,** $\ln c_i + \phi_i \frac{(T - h_i)^{1-\gamma}}{1-\gamma}$**,** $\varphi_i > 0, \gamma > 0$
- **Linear production depending efficiency units of labor:** $Y_j = E_j$**, j = {H,M,L}**
- **Individual i's efficiency units of labor nonlinear in hours:** $e_{ij} = a_{ij} h_{ij}^{1+\theta_j}$**,** $\theta_j > 0$**, with** $\theta_H \geq \theta_M \geq \theta_L$ **(linear when** $\theta_j = 0$**)**

- **Individual optimization:**

$$\max_{c_i,\{h_{ij}\}_{j=H,M,L}} \left\{ \ln c_i + \phi_i \frac{\left(T - \sum_{j=H,M,L} h_{i,j}\right)^{1-\gamma}}{1-\gamma} \right\}$$

$$\text{subject to } c_i = \sum_{j=H,M,L} a_{ij} h_{ij}^{1+\theta_j}, \quad \sum_{j=H,M,L}^{3} h_{ij} \leq T, \; h_{ij} \geq 0$$

- **Two-stage decision:**
  - **Stage 1: choose optimal hours conditional on an occupational choice**
  - **Stage 2: choose the optimal occupation under hours chosen in stage 1**
- **FOC of stage 1:** $\frac{1+\theta_j}{\phi_i} = h_{ij}(T-h_{ij})^{-\gamma} \equiv g(h_{ij})$ **=>** $h_{iH} > h_{iM} > h_{iL}$
- **Within-occupation hours distribution is driven by** $\varepsilon_{h_{ij},\phi_i} = \frac{dh_{ij}}{h_{ij}} / \frac{d\phi_i}{\phi_i} = -\frac{1}{1+\gamma \frac{h_{ij}}{T-h_{ij}}}$
  - **its absolute value depends negatively on hours – least responsive for H and most responsive for L**
  - **occupation H has highest mean hours and lowest dispersion of log hours and occupation L lowest mean hours and highest dispersion of log hours**
  - **negative relationship between mean & variance of log hours across j**
  - **a proportional decrease in φ_i within an occupation leads to an increase in mean hours and a decrease in the variance of log hours**

- **Calibration:**

Description	Parameter	Non-linear
non-linearity H	θ_H	0.4490
non-linearity M	θ_M	0.3576
non-linearity L	θ_L	0.2673
corr (a_H, ϕ)	$\rho_{a_H,\phi}$	0.0
corr (a_M, ϕ)	$\rho_{a_M,\phi}$	0.0
corr (a_L, ϕ)	$\rho_{a_L,\phi}$	0.0
corr (a_H, a_M)	ρ_{a_H,a_M}	0.9863
corr (a_H, a_L)	ρ_{a_H,a_L}	0.9392
corr (a_M, a_L)	ρ_{a_M,a_L}	0.9779
mean ab occ. H	μ_{a_H}	-1.3631
mean ab occ. M	μ_{a_M}	-1.3190
mean ab occ. L	μ_{a_L}	-0.6888
var ab occ. H	$\sigma^2_{a_H}$	0.4199
var ab occ. M	$\sigma^2_{a_M}$	0.3532
var ab occ. L	$\sigma^2_{a_L}$	0.2929
mean taste for leisure	μ_ϕ	25.0072
var taste for leisure	σ^2_ϕ	1.6371

Target	Data	Non-linear
log mean hours occ. H	7.705	7.707
log mean hours occ. M	7.590	7.591
log mean hours occ. L	7.456	7.454
log mean wages occ. H	2.611	2.611
log mean wages occ. M	2.277	2.276
log mean wages occ. L	1.931	1.931
share of emp. occ. H	0.333	0.333
share of emp. occ. M	0.333	0.333
var log hours occ. L	0.239	0.238
var log wages occ. H	0.334	0.332
var log wages occ. M	0.281	0.287
var log wages occ. L	0.294	0.290
var log hours occ. H	0.099	0.100
var log hours occ. M	0.146	0.147
Loss Function $\times$ (10^{-5})	—	6.47

