Lecture 3: Limited Enforcement in Exchange Economies

3.1 Constrained optimum with strong default penalties

First we solve the social planner's problem in a deterministic symmetric economy with two equally big groups of agents and endowments that fluctuate between two values $1+\alpha$ and $1-\alpha$. The planner's problem is to choose a stationary allocation

$$\begin{aligned} c_t^i &= c_H \in [1, 1+\alpha], \quad \text{if} \quad w_t^i = 1+\alpha \\ c_t^i &= c_L \in [1, 1-\alpha], \quad \text{if} \quad w_t^i = 1-\alpha \end{aligned}$$

to maximize some well-defined social welfare function (SWF) subject to appropriate resource constraints(RC's) and to participation constraints (PC's).

In this environment, we give agents the option of deviating from the planner's allocation at any point in time and going into perpetual autarky with probability one. The default option is consistent with a permanent ban from borrowing and lending in asset markets. The value of default for agent $i = 1, 2$ is

$$V_D^H = \frac{u(1+\alpha) + \beta u(1-\alpha)}{1-\beta^2}, \quad \text{if current income is high} \tag{1a}$$

$$V_D^L = \frac{u(1-\alpha) + \beta u(1+\alpha)}{1-\beta^2}, \quad \text{if current income is low} \tag{1b}$$

The value of solvency is (V^H, V^L) if the current state is (high, low) where

$$V^H = \frac{u(c_H) + \beta u(c_L)}{1-\beta^2} \tag{2a}$$

$$V^L = \frac{u(c_L) + \beta u(c_H)}{1-\beta^2} \tag{2b}$$

Assumption 3.1: *The high-income agent prefers autarky to the symmetric Arrow-Debreu allocation,* $(c_H, c_L) = (1, 1)$ *i.e.,*

$$(1+\beta)u(1) < u(1+\alpha) + \beta u(1-\alpha) \tag{3}$$

The planner's resource constraint is

$$c_H + c_L \leq 2$$

and the participation constraints for households are $V_D^H \leq V^H$ and $V_D^L \leq V^L$, or

$$u(c_L) + \beta u(c_H) \geq u(1-\alpha) + \beta u(1+\alpha) \quad \text{(PC1)}$$
$$u(c_H) + \beta u(c_L) \geq u(1+\alpha) + \beta u(1-\alpha) \quad \text{(PC2)}$$

If either constraint fails, the planner's decision will be ignored by some households. It is easy to show that (PC2) implies (PC1): Suppose not, that is, (PC2) holds while (PC1) does not. Then

$$u(c_H) + \beta u(c_L) - u(c_L) - \beta u(c_H) > u(1+\alpha) + \beta u(1-\alpha) - u(1-\alpha) + \beta u(1+\alpha)$$
$$\Rightarrow u(1-\alpha) - u(c_L) > u(1+\alpha) - u(c_H)$$

which contradicts $c_H \leq 1+\alpha$ and $c_L \geq 1-\alpha$.

Finally, the SWF with equal treatment is $V^H + V^L$ or equivalently $u(c_H) + u(c_L)$. Therefore, the planner's problem becomes
(**P**)

$$\max_{c_H, c_L} \quad \{u(c_H) + u(c_L)\}$$
$$s.t. \quad c_H + c_L \leq 2 \quad \text{(RC)}$$
$$u(c_H) + \beta u(c_L) \geq u(1+\alpha) + \beta u(1-\alpha) \quad \text{(PC2)}$$

The unique solution is $s(c_H, c_L) = (\hat{x}, 2-\hat{x})$ where $\hat{x} \in [1, 1+\alpha]$ is the smallest number satisfying

$$u(x) + \beta u(2-x) = u(1+\alpha) + \beta u(1-\alpha)$$

with implied interest yield

$$\hat{R} = \frac{u'(\hat{x})}{\beta u'(2-\hat{x})} \in (1, \frac{1}{\beta})$$

From Figure 3.1 it is clear that the allocation $(c_H, c_L) = (\hat{x}, 2-\hat{x})$ is

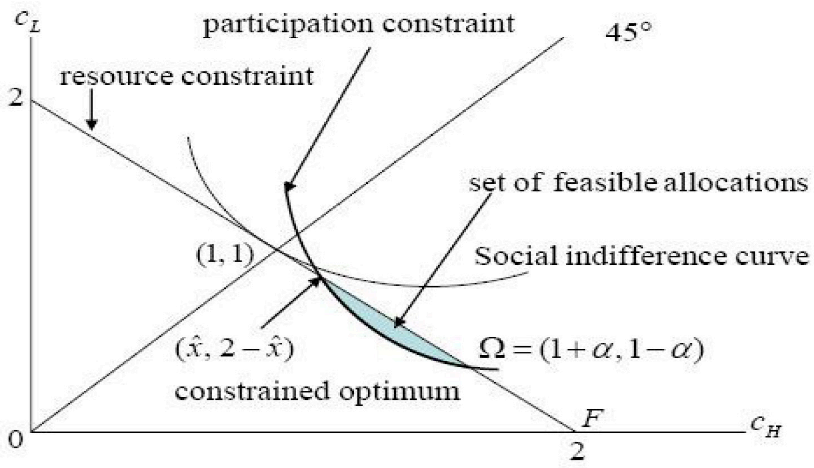
a) Constrained optimal (it solves the planner's problem)

b) Dynamically efficient (because the present value of the aggregate income stream is finite when discounted at the rate of time preference)

Question 3.1: Is autarky constrained optimal?

Ans: Yes, if $\frac{1}{\beta}\frac{u'(1+\alpha)}{u'(1-\alpha)} \equiv$ implied autarky yield ≥ 1, for all agents, that is, if autarky is *a high-interest-rate allocation.*
No, if $\frac{1}{\beta}\frac{u'(1+\alpha)}{u'(1-\alpha)} < 1$ (as in Figure 3.1), that is, if autarky is *a low-interest-rate allocation.*

Figure 3.1: The constrained optimal allocation.



<u>Problem 3.1:</u> Formulate and solve the planner's problem under the assumption that there is a continuum of agents indexed $\alpha \in [0,1]$ with mass 1 and endowment $\omega_t(\alpha) = \alpha$ if $t = 0, 2, \ldots$; and $\omega_t(\alpha) = 2 - \alpha$ if $t = 1, 3, \ldots$. Suppose α is distributed uniformly on $[0, 2]$. Assume the planner chooses deterministic allocations $(c_H(\alpha), c_L(\alpha))$ indexed on type and income state.

(a) Show that $c_H(\alpha) > c_L(\alpha)$ for all α, i.e., no agent enjoys completely smooth consumption. [Hint: Assume $c_H(\alpha) = c_L(\alpha)$ for some α and obtain a contradiction]

(b) Show that no household can be unconstrained in the planning optimum.

