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RESEARCH ARTICLE

Best-reply sets

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Abstract
We provide results concerning, in a given normal-form game, which sets of actions are best replies to some belief. Proposition 1 states that for any set S of actions, there is a belief under which all actions in S are simultaneously best replies if and only if no mixture of actions in S is strictly dominated. Similarly, Proposition 2 states that for any set S of actions, there is a full-support belief under which all actions in S are best replies if and only if no mixture of actions in S is weakly dominated. One important consequence is Corollary 1: a two-player game has a totally mixed Nash equilibrium if and only if neither player has a pair of mixed strategies such that one weakly dominates the other.

Keywords Game theory · Normal-form games · Best replies · Rationalizability

JEL Classification C72

1 Introduction

The best-reply correspondence of a game—the mapping from beliefs over one's opponents' actions to one's own optimal actions—is fundamental to game theory. Indeed, all major solution concepts are completely determined by the best-reply correspondences of a game: rationalizability, iterated admissibility, correlated equilibrium, and Nash equilibrium. In that sense, the best-reply correspondence extracts all strategically relevant information from a utility function. Surprisingly, some basic results on what sets of actions are best replies to some belief over opposing action profiles appear not to be known. Propositions 1 and 2 give simple characterizations of what sets may be best replies, respectively, without and with the restriction that beliefs have full support.

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Proposition 1 extends "Pearce's Lemma", the classic result that an action is a best reply to some belief if and only if it is not strictly dominated: it states that for any *set* S of actions, there is a belief under which *all* actions in S are best replies if and only if no *mixture* of actions in S is strictly dominated. Similarly, Proposition 2 states that for any set S of actions, there is a *full-support* belief under which all actions in S are best replies if and only if no mixture of actions in S is *weakly* dominated. One consequence is Corollary 1: a two-player game has a totally mixed Nash equilibrium if and only if neither player has a pair of mixed strategies such that one weakly dominates the other.

2 Preliminaries

A *finite complete-information normal-form game* (henceforth "*game*") is a triple $G = (N, A, u)$ where $N = \{1, \dots, n\}$ is a finite set of players, $A = \prod_{i \in N} A_i$ is a finite set of (pure) action profiles, and $u = (u_1, \dots, u_n)$ is a profile of VN-M utilities for each player over A, $u_i : A \to \mathbb{R}$. As usual, $A_{-i} = \prod_{j \neq i} A_j$, and $\Delta(S)$ is the set of probability distributions over any set S – with slight abuse of notation, we consider $\Delta(S) \subseteq \Delta(A_i)$ when $S \subseteq A_i$. Note that when we write $\Delta(A_{-i})$ for the set of possible beliefs for player i over opposing action profiles, correlated beliefs are allowed. We write $\Delta^0(A_i)$ or $\Delta^0(A_{-i})$ to denote full-support distributions on each set, also called *totally mixed* strategies. We extend the utility functions u_i to mixed strategies via expectation, as usual. The best-reply correspondence, $BR_i : \Delta(A_{-i}) \rightrightarrows A_i$, gives the set of pure best replies to a belief over opposing profiles. For $\sigma_i, \sigma_i' \in \Delta(A_i)$, we define the relations:[1]

- Strict dominance: $\sigma_i >> \sigma_i'$ if $u_i(\sigma_i, a_{-i}) > u_i(\sigma_i, a_{-i})$ for all a_{-i}.
- Weak dominance: $\sigma_i > \sigma_i'$ if $u_i(\sigma_i, a_{-i}) \geq u_i(\sigma_i, a_{-i})$ for all a_{-i}, with strict inequality for some a_{-i}.
- At least tied: $\sigma_i \geq \sigma_i'$ if $u_i(\sigma_i, a_{-i}) \geq u_i(\sigma_i, a_{-i})$ for all a_{-i}.

We recall some standard terminology and results for finite two-player *zero-sum* games: payoffs $(v, -v)$ are the same in every equilibrium, and we call v the *value* of the game (to Player 1). We call a mixed strategy in a two-player zero-sum game a *minimax strategy* if it guarantees a player at least his value- –equivalently, if it is played in some equilibrium. The set of minimax strategies is convex. We also need the following less-well-known result, which is Lemma 4 from Gale and Sherman (1950), translated into modern notation:

Lemma 1 (Gale and Sherman 1950)
Let $G = (\{1, 2\}, A, u)$ be any finite two-player zero-sum game, let $v \in \mathbb{R}$ be the value of G to Player 1, and let $\hat{a}_2 \in A_2$. Exactly one of the following is true:

1. *Player 1 has a minimax strategy σ_1 with $u_1(\sigma_1, \hat{a}_2) > v$.*
2. *Player 2 has a minimax strategy σ_2 with $\sigma_2(\hat{a}_2) > 0$.*

[1] Of course, these relations depend on the game G, but we suppress this dependence from our notation, as the relevant game will be clear from context.