- **Occupational differences:**

	Data	Non-linear
Mean Log Earnings		
Occ H	10.322	10.339
Occ M	9.872	9.889
Occ L	9.407	9.420
Log Earn Gap H-M	0.449	0.450
Log Earn Gap L-M	-0.466	-0.469
Var Log Earnings		
Occ H	0.464	0.480
Occ M	0.476	0.486
Occ L	0.598	0.621
Var log earn- Var log wages		
Occ H	0.130	0.147
Occ M	0.195	0.199
Occ L	0.304	0.331
Corr of log hours and log wages		
Occ H	0.075	0.130
Occ M	0.115	0.127
Occ L	0.120	0.177

  - **overall good fit, except variance of log earnings for L and cov of log hours and log wages for H and L**

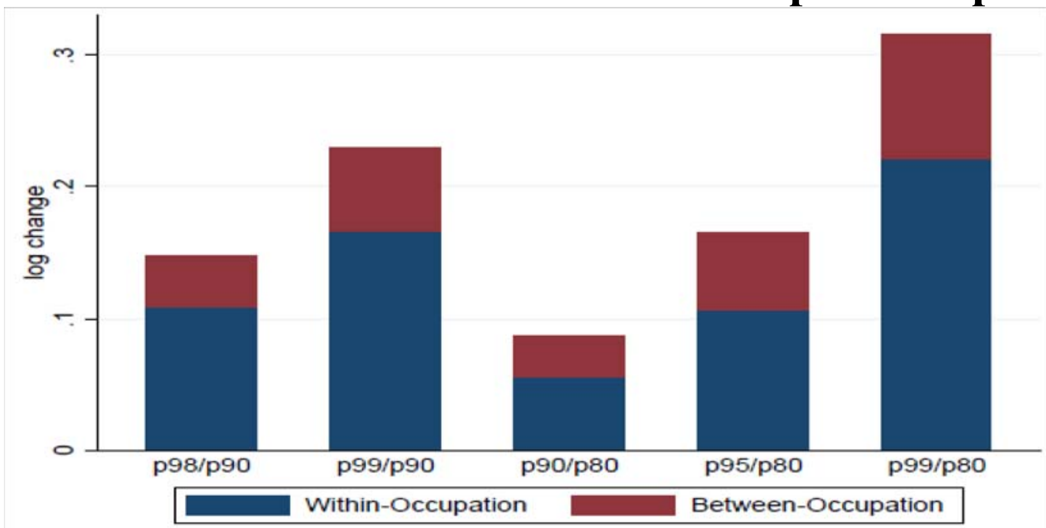
- **Non-targeted moments:**

	Data	Non-linear
Log Mean Hours		
Occ H - Occ M	0.114	0.116
Occ L - Occ M	-0.134	-0.137
Var Log Hours		
Occ H - Occ M	-0.047	-0.047
Occ L - Occ M	0.093	0.091
Log Mean Wages		
Occ H - Occ M	0.334	0.335
Occ L - Occ M	-0.346	-0.345
Var Log Wages		
Occ H - Occ M	0.053	0.045
Occ L - Occ M	0.013	0.003
Emp shares		
Occ H	0.333	0.333
Occ M	0.333	0.333
Occ L	0.333	0.334

  - **overall good fit, except variance of log wages**

L. Occupation Spillover and Top Inequality: Gottlieb-Hémous-Hicks-Olsen (2023)

- **A new trend since 1980: rise of within-occupation top income inequality**



- **Could inequality spill over across occupations?**
- **Consider two types of agents: widget makers (a continuum of mass 1) and potential doctors of mass μ_d**
  - **A widget maker of ability x can produce x widgets,** $P(X > x) = \left(\frac{x_{\min}}{x}\right)^{\alpha_x}$**, with α_x >1 and** $x_{\min} = \frac{\alpha_x - 1}{\alpha_x}\widehat{x}$ **s.t. mean is fixed at** $\widehat{x}$ **as α_x changes (mean preserving spread)**