(c) Prove that the optimum splits all agents into two groups. People with highly variable endowments will be rationed (that is, their participation constraint will be binding) when income is low, unrationed when income is high. Agents with low income variability will be completely autarkic [Hint: Find the critical value of α , say $\bar{\alpha}$, at which a high-income agent is on the verge of perpetual autarky but her consumption-Euler Equation still holds].

3.2 Equilibrium with strong default penalties

We continue with deterministic symmetric economies. In order to decentralize the constrained efficient allocation in the previous section as a competitive equilibrium, we continue to assume:

(1) Two-sided exclusion following default which is detected with probability one

(2) Assumption 3.1, that is,

$$(1+\beta)u(1) < u(1+\alpha) + \beta u(1-\alpha)$$

Recall also from the previous section that the planner's constrained optimal planner's allocation is

$$c_t^i = \begin{cases} \hat{x} & \text{if} \quad \omega_t^i = 1+\alpha \\ 2-\hat{x} & \text{if} \quad \omega_t^i = 1-\alpha \end{cases}$$

where $\hat{x} \in [1, 1+\alpha]$ is the smallest solution to

$$u(x) + \beta u(2-x) = u(1+\alpha) + \beta u(1-\alpha)$$

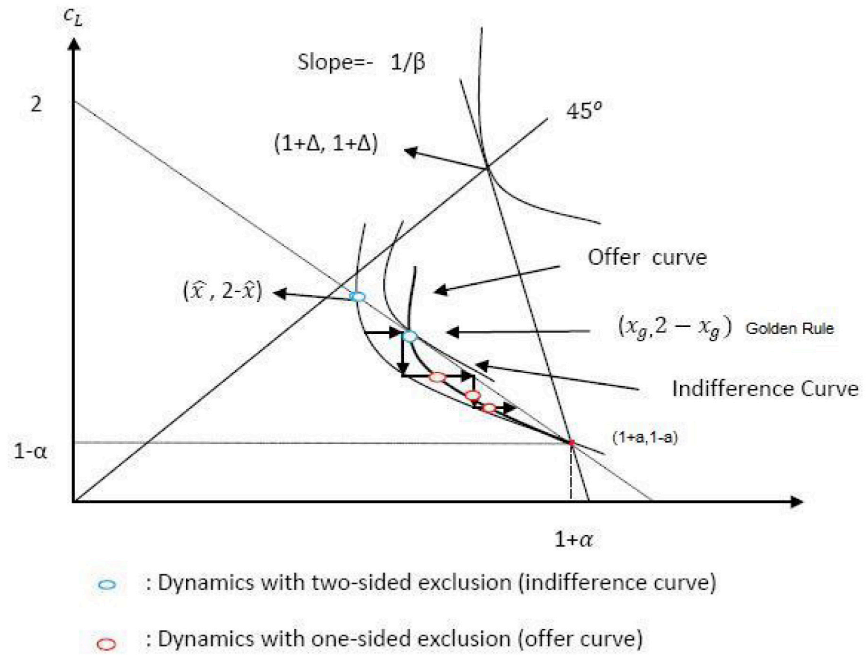


Figure 3.2: Dynamic Equilibria with Limited Enforcement

Note: $\hat{x} = 1+\alpha$ iff autarky is a high-interest-rate allocation. Equivalently, $\hat{x} = 1+\alpha$ iff $1+\alpha < x_g$ where the implied interest yield at x_g equals the zero growth rate of aggregate income or $u'(x_g) = \beta u'(2-x_g)$. This solution is called the golden rule allocation. Suppose next that the only asset is loans in zero supply with, b_t^i = loans hold by agent i which mature at t, and L_t^i = debt limit faced by agent i. Then we define competitive equilibrium as follows.

Definition 3.1: The sequence $(c_t^1, c_t^2, b_t^1, b_t^2, L_t^1, L_t^2, R_t)$ is a competitive equilibrium of the economy with two-sided exclusion if

a) Consumer i solves the following problem:

$$\max_{(c_t^i, b_t^i)} \quad \sum_{t=0}^{\infty} \beta^t u(c_t^i)$$
$$s.t. \quad c_t^i + b_{t+1}^i = \omega_t^i + R_t b_t^i$$
$$-b_t^i \leq L_t^i \qquad \forall\ (t,i)$$

taking (R_t, L_t^i) as exogenous.

b) Markets clear, that is,

(goods) $c_t^1 + c_t^2 = \omega_t^1 + \omega_t^2 = 2$

(assets) $b_t^1 + b_t^2 = 0$

c) Debt limits (L_t^1, L_t^2) are the largest values consistent with solvency payoffs never falling below default payoffs, that is,

$$V_t^S - V_t^D \equiv \sum_{s=0}^{\infty} \beta^s [u(c_{t+s}^i) - u(\omega_{t+s}^i)] \geq 0, \qquad \forall\ (i,t).$$

Next we show that *the constrained optimum and autarky are both stationary equilibria.* In particular, it is easy to show

Lemma 3.1: The constrained optimum allocation can be decentralized as a stationary competitive equilibrium with

$$c_t^i = \begin{cases} \hat{x} & \text{if} \quad \omega_t^i = 1+\alpha \\ 2-\hat{x} & \text{if} \quad \omega_t^i = 1-\alpha \end{cases}$$

$$R_t = \hat{R}, \text{ where } u'(\hat{x}) = \beta \hat{R} u'(2-\hat{x}), \text{ and } \hat{R} \in (1, \frac{1}{\beta})$$

$$b_t^i = \begin{cases} \hat{L} & \text{if} \quad \omega_t^i = 1+\alpha \\ -\hat{L} & \text{if} \quad \omega_t^i = 1-\alpha \end{cases}$$

where $\hat{L}$ satisfies the budget constraint of the high-income agent.

$$\hat{x} + \hat{L} = 1 + \alpha - \hat{R}\hat{L}$$

$$\Rightarrow \quad \hat{L} = \frac{1+\alpha-\hat{x}}{1+\hat{R}} \tag{4}$$

Problem 3.2: Derive eq.(4) for the debt limit $\hat{L}$ by using the budget constraint of the low-income agent.

The next result shows how to decentralize autarky.

Lemma 3.2: Autarky is always an equilibrium with

$$c_t^i = \omega_t^i, \quad \forall\ (i,t)$$

$$L_t^i = 0 \quad \text{if} \quad \omega_t^i = 1-\alpha$$

$$R_t = \frac{u'(1+\alpha)}{\beta u(1-\alpha)} \equiv \bar{R}, \quad \forall\ t$$

In general, dynamical equilibria are sequences (x_t) such that $(c_t^H, c_t^L) = (x_t, 2-x_t)$ and x_t satisfies

(a) $1 \le x_t \le 1 + \alpha$ (incomplete consumption smoothing)

(b) $u'(x_t) = \beta R_{t+1} u'(2 - x_{t+1})$ (the consumption-Euler equation for the high-income household)

(c) $u(x_t) + \beta u(2 - x_{t+1}) = u_A \equiv u(1 + \alpha) + \beta u(1 - \alpha)$ (the participation constraint, when high-income agents are indifferent between solvency and default)

(d) If autarky is a high-interest-rate allocation, that is, $\bar{R} = \frac{u'(1+\alpha)}{\beta u'(1-\alpha)} > 1$, then autarky $x_t = 1+\alpha, \forall\, t$ is the only equilibrium and it is Pareto optimal (see Álvarez and Jermann, 2000). In this case, both welfare theorems hold. Equilibrium is supported by the price $R_t = \bar{R}, \forall\, t$ and debt limits $L_t = 0$.