As Gale–Sherman, we call an action $\hat{a}_2$ *superfluous* if it satisfies (1) and *essential* if it satisfies (2).[2] The following is a straightforward consequence of Lemma 1.

Lemma 2 *For any finite two-player zero-sum game* $(\{1, 2\}, A, u)$ *with value* $v \in \mathbb{R}$, *exactly one of the following is true:*

1. *Player 1 has a minimax strategy* σ_1 *with* $u_1(\sigma_1, a_2) > v$ *for some* a_2.
2. *Player 2 has a totally mixed minimax strategy* σ_2.

Proof of Lemma 2 If both held, we would get $u_1(\sigma_1, \sigma_2) > v$ contradicting the minimax property of σ_2. If (1) fails, then Lemma 1 tells us that every a_2 is essential. That is for every a_2, Player 2 has a minimax strategy σ_2 with $\sigma_2(a_2) > 0$. Any strictly positive convex combination of these satisfies (2). □

3 Results

Our first result tells us when all elements of a set $S \subseteq A_i$ are simultaneously best replies to some belief. Note that when $S = A_i$, Proposition 1 states that there is a belief making a player indifferent to *all* of his actions exactly when there is no strict dominance relation between any pair of his mixed strategies. Note also that in the case that S is a singleton, Proposition 1 reduces to the classic result that an action is either strictly dominated or else a best reply. That result gained prominence in game theory as Lemma 3 in Pearce (1984), as it provides an alternate characterization of rationalizability.[3] Our proofs of Proposition 1 and later results are closely related to Pearce's clever application of the Minimax Theorem to prove his lemma.[4]

Proposition 1 (Best-reply sets)
For any game $G = (N, A, u)$, *player* $i \in N$, *and subset* $S \subseteq A_i$, *exactly one of the following is true:*

1. *There is a belief* $\sigma_{-i} \in \Delta(A_{-i})$ *such that* $S \subseteq BR_i(\sigma_{-i})$.
2. *There is a pair* $\sigma_i \in \Delta(A_i)$, $\sigma_i' \in \Delta(S)$ *such that* $\sigma_i >> \sigma_i'$.

When case 2 holds, the pair σ_i, σ_i' *can be selected such that* $\text{supp}(\sigma_i) \cap \text{supp}(\sigma_i') = \emptyset$.

Proof of Proposition 1 (1), (2) cannot both hold: let σ_{-i} be as in (1); then for any σ_i, σ_i', with $\sigma_i' \in \Delta(S)$, $u_i(\sigma_i', \sigma_{-i}) \geq u_i(\sigma_i, \sigma_{-i})$, contradicting $\sigma_i >> \sigma_i'$.

Failure of (1) implies (2): given $G = (N, A, u)$ and $i \in N$, consider the two-player zero-sum game $H = (\{1, 2\}, A^h, u^h)$ where

$$A_1^h = A_i \times S, A_2^h = A_{-i}, u_1^h((a_i, a_i'), a_{-i}) = u_i(a_i, a_{-i}) - u_i(a_i', a_{-i}), u_2^h = -u_1^h$$

Now suppose (1) is false. Then, for any mixture σ_{-i} of Player 2 in H, we have $u_i(a_i, \sigma_{-i}) > u_i(a_i', \sigma_{-i})$ for some $a_i \in A_i$, $a_i' \in S$. This means $u_1^h((a_i, a_i'), \sigma_{-i}) > 0$,

[2] Do not confuse essential with the stronger notion of being played in *every* minimax strategy. To my ear, the latter more closely matches the English meaning of essential, but I use the existing terminology.

[3] See that paper for references to several earlier works with closely related results.