- Each doctor of ability Z serves λ patients, $P(Z > z) = \left(\frac{z_{min}}{z}\right)^{\alpha_z}$, with $1/\lambda < \mu_d$ s.t. everyone can be served
  - those failing to become doctor having widget ability of x_{min}
  - a more capable doctor does not increase # of patients served but raises patients utility by improving their health more effectively
- Utility depends on widget consumption & healthcare quality: $u(z,c) = z^{\beta}c^{1-\beta}$
- Optimization:
  - widget maker: $\max_{z,c} u(z,c) = z^{\beta}c^{1-\beta}$ subject to $\omega(z) + c \leq x$ =>

    (FOC) $\omega'(z)\, z = \frac{\beta}{1-\beta}\left[x - \omega(z)\right]$
  - doctor: due to sufficient supply of doctors, some below a cutoff z_c would be better off by working as widget maker
    - market-clearing => $P(X > m(z)) = \lambda\mu_d P(Z > z), \ \forall z \geq z_c$
    - Pareto distr. => matching function $m(z) = x_{\min}(\lambda\mu_d)^{-\frac{1}{\alpha_x}}\left(\frac{z}{z_{min}}\right)^{\frac{\alpha_z}{\alpha_x}}$
- Combining => a differential eq $w'(z)\, z + \frac{\beta}{1-\beta} w(z) = \frac{\beta}{1-\beta} x_{\min}\left(\frac{\lambda^{\alpha_x - 1}}{\mu_d}\right)^{\frac{1}{\alpha_x}}\left(\frac{z}{z_{min}}\right)^{\frac{\alpha_z}{\alpha_x}}$

- **using boundary condition at z_c, the solution takes the following form:**

$$w(z) = x_{\min}\left[\frac{\lambda\beta\alpha_x}{\alpha_z(1-\beta)+\beta\alpha_x}\left(\frac{z}{z_c}\right)^{\frac{\alpha_z}{\alpha_x}} + \frac{\alpha_z(1-\beta)+\beta\alpha_x(1-\lambda)}{\alpha_z(1-\beta)+\beta\alpha_x}\left(\frac{z_c}{z}\right)^{\frac{\beta}{1-\beta}}\right]$$

- **it has a Pareto tail** $x_{\min}\frac{\lambda\beta\alpha_x}{\alpha_z(1-\beta)+\beta\alpha_x}\left(\frac{z}{z_c}\right)^{\frac{\alpha_z}{\alpha_x}}$
- **top-income inequality of doctors spill over to inequality of the widget makers and the entire population**

- **Model fit:**

Table 1: Wage income: Ratio 98/90: actual values and predicted values

	General Population			Physicians		
Year	α^{-1}	Actual	Predicted	α^{-1}	Actual	Predicted
1980	0.34	1.70	1.72	0.25	1.50	1.50
1990	0.38	1.87	1.85	0.40	1.89	1.90
2000	0.42	2.00	1.96	0.33	1.75	1.71
2012	0.42	1.99	1.96	0.34	1.72	1.72

- **Spillover estimates for physicians**

Table 3: Spillover estimates for Physicians

	OLS		1st Stage		IV	
Dependent variable	$\ln(\alpha_o^{-1})$ (1)	$\ln(\alpha_o^{-1})$ (2)	$\ln(\alpha_{-o}^{-1})$ (3)	$\ln(\alpha_{-o}^{-1})$ (4)	$\ln(\alpha_o^{-1})$ (5)	$\ln(\alpha_o^{-1})$ (6)
$\ln(\alpha_{-o}^{-1})$	0.16**	0.22***			1.74**	1.50**
	(0.08)	(0.06)			(0.75)	(0.70)
ln(Average Income)		−0.40***		0.17***		−0.60***
		(0.09)		(0.05)		(0.14)
ln(Population)		−0.02		−0.06		0.07
		(0.03)		(0.04)		(0.07)
$\ln(I)$			0.70***	0.70***		
			(0.24)	(0.26)		
LMA FE	Yes	Yes	Yes	Yes	Yes	Yes
Year FE	Yes	Yes	Yes	Yes	Yes	Yes
N	750	750	750	750	750	750
F-Statistic					8.65	7.43