(e) If autarky is a low-interest-rate allocation, that is, $\bar{R} < 1$, then we have two steady states $x_t \in \{\hat{x}, 1 + \alpha\}, \forall\, t$, plus a continuum of sequences (x_t) that start at $x_0 \in (1, 1 + \alpha)$ and converge to autarky, $x_\infty = 1 + \alpha$. Allocations are Pareto-ranked depending on the initial value x_0. In particular, the allocation $x_t = \hat{x}, \forall\, t$ is constrained optimal. All other equilibria are suboptimal. *The first welfare theorem fails in this case but the second welfare theorem holds.* Market outcomes may or may not be optimal, depending on rational expectations about future credit limits. If those expectations are high, then the market outcome is optimal.

Remark 3.1: Secured (or collateral) borrowing vs unsecured (or reputational)borrowing: "Autarky" should be understood as the complete collapse of unsecured loans. If we had allowed collateral, all actual borrowing would have been against collateral.

Remark 3.2: Why is unsecured borrowing indeterminate? A *dynamic complementarity* exists between current and expected future debt limits. Higher future debt limits raise the value of market participation today, and also the penalty for default which helps enlarge current debt limits. Dynamic equilibria are graphed in Figure 3.2.

Problem 3.2: Suppose autarky is a low-interest-rate allocation and that all households prefer the symmetric Arrow-Debreu outcome $c_t^i = 1, \forall\, (i, t)$ to autarky. Is autarky still a competitive equilibrium? If so, describe fully the set of all dynamic equilibria.

3.3 Equilibrium with a weak default penalty

Bulow-Rogoff (1989) and Hellwig-Lorenzoni (2009) explore equilibria in a limited enforcement economy with unsecured loans in which default is punished by banning offenders from ever borrowing again but permitting them to lend freely at the market interest rate. In this environment the Arrow-Debreu allocation $c_H = c_L = 1$ clearly violates the participation constraint. To see this, check that inequality (7) in lecture 2 fails for $\phi = 0$ and all parameter values (α, β).

Similarly, the allocation achievable by a defaulting high-income agent at the Arrow-Debreu interest rate $1/\beta$, $c_H = c_L = 1+\alpha(1-\beta)/(1+\beta)$, dominates $c_H = c_L = 1$, as shown in Figure 3.2.

Competitive equilibrium in this setting is defined analogously with the previous section: it is any sequence (x_t) such that $(c_t^H, c_t^L) = (x_t, 2 - x_t)$ with x_t satisfying

(a) $1 \le x_t \le 1 + \alpha$ (potentially incomplete consumption smoothing)

(b) $u'(x_t) = \beta R_{t+1} u'(2 - x_{t+1})$ (consumption-Euler equation for high-income persons)

(c) $u(x_t) + \beta u(2 - x_{t+1}) = \max_{s \in [0,1+\alpha]} \{u(1 + \alpha - s) + \beta u(1 - \alpha + R_{t+1} s)\}$ (indifference between default and solvency)

Assuming that autarky is a low-interest rate equilibrium, we have

$$x_t = 1 + \alpha \qquad \text{if} \qquad R_t = \bar{R} = u'(1 + \alpha)/[\beta u'(1 - \alpha)] < 1$$
$$x_t = x_g \qquad \text{if} \qquad R_t = 1$$

Thus, x_t lies in the offer curve through the initial endowment point, as shown in Figure 3.2. Dynamical equilibria are exactly like those of a pure exchange overlapping generations economy with constant population, constant money supply, and a two-period life-cycle endowment $\omega = (1 + \alpha, 1 - \alpha)$.

For example, there exists a *golden rule* steady state $(c_H, c_L) = (x_g, 2 - x_g)$ which permits less consumption smoothing than the constrained-efficient allocation $(c_H, c_L) = (\hat{x}, 2 - \hat{x})$ does under two-sided exclusion. Can you explain why? In addition, we have a constrained inefficient autarky state $(c_H, c_L) = (1 + \alpha, 1 - \alpha)$, and a continuum of equilibria that start with $x_0 \in (x_g, 1 + \alpha)$ and converge to autarky.

We will see in Lecture 4 how all of these allocations can be supported as equilibria in an economy where all loans must be secured by the value of an unproductive capital asset whose flow of dividends is always zero and whose price reflects its value as pure collateral. Figure 3.3 illustrates how equilibrium outcomes depend on default penalties.

The only difference between infinitely-lived agents with one-sided exclusion and overlapping generations is in the type of assets that are needed to support the same competitive allocation. Unproductive assets in positive net supply, like public debt, or equivalent social institutions, like social-security transfers between young and old households, are useful in the life-cycle economy because they encourage intergenerational transfers. Private debt achieves the same goal in an infinite horizon economy with one-sided exclusion *without any help from public institutions.*

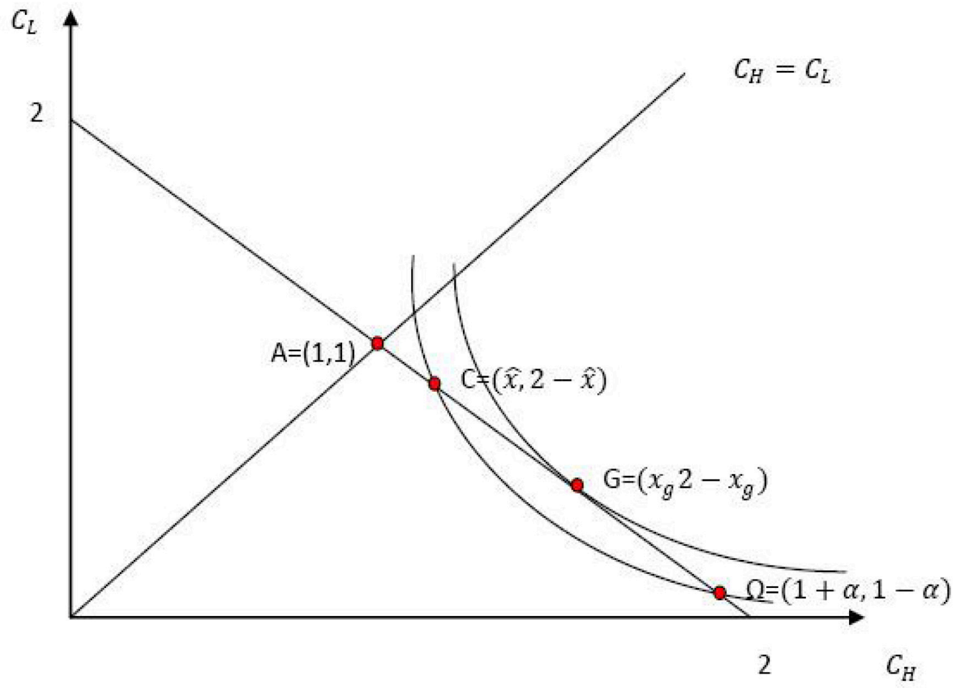


Figure 3.3: Financial Frictions in Exchange Economies

A $\rightarrow$ equal-treatment optimum & Arrow-Debreu equilibrium. Achievable with secured lending if enough collateralizable income