[4] Pearce mentions that this proof was developed in conversations with Dilip Abreu.

so we can conclude that the value of H to Player 1 is positive. Now, the Minimax Theorem tells us that Player 1 has a mixture σ_1 over $A_1^h = A_i \times S$ giving positive payoff in H against every a_{-i}. Letting σ_{11}, σ_{12} be the marginal distributions of σ_1 on each coordinate, we will find that σ_{11} dominates σ_{12} as strategies in G. Indeed, for every a_{-i},

$$\sum_{(a_i,a_i')} \sigma_1(a_i, a_i')[u_i(a_i, a_{-i}) - u_i(a_i', a_{-i})] > 0$$

$$\sum_{(a_i,a_i')} \sigma_1(a_i, a_i')u_i(a_i, a_{-i}) > \sum_{(a_i,a_i')} \sigma_1(a_i, a_i')u_i(a_i', a_{-i})$$

$$\sum_{a_i} \sigma_{11}(a_i)u_i(a_i, a_{-i}) > \sum_{a_i'} \sigma_{12}(a_i')u_i(a_i', a_{-i})$$

as desired, and of course $\sigma_{12} \in \Delta(S)$.

The final claim: let σ_i strictly dominate σ_i' and observe that for every a_{-i},

$$\sum_{a_i} \sigma_i(a_i)u_i(a_i, a_{-i}) > \sum_{a_i} \sigma_i'(a_i)u_i(a_i, a_{-i})$$

$$\sum_{a_i}(\sigma_i(a_i) - \min(\sigma_i(a_i), \sigma_i'(a_i)))u_i(a_i, a_{-i})$$
$$> \sum_{a_i}(\sigma_i'(a_i) - \min(\sigma_i(a_i), \sigma_i'(a_i)))u_i(a_i, a_{-i})$$

The sum of the coefficients on each side is

$$\kappa := 1 - \sum_{a_i} \min(\sigma_i(a_i), \sigma_i'(a_i))$$

which is positive because $\sigma_i \neq \sigma_i'$. We thus find that μ_i dominates μ_i', where

$$\mu_i(a_i) := \frac{\sigma_i(a_i) - \min(\sigma_i(a_i), \sigma_i'(a_i))}{\kappa}, \ \mu_i'(a_i) := \frac{\sigma_i'(a_i) - \min(\sigma_i(a_i), \sigma_i'(a_i))}{\kappa}$$

and clearly μ_i, μ_i' have disjoint support. □

Note that, for $n > 2$, it is necessary for Proposition 1, and indeed all of our results, that correlated beliefs about other players' actions are allowed. Both Bernheim (1984) and Pearce (1984) observed that if independent beliefs are required, Proposition 1 fails even in the case of singleton S. In our proofs, the possible presence of correlation allows us to regard the other players as a single player in the two-player zero-sum game H. For an argument as to why correlated beliefs "should" be allowed, see Aumann (1987).

In Proposition 2, we give conditions for existence of a full-support distribution such that all actions in S are best replies. Note that in the case that S is a singleton, Proposition 2 reduces to the classic result that an action is either weakly dominated or else *admissible*, i.e. a best reply to a full-support belief.[5]

Proposition 2 (Full-support best-reply sets)
For any game $G = (N, A, u)$, player $i \in N$, and subset $S \subseteq A_i$, exactly one of the following is true:

1. *There is a belief $\sigma_{-i} \in \Delta^0(A_{-i})$ such that $S \subseteq BR_i(\sigma_{-i})$.*
2. *There is a pair $\sigma_i \in \Delta(A_i)$, $\sigma_i' \in \Delta(S)$ such that $\sigma_i > \sigma_i'$.*

When case 2 holds, the pair σ_i, σ_i' can be selected such that $\text{supp}(\sigma_i) \cap \text{supp}(\sigma_i') = \emptyset$.

Proof of Proposition 2 It is immediate that both conditions cannot hold, because weak dominance implies $u(\sigma_i, \sigma_{-i}) > u(\sigma_i', \sigma_{-i})$ for any full-support σ_{-i}.

To show that one condition must hold: construct H as in the proof of Proposition 1. If H has positive value to player 1, as before there is a *strict* dominance relation for some pair σ_i, σ_i'. So we are left with the case that H has value 0, because Player 1's actions (a_i, a_i) ensure payoff 0. Now suppose (1) of Proposition 2 fails. This means that for every $\sigma_{-i} \in \Delta^0(A_{-i})$ there is a pair (a_i, a_i') with $a_i' \in S$ such that $u_i(a_i, \sigma_{-i}) > u_i(a_i', \sigma_{-i})$, i.e. $u_i^h((a_i, a_i'), \sigma_{-i}) > 0$, and this is precisely failure of condition (2) of Lemma 2 regarding H. So in H, Player 1 has a minimax strategy σ_1 with $u^h(\sigma_1, \hat{a}_2) > 0$ for *some* $\hat{a}_2$, and the minimax property gives $u^h(\sigma_1, a_2) \geq 0$ for *all* a_2. Following the same algebraic steps as in the proof of Proposition 1, we find that σ_{11} *weakly* dominates σ_{12} as strategies in G, as desired. The final claim follows as in the proof of Proposition 1. □