  - **same applies to dentists, real estate agents and system analysts and scientists, but not to financial managers, other managers, engineers, or other professionals**

M. Inequality and Growth: Oberfield (2023)

- **In conjunction with widening inequality, the US has experienced fallen productivity over the past two decades. Putting aside issues regarding the measurement of TFP (using working-age population, including labor and capital utilization, etc.), can one come up with a unified endogenous growth model explaining this much concerning observation?**
- **Oberfield (2023) proposed two The two key ingredients:**
  - **non-homothetic preferences**
  - **productivity improvements directed toward goods with larger market size**
- **Households: a continuum of mass one with identical preferences**
  - **each supplying labor inelastically differing in labor productivity ~ G(ℓ)**
  - **facing a tax function** $T(y) = y - \bar{y}^{\tau} y^{1-\tau}$
    - $\bar{y}$ **s.t. balanced GBC**
    - **τ = degree of progressiveness (=1 => uniform)**
    - **after-tax income y-T(y) is log-linear in pre-tax income y**
    - **GBC =>** $w\ell^{1-\tau}/\overline{\ell^{1-\tau}}$, where $\overline{\ell^{1-\tau}} \equiv \int \ell^{1-\tau} dG(\ell)$

- nonhomothetic preference: a nonhomothetic CES with a specific function of consumption weights $\sup_C C$ s.t. $\left[\int_{-\infty}^{\infty} h\left(i-\gamma\log C\right)^{\frac{1}{\sigma}}\left(\frac{c_i}{C}\right)^{\frac{\sigma-1}{\sigma}} di\right]^{\frac{\sigma}{\sigma-1}} \geq 1$
  - consumption wight function: $h\left(i-\gamma\log C\right)$
    - $\int_{-\infty}^{\infty} h(i)di = 1$
    - as C rises, weights toward higher ranked goods with higher i
  - γ = strength of nonhomotheticity (homothetic when $\gamma = 0$)
  - expenditure minimization followed by consumption bundle choice =>

    $c_i = p_i^{-\sigma}E^{\sigma}C^{1-\sigma}h\left(i-\gamma\log C\right)$ with C solving $C\left(\int_{-\infty}^{\infty} p_i^{1-\sigma}h\left(i-\gamma\log C\right)di\right)^{\frac{1}{1-\sigma}} = E$
  - special case (uniform p_i):
    - consumption bundle: $c_i = \frac{E}{p}h\left(i-\gamma\log\frac{E}{p}\right)$
    - weight: $h(u) = \frac{1}{\sqrt{2\pi\upsilon_h}}e^{-\frac{u^2}{2\upsilon_h}}$, $u = i-\gamma\log C$, υ_h = taste dispersion
    - labor productivity distribution: log-normal mean 1 (Gaussian)

      $G'(\ell) = \frac{1}{\ell}\frac{1}{\sqrt{2\pi\upsilon_\ell}}e^{-\frac{(\log\ell+\upsilon_\ell/2)^2}{2\upsilon_\ell}}$, υ_ℓ = labor productivity dispersion

- product concentration: Herfindahl-Hirschman Index of aggregate expenditures across goods

  $HHI = \int_{-\infty}^{\infty} \omega_i^2 di$, where $\omega_i \equiv \frac{p_i y_i}{\int_{-\infty}^{\infty} p_{\tilde{i}} y_{\tilde{i}} d\tilde{i}}$ **and** $y_i = \int c_{\ell i} dG(\ell)$ **=>**