C $\rightarrow$ constrained optimum and stationary competitive equilibrium with unsecured loans and strong default penalty

G $\rightarrow$ golden rule allocation and stationary competitive equilibrium with unsecured lending and weak default penalty or with loans backed by unproductive collateral asset

Ω $\rightarrow$ constrained inef?cient autarkic allocation and stationary competitive equilibrium for all institutional arrangements listed in both C and G

3.4 Stochastic exchange economies with sectoral shocks

In the remainder of this lecture, we extend the material in sections 3.1-3.3 to economies with stochastic individual endowments.

Assume once more that default is punished by perpetual autarky with probability one. Let
$c(i, s^t)$ =consumption for agent i in history s^t
$b(i, s^{t+1})$ =security holdings for i in history s^t contingent on the future state $s^{t+1} = 1, 2$
$p(s^t, s_{t+1})$ =Arrow security price at history s^t for 1 unit of consumption in future state s_{t+1}

$L(i, s^{t+1})$ =debt limit such that $-b(i, s^{t+1}) \leq L(i, s^{t+1})$ for all (i, s^{t+1})

Definition 3.2: A competitive equilibrium is a list $c(i, s^t)$, $b(i, s^t)$, $p(s^t)$, $L(i, s^t)$ such that

a) Given prices and debt limits, agent $i = 1, 2$ maximizes the expected payoff conditional on a given initial state s_0 and subject to budget constraints and debt limits. The formal problem is

$$\begin{aligned} \max_{(c,b)} \quad & V(i, s_0) = E\{\sum_{t=0}^{\infty} \beta^t u[c(i, s^t)] | s_0\} \\ s.t. \quad & -b(i, s^{t+1}) \leq L(i, s^{t+1}), \\ & c(i, s^t) + \sum_{s_{t+1}} p(s^t, s_{t+1}) b(i, s^{t+1}) \leq \omega(i, s_t) + b(i, s^t) \end{aligned}$$

b) Markets clear, that is,

$$\begin{aligned} &\text{(goods)} \qquad c(1, s^t) + c(2, s^t) = 2, \qquad \forall\, s^t \\ &\text{(assets)} \qquad b(1, s^{t+1}) + b(2, s^{t+1}) = 0, \qquad \forall\, s^{t+1} \in (s^t, 1) \times (s^t, 2) \end{aligned}$$

c) Debt limits are the largest values satisfying $V(i, s^t) \geq V_A(i, s^t)$, that is, ones that prevent the solvency payoff from falling below the default payoff for all agents and all possible event histories.

3.4.1 Autarkic equilibria

Idea: financial autarky is always an equilibrium supported by

(i) prices matching the MRS of the most patient agent (highest β, lowest rate of income growth); and

(ii) zero debt limits for all other agents

Theorem 3.1: Autarky is always an equilibrium supported by security prices

$$p(s, s') = \max_i \{\beta \pi(s, s') \frac{u'(\omega(i, s'))}{u'(\omega(i, s))}\}$$

and debt limits

$$L(i, s^{t+1}) = \begin{cases} 0 & \text{if} \quad s_{t+1} = i \\ \infty & \text{if} \quad s_{t+1} \neq i \end{cases}$$

Autarky security prices $p(s, s')$ *are generally higher than the prices* $p^*(s, s')$ *supporting the Arrow/Debreu equilibrium* because security suppliers (i.e., agents desiring short positions) are rationed. In particular, if we assume a symmetric Markov chain, we obtain

$$p(1,1) = p(2,2) = \beta\pi = p^*(1,1) = p^*(2,2)$$
$$p(1,2) = p(2,1) = \beta(1-\pi)\frac{u'(1-\alpha)}{u'(1+\alpha)} > \beta(1-\pi) = p^8(1,2) = p^*(2,1)$$

Therefore, $p(s,s') \geq p^*(s,s')$, $\forall$ (s,s').

Remark 3.1: Once more, we should interpret "autarky" as borrowing against collateral only, that is, as the complete absence of unsecured borrowing and lending.

Remark 3.2: The autarky risk-free yield is the reciprocal of the total cost of a safe portfolio paying off one consumption unit in every future state. Thus autarky reduces safe asset yields below their Arrow-Debreu values.

$$R_F^a \equiv \frac{1}{p(1,1)+p(1,2)} = \frac{1}{\beta[\pi + (1-\pi)\frac{u'(1-\alpha)}{u'(1+\alpha)}]} < \frac{1}{\beta}$$

For $\alpha = 0.1$, $u(c) = log(c)$, $\beta = 0.96$,

$$R_F^a = \begin{cases} 1.02 & \text{if} \quad \pi = 0.9 \\ 0.93 & \text{if} \quad \pi = 0.5 \\ 0.85 & \text{if} \quad \pi = 0 \end{cases}$$

If α is a continuous parameter on $[0,1]$, then $p(1,2) = p(1,2) \to \infty$, and $R_F^a \to 0$ as $\alpha \to 1$. The rate of return on a riskless portfolio falls to -100% . We note that real pre-tax yields an safe U.S. government debt (that is, treasury bills and short-term bonds) have historically averaged 1% per year, far below the sum of the rates of time preference and growth in per capita consumption.

3.4.2 Non-autarky equilibria

In the deterministic economy with two-sided exclusion, these equilibria were solutions to the participation constraint

$$u(x_t) + \beta u(2 - x_{t+1}) = u(1-\alpha) + \beta u(1-\alpha) \qquad \text{for} \qquad 1 \leq x_t \leq 1+\alpha$$

We generalize this result to a symmetric Markov economy with two types of agents and probability $\pi \in [0,1]$ of remaining in the same state.