Note that when $S = A_i$, Proposition 2 states that there is a full-support belief making a player indifferent to *all* of his actions exactly when there is no weak-dominance relationship between any pair of his mixed strategies. In a two-player game, clearly there is a totally mixed Nash equilibrium precisely when each player has a full-support mixture which makes his opponent indifferent to all of his actions. This gives the following:

Corollary 1 *A two-player game has a totally mixed Nash equilibrium if and only if there is no weak-dominance relationship between any pair of mixed strategies of either player.*

For games with $n > 2$, Corollary 1 does not hold (and in fact the problem of finding Nash equilibria with given supports is considerably more complex). For example, consider:

	L	R		L	R		L	R
U	1,1,2	0,0,0		1,1,0	0,0,0		1,1,1.1	0,0,-10
D	0,0,0	1,1,0		0,0,0	1,1,2		0,0,-10	1,1,1.1

where Player 3 chooses the matrix. Here, Players 1 and 2 are playing a coordination game, while Player 3, who does not affect their payoffs, tries to guess their profile. P3's third action is not a best reply to any independent belief, so is not played in any

[5] This case of the result appears, for instance, as Lemma 4 in Pearce (1984).

Nash equilibrium, but it is the unique best reply to the correlated belief placing .5 probability on each of (U,L) and (D,R), and, therefore, it is not dominated. It is easy to check that no mixed strategy is dominated either, so we violate Corollary 1.

We can view this corollary as a special case of a result characterizing the existence of an equilibrium with any given support. We will need notation for one additional relation: given a subset T of A_{-i}, write $\sigma_i >_T \sigma_i'$ if $u_i(\sigma_i, a_{-i}) \geq u_i(\sigma_i, a_{-i})$ for all $a_{-i} \in T$, with strict inequality for some $a_{-i} \in T$; that is, if σ_i weakly dominates σ_i' when opposing profiles are restricted to T. Applying Proposition 2 to a modification G' of G in which players A_{-i} are restricted to actions in T tells us that there is a belief $\sigma_{-i} \in \Delta^0(T)$ with $S \subseteq BR_i(\sigma_{-i})$ if and only if there is no pair $\sigma_i \in \Delta(A_i), \sigma_i' \in \Delta(S)$ such that $\sigma_i >_T \sigma_i'$. This gives:

Corollary 2 *A two-player game has a Nash equilibrium with supports $S \subseteq A_1, T \subseteq A_2$, if and only if there is no pair $\sigma_1 \in \Delta(A_1), \sigma_1' \in \Delta(S)$ such that $\sigma_1 >_T \sigma_1'$ and no pair $\sigma_2 \in \Delta(A_2), \sigma_2' \in \Delta(T)$ such that $\sigma_2 >_S \sigma_2'$.*

There is also a related corollary to Proposition 1. Call an equilibrium *quasi-totally mixed* if every action is a best reply to the opposing profile.[6] Then, from the $S = A_i$ case of Proposition 1 we conclude:

Corollary 3 *A two-player game has a quasi-totally mixed Nash equilibrium if and only if there is no strict-dominance relationship between any pair of mixed strategies of either player.*

We might also be interested in when a proper subset S of A_i is the *exact* set of best replies to some belief $\sigma_{-i} \in \Delta(A_{-i})$, and the next result addresses that issue. Note that in the case that S is a singleton, Proposition 3 reduces to the result that an action can be a *unique* best reply precisely if it is not weakly dominated or tied by any mixed strategy.

Proposition 3 (Exact best-reply sets)
For any game $G = (N, A, u)$, player $i \in N$, and strict subset $S \subset A_i$, exactly one of the following is true:

1. *There is a belief $\sigma_{-i} \in \Delta(A_{-i})$ such that $S = BR_i(\sigma_{-i})$.*
2. *There is a pair $\sigma_i \in \Delta(A_i) - \Delta(S), \sigma_i' \in \Delta(S)$ such that $\sigma_i \geq \sigma_i'$.*

When case 2 holds, the pair σ_i, σ_i' can be selected such that $\text{supp}(\sigma_i) \cap \text{supp}(\sigma_i') = \emptyset$.