  $HHI = \frac{1}{2\sqrt{\pi}\sqrt{v_h + (1-\tau)^2\gamma^2 v_\ell}}$ **which depends negatively on taste and labor productivity dispersion with the latter effect more prominent when the strength of nonhomotheticity (γ) is higher**

- **Production: each i is produced labor under a general technology A and a goods specific technology B_i**
  - **production function:** $Y_{it} = A_t B_{it} L_{it}$
  - **evolution of goods specific technology:** $\frac{\dot{B}_{it}}{B_{it}} = \phi L_{it}$ **(learning by doing)**
- **Equilibrium:**
  - **labor market clearing:** $L_{it} = \frac{Y_{it}}{A_t B_{it}} = \frac{1}{A_t B_{it}} \int c_{\ell it} dG(\ell)$
  - **balanced growth (BGP): constant tax function and** $\frac{\dot{A}_t}{A_t} = g$ **under which all growing variables grow at g and all non-growing variables are constant and** $\frac{\dot{B}_{it}}{B_{it}} = \phi L_{it} = \phi L \omega_{it}$ **, where the unique BGP exists if** $e^{(\sigma-1)\frac{\phi L}{\gamma g}} < 2$

- **TFP growth:** $\frac{d\log\widehat{TFP_t}}{dt} = \int_{-\infty}^{\infty} \omega_{it}\frac{\dot{Y}_{it}}{Y_{it}}di$ **,**
  - $\omega_{it} \equiv \frac{p_{it}Y_{it}}{\int_{-\infty}^{\infty} p_{\tilde{i}t}Y_{\tilde{i}t}} = \frac{w_t L_{it}}{w_t L} = \frac{L_{it}}{L}$ **can be measured by expenditure share**
  - **thus,** $\frac{d\log\widehat{TFP_t}}{dt} = \frac{\dot{A}_t}{A_t} + \phi L \underbrace{\int_{-\infty}^{\infty} \omega_{it}^2 di}_{HHI}$**, implying more concentrated product market driven by demand (expenditure) can lead to higher TFP and economic growth**
  - **with uniform price $p_i = p$,** $\frac{d\log\widehat{TFP_t}}{dt} \approx g + \phi L \frac{1}{2\sqrt{\pi}\sqrt{v_h + (1-\tau)^2\gamma^2 v_\ell}}$ **=> more equitable distribution of after-tax income serves as a driver of TFP and economic growth => negative relationship between inequality and growth**
- **in general,** $p_{it} = \frac{w_t}{A_t B_{it}}$**, so** $\frac{\dot{p}_{it}}{p_{it}} = \frac{\dot{w}_t}{w_t} - \frac{\dot{A}_t}{A_t} - \frac{\dot{B}_{it}}{B_{it}}$ **and inflation dynamics is** $\widehat{Inflation}_{\ell t} = \frac{\dot{w}_t}{w_t} - \frac{\dot{A}_t}{A_t} - \int_{-\infty}^{\infty} \omega_{\ell it}\frac{\dot{B}_{it}}{B_{it}}di$ **=** $\frac{\dot{w}_t}{w_t} - g - \phi L \int_{-\infty}^{\infty} \omega_{\ell it}\omega_{it} di \approx \frac{\dot{w}_t}{w_t} - \frac{\dot{A}_t}{A_t} - \phi L \int_{-\infty}^{\infty} \omega_{\ell it}^0 \omega_{it}^0 di$ **with** ω_{it}^0 and $\omega_{i\ell t}^0$ **denoting aggregate and individual expenditure shares without LBD (φ = 0)**