Define (V_t^H, V_t^L) to be solvency payoffs if $\omega_t^i = 1+\alpha$ and $\omega_t^i = 1-\alpha$, respectively. Also define the corresponding default (or autarky) payoffs to be V_D^H, V_D^L. Suppose also that the high-income agent is indifferent between solvency and default, i.e., $V_t^H = V_D^H$ for all t. Next we let $(c_t^H, c_t^L) = (x_t, 2 - x_t)$ and obtain the recursive relations:

$$\begin{aligned}
V_{t+1}^L &= u(2 - x_{t+1}) + \beta\pi V_{t+1}^L + \beta(1-\pi)V_D^H;\\
V_t^H &= V_D^H = u(x_t) + \beta\pi V_D^H + \beta(1-\pi)V_{t+1}^L\\
&= u(x_t) + \beta\pi V_D^H + \beta(1-\pi)u(2-x_{t+1}) + \beta(1-\pi)\beta\pi V_{t+2}^L + \beta^2(1-\pi)^2 V_D^H\\
&= u(x_t) + \beta\pi V_D^H + \beta(1-\pi)u(2-x_{t+1}) + \beta\pi[(1-\beta\pi)V_D^H - u(x_{t+1})] + \beta^2(1-\pi)^2 V_D^H\\
&= \frac{1}{1-\beta\pi-\beta\pi(1-\beta\pi)-\beta^2(1-\pi)^2}[u(x_t) + \beta(1-\pi)u(2-x_{t+1}) - \beta\pi u(x_{t+1})]
\end{aligned}$$

Similarly, let $(c_D^H, c_D^L) = (1+\alpha, 1-\alpha)$ and obtain

$$\begin{aligned}
V_D^L &= u(1-\alpha) + \beta\pi V_D^L + \beta(1-\pi)V_D^H;\\
V_D^H &= u(1+\alpha) + \beta\pi V_D^H + \beta(1-\pi)V_D^L\\
&= u(1+\alpha) + \beta\pi V_D^H + \beta(1-\pi)u(1-\alpha) + \beta(1-\pi)\beta\pi V_D^L + \beta^2(1-\pi)^2 V_D^H\\
&= u(1-\alpha) + \beta\pi V_D^H + \beta(1-\pi)u(1-\alpha) + \beta\pi[(1-\beta\pi)V_D^H - u(1+\alpha)] + \beta^2(1-\pi)^2 V_D^H\\
&= \frac{1}{1-\beta\pi-\beta\pi(1-\beta\pi)-\beta^2(1-\pi)^2}[(1-\beta\pi)u(1+\alpha) + \beta(1-\pi)u(1-\alpha)]
\end{aligned}$$

Hence, we conclude from (2a) and (2b) that $V_t^H = V_D^H$ leads to

$$u(x_t) + \beta(1-\pi)u(2-x_{t+1}) - \beta\pi u(x_{t+1}) = (1-\beta\pi)u(1+\alpha) + \beta(1-\pi)u(1-\alpha) \qquad (*)$$

Stochastic equilibria are characterized in the following statement:
Theorem 3.2: Let $(c_t^H, c_t^L) = (x_t, 2 - x_t)$ describe the consumption of the two agents at t. Then competitive equilibria for an economy with two-sided exclusion are solutions to the equation

$$u(x_t) + \beta(1-\pi)u(2-x_{t+1}) - \beta\pi u(x_{t+1}) = (1-\beta\pi)u(1+\alpha) + \beta(1-\pi)u(1-\alpha) \qquad (*)$$

with the following properties

(a) $x_t = 1+\alpha$, $\forall\ t$ is an equilibrium

(b) If autarky is a constrained optimal (CO) allocation, then $x_t = 1+\alpha$, $\forall\ t$ is the only solution to (*)

(c) If autarky is not a CO allocation, then there is another steady state of equation (*), $x_t = \hat{x} \in (1, 1+\alpha)$ which is CO and unstable. The value $\hat{x}$ is the same as in the deterministic economy.

(d) If autarky is not a CO allocation, then there is a continuum of equilibria (x_t) indexed on $x_0 \in (1, 1+\alpha)$ all of which converge to autarky. All such equilibria satisfy $x_{t+1} \geq x_t$ and $x_t + x_{t+1} \geq 2$. They can be Pareto-ranked by the initial value x_0.

(e) Security prices reflect the valuation of unrationed agents, i.e., of those households with the strongest desire for future consumption. Specifically,

$$p_t(1,1) = p_t(2,2) = \beta\pi \frac{u'(\hat{x}_{t+1})}{u'(\hat{x}_t)} \le \beta\pi$$
$$p_t(1,2) = p_t(2,1) = \beta(1-\pi) \frac{u'(2-\hat{x}_{t+1})}{u'(\hat{x}_t)} \le \beta(1-\pi)$$

To prove this theorem, readers are asked to solve the following problems.

Problem 3.3:

(a) Define the planner's problem for a stochastic exchange economy with a symmetric two-state Markov matrix and two types of agents.

(b) Suppose the Arrow-Debreu outcome violates the participation constraint. Prove part (b) of Theorem 3.2. Under what conditions is autarky constrained optimal?

(c) Suppose autarky is not constrained optimal. Prove part (c) of Theorem 3.2.

Problem 3.4: Prove parts (d) and (e) of Theorem 3.2 and show that the security prices described in Theorem 3.2 (e) converge to the autarkic prices of Theorem 3.1 for all initial values $x_0 \in (\hat{x}, 1+\alpha)$.

3.5 Asymmetric Economies

We have analyzed in this lecture the behavior of exchange economies with various types of financial frictions that arise from limited loan enforcement. In almost all cases we have assumed some form of *symmetry* in the characteristics of households and sectors: sector 1 in state 1 (or in odd-numbered dates) looks exactly like sector 2 in state 2 (or in even-numbered dates). Symmetry simplifies the analysis of financial frictions, allowing the payoffs from default to be the same for all high-income agents no matter what type or sector they belong to. By-products of this symmetry are endogenous debt limits that may depend on the history of events but not on agent type. Another by-product is the existence of *simple invariant states*, that is, deterministic steady states like those of section 3.2 and 3.3, and stochastic ones like those of sections 3.5 and 3.6.

As we find out next, *steady-state equilibria do not generally exist in asymmetric economies* with binding debt limits for the simple reason that debt limits differ from one household to the next. The easiest way to see this is to inject some taste asymmetry into the deterministic economy of section 3.2 with strong default penalties. Suppose everything is as described in that section except payoffs are heterogeneous

$$V_i = \sum_{t=0}^{\infty} \beta^t u_i(c_t^i)$$

with a common discount rate $\beta \in (0,1)$ and different utility flows u_i. Let each agent prefer the consumption vector $(1+\alpha, 1-\alpha)$ to $(1,1)$, that is,

$$(1+\beta)u_i(1) < u_i(1+\alpha) + \beta u_i(1-\alpha) \qquad \forall\, i$$

Equilibria are defined as in Definition 3.1, by the consumption sequence (x_t) of the high-income household which satisfies:

i) incomplete consumption smoothing, $x_t \in [1, 1+\alpha]$

ii) the Euler equation

$$\begin{aligned} u_2'(x_t) &= \beta R_{t+1} u_2'(2 - x_{t+1}) \qquad \text{if} \qquad t = 0, 2, \ldots \\ u_1'(x_t) &= \beta R_{t+1} u_1'(2 - x_{t+1}) \qquad \text{if} \qquad t = 1, 3, \ldots \end{aligned}$$

iii) the participation constraint as an equality

$$\begin{aligned} u_i(x_t) + \beta u_i(2 - x_{t+1}) &= u_i(1+\alpha) + \beta u_i(1-\alpha) \\ i &= 2, \qquad \text{for} \qquad t = 0, 2, \ldots \\ i &= 2, \qquad \text{for} \qquad t = 1, 3, \ldots \end{aligned} \tag{2}$$

