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Semi-Continuity[1]

1 Definition.

Let (X, d) be a metric space. For a function $f : X \to \mathbb{R}$ and a point $y \in \mathbb{R}$, the upper contour set defined by y is

$$U(y) = f^{-1}([y, \infty)) = \{x \in X : f(x) \geq y\}.$$

The lower contour set defined by y is

$$L(y) = f^{-1}((-\infty, y]) = \{x \in X : f(x) \leq y\}.$$

The next result establishes that a number of properties are equivalent.

Theorem 1. *Let* $f : X \to \mathbb{R}$.

1. *The following are equivalent.*
   (a) *For any* $y \in \mathbb{R}$, $U(y)$ *is closed.*
   (b) *For any* $y \in \mathbb{R}$, $f^{-1}((-\infty, y)) = [U(y)]^c$ *is open.*
   (c) *For any* $x \in X$, *if the sequence* (x_t) *in* X *converges to* x, *then for any* $\varepsilon > 0$ *there is a* T *such that for all* $t > T$, $f(x) > f(x_t) - \varepsilon$.
2. *The following are equivalent.*
   (a) *For any* $y \in \mathbb{R}$, $L(y)$ *is closed.*
   (b) *For any* $y \in \mathbb{R}$, $f^{-1}((y, \infty)) = [L(y)]^c$ *is open.*
   (c) *For any* $x \in X$, *if the sequence* (x_t) *in* X *converges to* x *then for any* $\varepsilon > 0$ *there is a* T *such that for all* $t > T$, $f(x) < f(x_t) + \varepsilon$.

Proof. I provide the proof of equivalence for the first set of conditions. The proof for the second set of conditions is analogous.

- 1(a) $\Rightarrow$ 1(b). Almost immediate, since $U(y)$ is closed iff $[U(y)]^c$ is open.

- 1(b) $\Rightarrow$ 1(c). By contraposition. Suppose that there is an $x \in X$ and a sequence (x_t) in X that converges to x such that for for some $\varepsilon > 0$ there are infinitely many t such that $f(x) \leq f(x_t) - \varepsilon$. Choose any $y \in (f(x), f(x) + \varepsilon)$. Then there are infinitely many t such that $x_t \in U(y)$. These x_t constitute a sequence in $U(y)$ that converges to x, but $x \notin U(y)$, hence $U(y)$ is not closed, hence $[U(y)]^c$ is not open.

- 1(c) $\Rightarrow$ 1(a). Take any $y \in \mathbb{R}$. If $U(y) = \emptyset$ then I am done. Otherwise, take any convergent sequence (x_t) in $U(y)$ let $x = \lim x_t$. I need to show that $x \in U(y)$. By 1(c), for any $\varepsilon > 0$ there is a T such that for all $t > T$, $f(x) > f(x_t) - \varepsilon$. Since $x_t \in U(y)$, $f(x_t) \geq y$, hence $f(x) > y - \varepsilon$. Since this must hold for any $\varepsilon > 0$, $f(x) \geq y$, which implies $x \in U(y)$.

■

Since conditions listed under 1 and 2 are equivalent, I can choose any pair of them to define upper and lower semicontinuity. To underscore the analogy with continuity, I use the "b" conditions.

Definition 1. *Let $f : X \to \mathbb{R}$.*

1. *f is* upper semicontinuous *(USC) iff for any $y \in \mathbb{R}$, $f^{-1}((-\infty, y))$ is open.*

2. *f is* lower semicontinuous *(LSC) iff for any $y \in \mathbb{R}$, $f^{-1}((y, \infty))$ is open*

Informally, a function is upper semicontinuous if it is continuous or, if not, it only jumps up; a function is lower semicontinuous if it is continuous or, if not, it only jumps down.

Example 1. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1/x & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ -1/x & \text{if } x > 0. \end{cases}$$

Then f is upper semi-continuous. In particular, if $x_t \to 0$, then $f(x_t) \to -\infty < 0 = f(0)$. □

Theorem 2. *f is continuous iff it is both upper and lower semi-continuous.*

Proof. Almost immediate form property 1(c) and 2(c), which together are equivalent to requiring that if (x_t) converges to x then for any $\varepsilon > 0$, there is a T such that for all $t > T$, $f(x_t) \in N_\varepsilon(f(x))$, hence $f(x_t)$ converges to $f(x)$. ■