Direct Complementarity

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How should we define complementarity?

- Let preferences $\succeq$ on $\mathbb{R}^n$ (bundle space) be represented by a smooth function $u : \mathbb{R}^n \to \mathbb{R}$. Denote partial derivatives by u_i, u_{ij}, etc.
- Naively, we might try classifying goods i and j as complements or substitutes according to the sign of u_{ij}. (Appears in Auspitz-Lieben (1889), also Edgeworth, Pareto.)

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- Naively, we might try classifying goods i and j as complements or substitutes according to the sign of u_{ij}. (Appears in Auspitz-Lieben (1889), also Edgeworth, Pareto.)
- **Problem**: This is sensitive to the choice of representation: if $u_i u_j \neq 0$, we can make the sign of u_{ij} whatever we want by replacing u with $f \circ u$ for smooth increasing f. (Noticed at least as early as Slutsky (1915).)
- If $v = f \circ u$ then

$$v_{ij} = f' u_{ij} + f'' u_i u_j$$

- Interestingly, if(f) $u_i u_j = 0$, then $\operatorname{sgn}(u_{ij})$ *is* invariant to representation. More on this later.

Demand-Based Definitions

To fill the vacuum, we have:

- **Gross Complementarity** of goods i and j: Negative uncompensated cross-price effect:

$$\partial x_i / \partial p_j < 0$$

  with prices p_{-j} and nominal income y fixed.

- **Hicks-Allen Complementarity** of goods i and j (1934): Negative compensated cross-price effect:

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- *Samuelson's Complaint* (1974): These definitions don't feel like they are about complementarity, except indirectly.
- Stigler (1950) said it was "difficult to see the purpose" in the Hicks-Allen definition. A little harsh.
- If possible, we would like a definition more closely tied to preference. Maybe by choosing a distinguished representation?

Definition of Direct Complements, Quasilinear Case

Consider quasi-linear utility function

$$u(x) = x_0 + f(x_1, \ldots, x_k)$$

Let H be the Hessian matrix for f; assume H invertible.
Cross-price effects on goods $1, \ldots, k$ are given by the matrix H^{-1}.

- Goods i, j are *direct complements* iff $H_{ij} > 0$.

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- Goods i, j are Hicks-Allen/gross complements iff $H^{-1}_{ij} < 0$.[1]

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- **Idea**: The vector space of possible bundles is fundamental. "Goods" are just one choice of basis for this space.
- There is a unique definition of *direct complementarity* which...
  1. Matches the definition we just made in the quasilinear case.
  2. Is determined by first and second derivatives of utility at a given point.
  3. Is invariant to changes of basis.
- Also, it has other appealing equivalent definitions.

Common Ground – The Three-Good Quasilinear Case

Let

$$u(x_0, x_1, x_2) = x_0 + f(x_1, x_2)$$

At each point where preferences are locally convex, these are equivalent:

- Gross complementarity of Goods 1 and 2
- Hicks-Allen complementarity of Goods 1 and 2
- Direct complementarity of Goods 1 and 2, i.e. $u_{12} > 0$

This family of examples confirms the intuition which motivates the demand-theory definitions of complementarity. But it is very special:

Appearance of indirect demand effects – The Four-Good Quasilinear Case

Let

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- Hicks-Allen complementarity of goods i, j is *not* equivalent to $u_{ij} > 0$
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- Intuiton: The market for Good 3 allows for "indirect" cross-price effects between Goods 1 and 2
- If we let

$$H = \begin{pmatrix} -1 & -\varepsilon & \gamma \\ -\varepsilon & -1 & \delta \\ \gamma & \delta & -1 \end{pmatrix}$$

  where $\gamma\delta > \varepsilon > 0$, we find 1,2 are direct substitutes but Hicks-Allen complements.

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- Restauarant M': Three goods: drinks, fries, "meal deal". Quantities $z = (x_1 - x_3, x_2 - x_3, x_3)$, prices $q = (p_1, p_2, p_1 + p_2 + p_3)$. Identical menus, represented differently.

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- Cross-price effects on drinks-fries differ at restaurants M and M':

$$\frac{\partial z_2}{\partial q_1} = \frac{\partial z_2}{\partial p_1} - \frac{\partial z_2}{\partial p_3}$$

$$= \frac{\partial x_2}{\partial p_1} - \frac{\partial x_2}{\partial p_3} - \frac{\partial x_3}{\partial p_1} + \frac{\partial x_3}{\partial p_3} \neq \frac{\partial x_2}{\partial p_1}$$

- ∂q_1 is different from ∂p_1; different things are fixed!
- Similarly, "Effect on z_2" has different meaning from "Effect on x_2".

Basis-Sensitivity: What's going on?

- Recall that cross-price effects are also second derivatives of the expenditure function:

$$\frac{\partial x_2}{\partial p_1} = \frac{\partial x_1}{\partial p_2} = \frac{\partial^2 E}{\partial p_1 \partial p_2}$$

where $E(p, u)$ is the minimum expenditure to achieve u at prices p.