Autarky is the only steady state that satisfies equations (2). Besides autarky, there is also constrained optimal periodic state

$$\begin{aligned} x_t &= x_2 \quad \text{if} \quad t = 0, 2, \ldots \\ &= x_1 \quad \text{if} \quad t = 1, 3, \ldots \end{aligned}$$

This state is the unique solution to the equations

$$u_2(x_2) + \beta u_2(2 - x_1) = u_2(1+\alpha) + \beta u_2(1-\alpha) \tag{3a}$$

$$u_1(x_1) + \beta u_1(2 - x_2) = u_1(1+\alpha) + \beta u_1(1-\alpha) \tag{3b}$$

It is shown in Figure 3.4

Another example of asymmetric equilibrium is to keep tastes identical for all households but allow asymmetric transition probabilities $\pi(1,1) \neq \pi(2,2)$. In that case, the sector with the more persistent shocks, that is, with the highest $\Pi(i,i)$, will be more tempted to default, and will be assigned a lower debt limit. For more details an asymmetric economies, consult Azariadis and Kaas (2007), "Asset price fluctuations without aggregate shocks", JET, 136, 334-349.

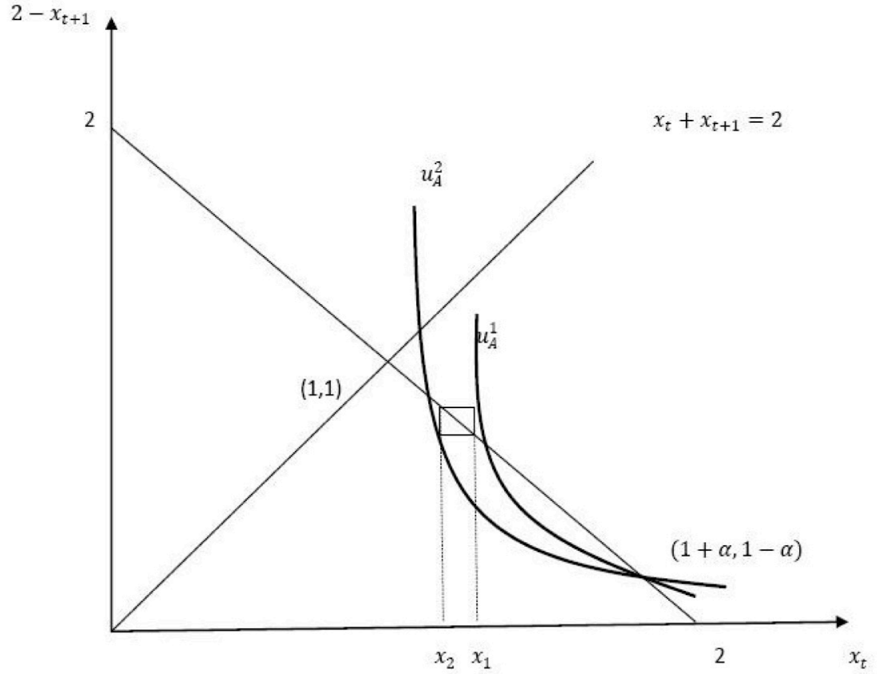


Figure 3.1

3.6 Can policy prevent financial panics and other bad equilibria?

We return to the deterministic economy of section 3.2 where the first welfare theorem fails. In this environment markets may duplicate the constrained optimal outcome as a best-case scenario or may lead society towards "autarky", that is, toward an inefficient situation with no unsecured lending whatsoever which resembles financial panics. This worst-case scenario seems more probable because constrained optimality presupposes a shared belief that credit limits will retain at their highest possible value *in perpetuity.* If at any point in time, households come to expect that debt limits will stray below this maximal value at any point in the future, however briefly or little, unsecured lending will unravel towards zero and the economy will head for "autarky".

Good outcomes in these economies require private beliefs about future credit conditions to be perpetually at their most optimistic. Why should they be? We can ask this question another way: Are there policy rules or possible interventions that will convince households to be optimistic about future credit conditions? If so, these policies help co-ordinate expectations and select good outcomes over bad ones.

Policymakers with a lot of power, information and credibility will easily defeat suboptimal equilibria. For example, a transfer $\hat{\tau} = 1 + \alpha - \hat{x}$ from high-income agents to low-income ones will change the pre-transfer endowment vector in Figure 3.1 to the constrained optimal allocation $(\hat{x}, 2-\hat{x})$. This intervention makes post-transfer "autarky" $(\hat{x}, 2-\hat{x})$ a high-implied interest rate allocation and a unique competitive equilibrium with no borrowing or lending. Credit markets become unnecessary and inactive.

Another intervention is for the government or capital banks to act as the lender of last resort. Following the advice of the British economist Walter Bagehot in Lombard Street (1873), the central bank should endeavor to "stay the panic" by "loans ... made at a very high rate of interest" against securities that in normal times and regarded as "commonly pledged and easily convertible". Suppose, in particular, that the government declares itself ready to trade private securities against its own liabilities at the "very high" interest yield $\hat{R}$ which supports the constrained optimum. If households take this announcement seriously, they will trade with the central bank at $\hat{R}$ and keep the economy at its constrained best. Lenders will volunteer loans to the central bank, because that bank is unlikely to default for various reasons. Borrowers will be happy to trade, too, because the high interest yield $\hat{R}$ comes together with high debt limits $\hat{L}$. In this scenario, the central bank crowds out all need for private intermediation and private banking, and successfully forestalls all deviations from optimal outcomes. This policy is *closer to lending of first rather than last resort.*

A realistic description of monetary/fiscal policy should regard it as a complement to private borrowing and lending and not as a replacement for private intermediaries. If the central bank could intermediate as efficiently as the private sector, it would defeat instantly all market departures from socially optimal outcomes. In order to understand last-resort lending, the first order of business seems to be a listing of feasible central bank or government policies, that is, of the resources, information, and instruments available to the policymaker depending on the state of the economy. Policy choice is then a selection from the set of available policies that maximizes the policymaker's payoff subject to the incentive or participation constraints that make policy choices sustainable or self-enforcing.

Rational policies designed to avoid the collapse of unsecured lending are a difficult and interesting subject. As a preliminary attempt that falls short of the plan laid out in the previous paragraph, let us assume that feasible government policies are described by a set of values connecting the current state of the economy, x_t, with changes in the stock of currency or public debt, M_t. Consider, for example, the family

$$(M_{t+1} - M_t)/M_t = \mu(1 + \alpha - x_t) \tag{4}$$

where $\mu : [0, 1 + \alpha - \hat{x}] \to R_t$ is a policy that maps the flow of private loans, into the rate of growth of nominal currency outstanding. One way to interpret these policies is to suppose that the central bank issues liabilities to purchase goods from high-income households or "lenders", and gives those goods away to low-income households or "borrowers".