Proof If both conditions held, all actions in the support of σ_i' would be best replies to σ_{-i}, hence also all actions in the support of σ_i, but some such action is in $A_i - S$, contradicting $S = BR_i(\sigma_{-i})$.

[6] For a generic set of games, in particular those where the set of best replies to a belief is never larger than its support, all quasi-totally mixed equilibria are totally mixed. The converse, of course, is always true.

To show that one condition must hold: let H be as in the proof of Lemma 1. Consider three cases:

Case 1: H has positive value to player 1. As before there is a *strict* dominance relation $\sigma_i >> \sigma_i'$ for some pair $\sigma_i \in \Delta(A_i), \sigma_i' \in \Delta(S)$. Because the relation is strict, we can pick any $a_i \in A_i - S$ and for small enough ε, $(1-\varepsilon)\sigma_i + \varepsilon a_i >> \sigma_i'$, satisfying (2).

Case 2: H has value 0 and all pairs $(a_i, a_i') \in (A_i - S) \times S$ are superfluous for Player 1 in H. Taking a positive convex combination of the strategies described in (1) of Lemma 1 (with players 1 and 2 reversed) gives a $\sigma_{-i} \in \Delta(A_{-i})$ which is a minimax strategy and satisfies $u_i^h((a_i, a_i'), \sigma_{-i}) < 0$ for all $(a_i, a_i') \in (A_i - S) \times S$. The minimax property translates to all elements of S being best replies to σ_{-i} and the second property translates to no other actions being best replies, so (1) is satisfied.

Case 3: H has value 0 and some pair $(a_i, a_i') \in (A_i - S) \times S$ is essential for Player 1 in H. Let σ_1 be a minimax strategy in H with $\sigma_1(a_i, a_i') > 0$ and let σ_{11}, σ_{12} be the marginals of σ_1 as in the proof of Propositions 1 and 2. As there we find that the minimax property gives $\sigma_{11} \geq \sigma_{12}$ (in G), and furthermore $\sigma_{11}(a_i) > 0$ where $a_i \in (A_i - S)$, so (2) is satisfied.

The final claim follows as in the proof of Proposition 1. □

4 Discussion and examples

Most readers will be familiar with the easily checked special case of Corollary 1 that in a 2×2 game, if no action is weakly dominated there is a totally mixed Nash equilibrium. Many will also know in larger games, even if no action is weakly dominated there may be no totally mixed equilibrium, and may even be actions which are not played in any Nash equilibrium. Corollary 1 makes these phenomena much more transparent: for there to be no weakly dominated pure action, yet also no totally mixed equilibrium, it is necessary that some mixed strategy is dominated. Clearly, this requires at least three actions for some player. A minimal-size example is this 3×2 game:

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$
TT	$-4, 4$	$2, -2$

In this game, which we might call "Extended Matching Pennies", no pure action is dominated, so all actions are rationalizable. However, an equal mixture of H and TT is dominated by T, blocking any Nash equilibrium from including both H and TT. Indeed, this game has a unique Nash equilibrium, the standard Matching Pennies solution with equal mixtures of H and T, and so TT is never played in equilibrium. However, note that the dominated mixed strategy only blocks a *single* equilibrium from containing *both* H and TT, and it could still happen that each action is played in *some* equilibrium despite the presence of a dominated mixed strategy. For instance, change the -2 to 4 in the example above, leaving the dominance relation intact (and making the game non-zero-sum). The original mixed equilibrium remains, and now

(TT,T) is an equilibrium also, as well as there being another mixed equilibrium with P1's support being T and TT, so indeed each action is played in some equilibrium.

5 Concluding remarks

Historically, the Minimax Theorem of Von Neumann and the associated theory of zero-sum games mark the beginning of modern game theory. Hence, when establishing fundamental results in the theory of non-zero-sum games, there is poetic justice in appealing to the Minimax Theorem rather than to linear programming or separating hyperplane results. These proofs are also intuitive and non-technical, hence attractive for teaching, and offer closely related arguments for results on strong and weak dominance. I believe that we gain more insight from seeing relationships between game-theoretic results than from other methods of proof. Algorithms for deciding the existence problems discussed here can be obtained by the usual translation of minimax problems into linear programming [see, for instance, Vohra (2005), p. 69]. In fact, the proofs here could all be written, rather less intuitively, in terms of linear-programming duality.

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