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- Crucially, price vectors do not lie in bundle space; they lie in its *dual*, i.e. price is a linear functional from bundles to $\mathbb{R}$
- Standard complementarity *really* looks at complementarity between *dual* vectors (in their effect on E), then relies on an isomorphism between a vector space and its dual...but this isomorphism is *non-canonical*, i.e. basis-dependent.

Basis-Sensitivity: What's going on?

- Intuitively "Increase the price of fries by 1¢" does not have definite meaning, because you need to specify what you hold fixed (the basis).
- Even more obviously, "increase the price of a meal deal" is completely unclear as to what's held fixed. But complementarity should have definite meaning for "composite goods" as well.
- NB the basis-dependence here is not mere dependence on what goods are available (the span of all goods); it is dependence on how available goods are *expressed*. This is *ugly*. (*Weinstein's Complaint*)
- On the other hand, "I'll have another fry" has *basis-free* meaning. To give basis-free meaning to complementarity of a marginal fry with a marginal drink, we must work in bundle-space, not its dual, price-space.

The Advantage of Generality

- Instead of defining complementarity only for pairs of goods i, j, it's cleaner to do so for all pairs of vectors $(v, w) \in V \times V$
- Or, even better, for any element of the tensor space $V \otimes V$
- Also, instead of looking at one utility function, we'll look simultaneously at the set of all functions representing the same preference

Basis-Free Notation: First Derivatives

- All derivatives are taken at a fixed point x, which is often suppressed in notation
- $Du : V \to \mathbb{R}$ denotes the linear functional for which $Du(v)$ is the directional derivative in direction v
- $Du \in V^*$ is the basis-free analogue of the gradient

The Basis-Free Point of View: Second Derivatives

- The second derivative, $D^2u : V \times V \to \mathbb{R}$ is a (symmetric) bilinear form such that $D^2u(v, w)$ is the cross-partial taken in directions v, w.
- Equivalently, $D^2u \in (V \otimes V)^*$ is a linear functional on the tensor product $(V \otimes V)$
- For any given basis, D^2u is represented by the Hessian matrix –

$$D^2u(v, w) \equiv vHw^T$$

– but it is fundamentally a basis-free object.

Refresher on Tensor Spaces

- $(V \otimes V)$ is a vector space generated by expressions $v_1 \otimes v_2$ for any $v_1, v_2 \in V$
- Bilinearity relations hold, of the form $v \otimes (\alpha w + \beta z) = \alpha v \otimes w + \beta v \otimes z$
- $(V \otimes V)$ has dimension n^2; for any basis $\{e_1, ..., e_n\}$ of v, the set $\{e_i \otimes e_j\}$ is a basis for $(V \otimes V)$.
- A bilinear form on V is equivalent to a linear functional on $(V \otimes V)$.

Calculus on Ordinal Functions – First Derivatives

- Let $u : V \to \mathbb{R}$ be a C^∞ function on a finite-dimensional real vector space V
- Write $u \sim \hat{u}$ if $\hat{u} = f \circ u$ for a C^∞ function $f : \mathbb{R} \to \mathbb{R}$ with $f' > 0$ everywhere. Write $[u]$ for the associated equivalence class (an "ordinal C^∞ function").

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- Chain rule says $D\hat{u}(x) = f'(u(x))Du(x)$. So

$$D[u](x) = \{\alpha Du(x) : \alpha \in \mathbb{R}^+\} \in V^*/\mathbb{R}^+$$

  i.e. the derivative is defined up to positive scalar.
- $D[u](x)$ corresponds, canonically, to a half-space in V, the "goods," bounded by the "indifference plane," $I = \mathrm{Ker}(Du(x))$

Calculus on Ordinal Functions – Second Derivative

- Again, let $\hat{u} = f \circ u$, then (chain rule)

$$D^2\hat{u}(v,w) = f'(u(x))D^2u(v,w) + f''(u(x))Du(v)Du(w)$$

$$D^2\hat{u} = f'(u(x))D^2u + f''(u(x))[Du \otimes Du]$$

$$D^2[u] = \{\alpha D^2u + \beta(Du \otimes Du) : \alpha \in \mathbb{R}^+, \beta \in \mathbb{R}\}$$

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- Define the "first-order-indifferent tensors"

$$I^2 := \mathrm{Ker}(Du \otimes Du) = \mathrm{Span}(I \otimes V \cup V \otimes I) \subseteq (V \otimes V)$$

- On I^2, $D^2[u]$ is well-defined up to the positive scalar α
- So $D^2[u]$ corresponds, canonically, to a half-space in I^2, bounded by $N := I^2 \cap \mathrm{Ker}(D^2u)$

Calculus on Ordinal Functions: Summarizing First and Second-Order Information

- First-order: $D[u]$ labels each $v \in V$ as good, indifferent, or bad. It can be summarized by the half-space of "goods" bounded by the indifference plane $I \in V$.
- $D[u]$ also defines "first-order-indifferent tensors" $I^2 \subseteq (V \otimes V)$

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- First-order: $D[u]$ labels each $v \in V$ as good, indifferent, or bad. It can be summarized by the half-space of "goods" bounded by the indifference plane $I \in V$.
- $D[u]$ also defines "first-order-indifferent tensors" $I^2 \subseteq (V \otimes V)$
- Second-order: $D^2[u]$ labels each tensor in I^2 as complementary, neutral, or substitutive. $D^2[u]$ can be summarized by half-space of complementary tensors $C \subseteq I^2$, bounded by the neutral plane $N \subseteq I^2$.
- N is the set of tensors which are first-and-second-order neutral: a space of codimension 2 in $(V \otimes V)$
- This is *all* the first and second-order information preserved by equivalence.