Transfers are a pure gift, and the government expects no payment of principal or interest in return. The idea behind these policies is to facilitate the flow of resources from people with an income surplus to those with an income shortage. The central bank does not act as a financial intermediary or lender of last resort; it just levies on inflation tax on the “haves” to subsidize the “have nots”.

The subsidy to each low-income agent at t is

$$\tau_t = q_t(M_{t+1} - M_t) \tag{5}$$

where q_t is the price of money in term of consumption goods. Suppose also that default on private loans does not compromise the flow of subsidies; defaulters are excluded forever from both sides of the private loan market but can still use currency as a store of value. The payoff from default for a high-income agent at t is

$$\begin{aligned} V_t^D &= \max_{\hat{M}_s^i} \sum_{s=t}^{\infty} \beta^{s-t} u(\hat{c}_s^i) \\ s.t. \qquad & \hat{c}_s^i + q_t \hat{M}_{s+1}^i = \omega_s^i + q_t \hat{M}_s^i + \tau_s^i \end{aligned} \tag{6a}$$

$$\begin{aligned} &\omega_t^i = 1 + \alpha, \qquad \hat{M}_t^i = M_\tau^i, \qquad \text{given} \\ &\tau_s^i = q_s(M_{t+1} - M_t) \qquad \text{if} \qquad \omega_s^i = 1 - \alpha \\ &\quad = 0 \qquad \text{if} \qquad \omega_s^i = 1 + \alpha \end{aligned} \tag{6b}$$

In economies without private lending, all households solve this problem and market equilibrium implies

$$\hat{M}_t^1 + \hat{M}_t^2 = M_t \tag{7}$$

3.7 Policy in a pure currency economy

Private lending is socially essential to support a constrained optimal allocation $(\hat{x}, 2 - \hat{x})$ if that allocation is not an equilibrium of the pure currency economy laid at the end of previous section. In fact, it is easy to show that the best stationary consumption allocation in a pure-currency economy is the golden rule $(x_g, 2 - x_g)$ which provides less smooth consumption than the constrained optimum. In this particular sense, private loans are desirable. To see this, we start from the first-order condition for the household problem laid out earlier:

$$u'(\hat{c}_t^i) \geq \beta(q_{t+1}/q_t)u'(\hat{c}_{t+1}^i), \qquad = \text{if} \qquad \hat{M}_{t+1}^i > 0 \tag{8}$$

where q_{t+1}/q_t is the gross yield on currency purchased at time t. If both sectors purchase currency at each t, then all households have constant consumption in equilibrium, that is,

$$\hat{c}_t^i = c_i^* \qquad \forall\, t$$

because aggregate consumption is constant. We know that complete consumption smoothing violates the participation constraint

$$(1+\beta)u(c_i^*) \geq u(1+\alpha) + \beta u(1-\alpha) \qquad \forall\, i$$

Therefore we assume that the F.O.C. in (8) holds as an equality for the high-income agent only who buys the entire stock of currency from the low-income agent. We then rewrite (8) in the form

$$q_t u'(c_t^H) = \beta q_{t+1} u'(c_{t+1}^L) \tag{9a}$$

and the budget constraint (6a) as

$$c_t^H + q_t M_{t+1} = 1 + \alpha$$

This one implies

$$c_t^H = 1 + \alpha - q_t M_{t+1} \tag{9b}$$
$$c_{t+1}^L = 1 - \alpha + q_{t+1} M_{t+2} \tag{9c}$$

Putting (9b,c) into (9a) we obtain

$$q_t u'(1+\alpha - q_t M_{t+1}) = \beta q_{t+1} u'(1-\alpha + q_{t+1} M_{t+1}) \tag{10}$$

In fact, passive policies of the form $M_t = 1$, $\forall\, t$ are consistent with equilibrium price sequences that solve

$$q_t u'(1+\alpha - q_t) = \beta q_{t+1} u'(1-\alpha + q_{t+1}) \tag{11}$$

The resulting allocations $(x_t, 2 - x_t)$ are exactly the same as those described in section 3.5 for an economy with private lending and weak default penalties.

Dynamical equilibria in this economy are described by the offer curve for the high-income household, as shown in Figure 3.5.

Given an active policy rule $\Delta : [0, 1 + \alpha - \hat{x}] \to \mathcal{R}_+$, equilibrium is a solution sequence (M_t, q_t) to the system of equations (4) and (10) where (4) may be recast as

$$M_{t+1} = M_t[1 + \mu(q_t M_t)] \tag{4'}$$

with $M_0 \geq 0$ given. Solutions will depend on the choice of policy rule.

Let's start by noting that (4') and (10) have two stationary solutions: autarky with zero price of money is supported by $(q_t, M_t) = (0, M)$, $\forall\, t$ for any $M \geq 0$; and the golden rule

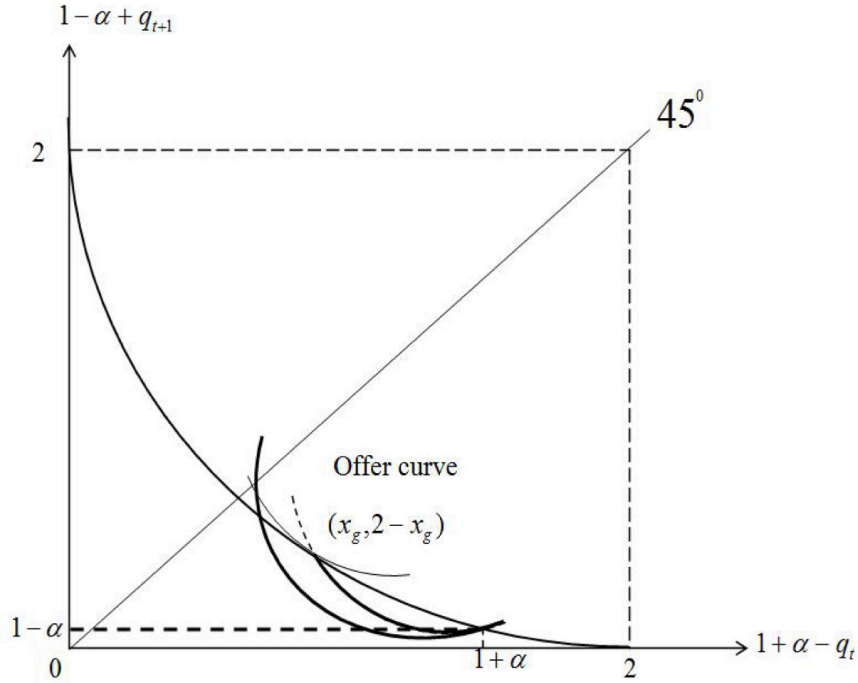


Figure 3.5: Equilibria in a Pure-Lending Economy with a constant stock of money

allocation $(x_g, 2 - x_g)$ with positive price of money is supported by $(q_t, M_t) = (q, M) > 0$ for all t for any (q, M) values satisfying:

$$qM = 1 + \alpha - x_g \tag{11}$$

Both states are consistent with our policy rule if we require:

$$\mu(1 + \alpha - x_g) = \mu(0) = 0 \tag{12}$$

The constrained efficient allocation $(\hat{x}, 2 - \hat{x})$ would require exploding currency prices q_t that satisfy:

$$q_{t+1}/q_t = \hat{R} \equiv u'(\hat{x})/[\beta u'(2 - \hat{x})] > 1 \quad \forall\, t \tag{13}$$