- thus, when φ is small and h and G are Gaussian, if log ℓ is k-sd above the mean and log ℓ′ k-sd below, then $\widehat{Inflation}_{\ell t} < \widehat{Inflation}_{\ell' t}$ => the poor got hurt more
- one may also compute the price index facing household ℓ by rewriting $E_{\ell t} = C_{\ell t} P_{\ell t}$ and $P_{\ell t} = \left[\int_{-\infty}^{\infty} h\left(i - \gamma \log C_{\ell t}\right) p_{it}^{1-\sigma} di\right]^{\frac{1}{1-\sigma}}$ => $\frac{\dot{P}_{\ell t}}{P_{\ell t}} = \frac{\dot{w}_t}{w_t} - \frac{\dot{A}_t}{A_t} = \frac{\dot{w}_t}{w_t} - g$ => $\frac{P_{\ell t}}{w_t / A_t}$ is constant for all ℓ, i.e., to all households, their consumption price indexes relative to effective wages remain stable over time

- Taking stock,
  - non-homothetic preferences together with productivity improvements (LBD) directed toward goods with larger market size (demand shifts) can induce a negative relationship between inequality and growth, so the observation of fallen TFP and rising income inequality can be explained
  - overall, the poor got hurt more due to suffering unfavorable inflation bias
  - nonetheless, price of consumption bundle relative to effective wages remain stable over time, so individual welfare measured by w/P_ℓ improves at the same constant rate g regardless of labor productivity ℓ

Appendix: On Modeling Top Income or Wealth Distributions

- **To model the distribution of labor/non-labor earnings by the super rich (top 1%) or their financial/non-financial wealth, we must source to the class of univariate extreme value distributions (ExVDs), which can only be one of the three types (cf. Fisher-Tippett 1928):**
  - **Type 1, Gumbel (1958): $\Pr(X \leq x) = \exp[-e^{(x-\mu)/\sigma}]$, or double exponential**
  - **Type 2, Fréchet (1927): $\Pr(X \leq x) = \exp\{-[(x-\mu)/\sigma]^{-\xi}\}$ for $x \geq \mu$, o.w. = 0**
  - **Type 3, Weibull (1939): $\Pr(X \leq x) = \exp\{-[(x-\mu)/\sigma]^{\xi}\}$ for $x \leq \mu$, o.w. = 0**

  where X is the random variable of interest (income or wealth) and μ, $\sigma > 0$ and $\xi > 0$ are location, scale and shape parameters
- **Key properties:**
  - **These ExVDs are limiting distributions of the greatest value among n independent random variables with each following the same distribution when $n \to \infty$**
  - **X follows an ExVD $\Rightarrow$ -X follows an ExVD as well**
  - **Type 2 and 3 can be transformed to type 1 with $Z = \log(X-\mu)$ and $Z = \log(\mu-X)$, respectively**

- **Combining all 3 ⇒ $Pr(X \le x) = \{1+\xi[(x-\mu)/\sigma]\}^{-1/\xi}$, with $1+\xi[(x-\mu)/\sigma] > 0$, $\sigma > 0$ and $\xi \in (-\infty,\infty)$:**
  - **$\xi \to -\infty$ or ∞ ⇒ type 1**
  - **$\xi > 0$ ⇒ type 2**
  - **$\xi < 0$ ⇒ type 3**
- **A special case of type 2 ExVD is Pareto: $Pr(X \le x) = 1-(x/x_{min})^{-\xi}$ with $x_{min} \ge 1$ where $1/\xi$ measures the thickness of the (right) tail – $\xi > 1$ ⇒ finite mean and $\xi > 2$ ⇒ finite variance (may not hold in practice)**
- **Pareto distribution is useful for income/wealth distribution because of the following property:**
  - **named after Pareto (1986) for his insight toward income heterogeneity**
  - **by setting $x_{min}=1$, $Pr(income > x) = (x)^{-\xi}$, a simple power law**
  - **Piketty-Saez (2003) top p percentile share = $(100/p)^{1/\xi-1}$ with top 1% share = $(100)^{1/\xi-1} \to$ 10% if $\xi \to 2$ and $\to$ 3.2% if $\xi \to 4$**
  - **in practice, many thick tail distributions have a Pareto tail – in most countries, top-20% income distribution follows Pareto**
  - **the entire distribution may be a combination of log-normal or logistic with a Pareto tail (use percentile chart to approximate the distribution and check precision by χ^2 test)**