Complementarity of Neutrals

$$\operatorname{sgn}(D^2[u](x)(v_1, v_2)) \text{ is well-defined} \Leftrightarrow (Du(x)(v_1))(Du(x)(v_2) = 0$$

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- That is, the sign of a cross-partial is well-defined iff one of the "goods" is actually a neutral
- Intuition: Taking $Du(x)(v_1) = 0$, $D^2u(x)(v_1, v_2) > 0$ means that heading in direction v_2 converts v_1 from a neutral to a good. Logically, this property refers to preference, not representation.

An alternate summary of $D^2[u](x)$

In generic cases, $D^2[u](x)$ can be represented as

1. A bilinear form on I, defined up to positive scalar, together with
2. A vector $v_x^* \notin I$, the vector of "income effects," or "numeraire," defined up to scalar, satisfying $D^2[u](x)(v_x^*, w) = 0$ for all $w \in I$

An equivalent definition of v_x^*: Movement in direction v_x^* has no first-order effect on MRSs, i.e. leaves $D[u](x)$ unchanged up to a scalar

Summarizing $D^2[u](x)$, continued (time permitting)

- Locally, preferences are convex iff $D^2u(x)(v,v) < 0$ for all $v \in I$, i.e. D^2u is negative-definite on I. In this case $-D^2u$ is an inner product on I, unique up to scalar.
- In a corresponding orthonormal basis, any two distinct basis elements are second-order neutral.
- Viewed in this basis, elements of I are direct substitutes if the "angle" between them is less than $\pi/2$, direct complements otherwise.

Direct Complements, General Case: Definition A

- Let v_x^* be the numeraire as before
- There is then a "locally quasilinear" representation u^x such that $D^2u^x(v_x^*, v) = 0$ for all v
- By analogy with the quasilinear case, we call w, z *direct complements* at x if

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- The choice among such representations doesn't matter
- One choice of such representation is "money-metric" utility for prices p which induce demand x. This is one proposal of Samuelson (1974).

Direct Complements: Equivalent Definition B

- Any bundle w can be decomposed as

$$w = \lambda_w v_x^* + w^n$$

where v_x^* is the numeraire as before and w^n is a neutral, meaning $Du(x)(w^n) = 0$.

- Bundles are composed of **nutrients** (utility-rich at first-order, second-order-neutral) and *flavor* (first-order-neutral, with second-order impact).
- Direct complementarity of (w, z) at x is equivalent to

$$D^2u(w^n, z^n) > 0 \Leftrightarrow D^2u(w, z^n) > 0 \Leftrightarrow D^2u(w^n, z) > 0$$

for *any* representation u.[2]

- It is the flavors which are complements (or substitutes.)

[2] Derivatives taken at x as usual.

Direct Complements: Equivalent Definition C

There is a unique way of partitioning the set of pairs of bundles, V^2, into three classes C, N, S (complement, neutral, substitute) such that:

1. **Symmetry**: (v, w) and (w, v) are always in the same class.
2. **Respects unanimity**: If
$$D^2u(v, w) > 0$$
for *all* smooth representations u, then $(v, w) \in C$. Similarly, if the cross-partial is 0 for all u then $(v, w) \in N$, and if negative then $(v, w) \in S$.
3. **Convexity and scale invariance**: If $(v, w) \in C$ and $(v, z) \in C \cup N$, then $(v, \alpha w + \beta z) \in C$ for any $\alpha > 0, \beta \geq 0$. Similarly for S.
4. **Neutrality**: If v^* satisfies
$$\frac{\partial MRS_{v,w}}{\partial v^*} = 0$$
for all v, w, then $(v^*, v) \in N$ for all v.

Direct Complements: Equivalent Definition D

- Direct complementarity of (w, z) is equivalent to

$$D(MRS_{w,v_x^*})(z) > 0 \Leftrightarrow D(MRS_{z,v_x^*})(w) > 0$$

- Note that if we used some other numeraire in place of v_x^*, we would not get a symmetric definition.

A few more advertisements

- Assuming convexity, each good is a direct substitute for itself, as is logical. (Under the Hicksian definition, every good is a complement for itself, a point normally avoided by refusing to apply the definition in this case.)
- Even if you are comfortable fixing a set of basis goods, and mostly care about cross-price effects, the concept of direct complementarity helps you understand how complementarity in *preferences* leads to cross-price effects.
- After estimating a matrix of cross-price effects, inverting the matrix to find out about direct complementarity should give insight into which goods are "really" complementary.