To achieve such prices, the stock of currency should shrink over time by taxing low-income agents and destroying the revenue. The law of motion for the stock of currency can be read from eq(10), i.e.,

$$q_t M_{t+1} = 1 + \alpha - \hat{x}$$

This solution implies an infeasible negative subsidy $\tau < 0$ on the endowment of low-income households.

We conclude that good sustainable monetary policies in a pure currency economy should promote the golden-rule outcome as a stable steady state and rule out equilibria that converge to autarky. Policies in a pure-currency economy can be ranked according to the speed of convergence to the target outcome; the faster we get away from autarky, the better off households are. The best policy in this setting of no private loans is to move to the golden rule in one step from any initial state (q_t, M_t) with $q_t > 0$.

Is immediate convergence a feasible policy? If so, what μ function do we need for this outcome? The answer is yes, and the required policy rule

$$\frac{M_{t+1}}{M_t} = 1 + \Delta(1 + \alpha - x_t) = \frac{(1 + \alpha - x_g)}{(1 + \alpha - x_t)} \frac{u'(x_g)}{u'(x_t)} \tag{14a}$$

is a non-negative currency growth rate that accelerates with the distance of consumption from the golden rule, and drops to zero once we reach that point. The implied path for the stock of currency stock is constant and policy is inactive after an initial intervention at time $t = 0$.

$$\begin{aligned} &M_t = M_0 \quad t \geq 0 \quad \text{if} \quad x_0 = x_g \\ &M_t = M_0[1 + \mu(1 + \alpha - x_0)] \quad t \geq 1 \quad \text{if} \quad x_0 \in (x_g, 1 + \alpha) \end{aligned} \tag{14b}$$

(Proof) Fix (q_t, M_t) arbitrarily s.t. $(c_t^H, c_t^L) > 0$ and find (q_{t+1}, M_{t+1}) that obey the budget equations (9a), (9b) and the first order condition (10). In particular, put

$$c_t^H \equiv x_t, \quad c_{t+1}^L = 2 - x_g$$

eliminate (q_t, q_{t+1}) from the resulting system of equations and solve for M_{t+1}. The policy suggested in equation (13) guarantees instant convergence to the golden rule from any initial position $x_0 \in (x_g, 1 + \alpha)$, and effectively reduces the set of rational expectation equilibria to just autarky and the the golden rule. All that is needed is a credible commitment to the rule of eq(4). Successful as this policy may be in eliminating equilibrium paths that converge to autarky, it does not deliver a constrained optimum in economies with strong default penalties. For that purpose, we need a robust amount of private lending working under an active policy assuming households that favorable credit conditions will prevail in the future.

3.8 Policy in economies with private lending

We return to the issues raised in the last paragraph of section 3.8 and explain the use of public liabilities in starving off a potential collapse of private lending. Suppose the government or central bank wants to issue a positive amount of currency or public debt to keep the economy away from an undesirable path toward autarky. Public liabilities would be sold, as in section 3.9, to potential lenders and the proceeds distributed as lump-sum transfers to potential borrowers. To compete with private debt, government liabilities would have to pay the same return. If those liabilities are currency, there would be no private demand for it unless

$$q_{t+1}/q_t = R_{t+1} \qquad \text{if} \qquad M_{t+1} > 0 \tag{15}$$

that is, unless private loans and money paid the same yield. Assume in addition that the central bank can not distinguish solvent borrowers of private loans from defaulters, paying each and every borrower a non-negative subsidy τ_t as described in equation (5) and permitting everyone to buy and sell assets.

This environment does not encourage mixed portfolios of private and government liabilities. One problem is that the no-arbitrage condition (15) cannot hold at the constrained optimal allocation $(\hat{x}, 2-\hat{x})$ because the implied interest yield $R_t = \hat{R} > 1$ for all t would require a geometric increase in the price of money to the point where currency is worth more than the aggregate resources of all lenders. This problem goes away at any low yield $R_t \leq 1$ but it is replaced by the specter of default on a private debt. Defaulters receive a subsidy like everyone else, and may use currency holdings as a consumption smoothing buffer shock. Private lending collapses and it is completely replaced by fiat money.

It is not clear that what policies are optimal in this setting. One sensible thing to do is to issue no currency when the economy is doing better than the golden rule, allowing private loans to smooth private consumption. In that case, the no-arbitrage condition (15) is irrelevant. When the state of economy becomes worse than the golden rule, the central bank prints a one-time addition to the pre-existing stock following the policy rule similar but not identical to eq (14b). The goal in this case is to jump from any $x_t > x_g$ to the constrained optimal outcome $x_{t+1} = \hat{x}$. Proceeding as in section 3.9, we obtain the policy rule:

$$
\begin{aligned}
&M_{t+1}/M_t = 1 \qquad \text{if} \qquad x_t \in (\hat{x}, x_g) \\
&M_{t+1}/M_t = \frac{1+\alpha-\hat{x}}{1+\alpha-x_t}\frac{u'(\hat{x})}{u'(x_t)}\frac{1}{\hat{R}} \qquad \text{if} \qquad x_t \in (x_g, 1+\alpha)
\end{aligned}
$$

The law of motion for the state $c_t^H = x_t$ becomes:

$$
\begin{aligned}
x_{t+1} &= x_t \quad \text{if} \quad x_t = \hat{x} \\
x_{t+1} &= \hat{x} \quad \text{if} \quad x_t \in (x_g, 1+\alpha) \\
u(x_t) + \beta u(2 - x_{t+1}) &= u(1+\alpha) + \beta u(1-\alpha) \quad \text{if} \quad x_t \in (\hat{x}, x_g)
\end{aligned}
$$

It is illustrated by the bold lines in Figure 3.6 where there are two steady states: autarky, which corresponds to a low value of currency or public debt, and the constrained optimum value $\hat{x}$. As in earlier sections, a good activist policy has made autarky an unstable outcome and the constrained optimum into a stable one. Deviations from optimality vanish over several periods. In particular, large initial deviations are corrected by the government issuing liabilities on behalf of households that cannot find private loans. The central bank acts now as a borrower of last resort.

Figure 3.6: The government as last-resort borrower