Two Elementary Proofs of the Minimax Theorem

Jonathan Weinstein*

Washington University in St. Louis

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Abstract

We give a proof of the Minimax Theorem where the key steps involve reducing the strategy sets. The proof is self-contained and elementary, avoiding appeals to theorems from geometry, analysis or algebra, such as the separating hyperplane theorem or linear-programming duality. The argument is valid with any ordered field in place of the usual field of real numbers. We give a second proof with similar merits which is closely modeled on an argument from Loomis (1946). This one is a bit simpler, but uses the Maximum Theorem for compact domains and hence is only valid in $\mathbb{R}$.

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*Email: j.weinstein@wustl.edu. ORCID: 0000-0001-8245-2067. I am grateful for conversations and correspondence with.
1 Introduction

The Minimax Theorem was the first major existence theorem in game theory. Perhaps it can best be described to a modern audience as saying that a Nash Equilibrium exists for any finite two-player zero-sum game. For a statement closer to the original, which avoids the anachronistic reference to Nash, see the next section. Since the Minimax Theorem was first proved in von Neumann (1928), many other proofs have appeared. Perhaps best-known is the argument which is used in von Neumann & Morgenstern (1947), using the separating hyperplane theorem. This approach, as well as von Neumann’s original proof, used properties special to the real numbers, specifically the Maximum Theorem for continuous functions on compact sets. But the Minimax Theorem actually holds when payoffs lie in any ordered field $\mathbb{F}$, and probabilities in the mixed strategies are allowed to range over $\mathbb{F}$, as was shown in Kuhn & Tucker (1950). The proof there relies on a result from Stiemke (1915), which was a forerunner of duality results in linear programming. Indeed, linear-programming duality (or related results such as Farkas’ Lemma) leads to a quick proof of the Minimax Theorem – see for instance Vohra (2004).

I have often felt a bit unsatisfied by these proofs. Because the Minimax Theorem is so fundamental to our understanding of strategic interaction, I wanted to create a proof that reasons directly about strategies. That is, an argument that harnesses strategic intuition, avoiding the use of theorems from geometry, algebra or analysis as the primary engine. Such a proof is provided here, with some preliminaries in Section 2 and the proof of the workhorse lemma in Section 3. The proof is valid for any ordered field, because it avoids any arguments based on continuity or compactness. I can only hope that others feel as I do that it contributes understanding beyond the existing proofs.

During my background reading, I encountered a lovely short paper on this subject, Loomis (1946). Loomis comments that von Neumann had given a recent lecture where he challenged the audience to give an elementary proof of the Minimax Theorem, or, more precisely, of an algebraic theorem which was a slight generalization of Minimax. So von Neumann, too, thought there was value in finding a more elementary approach than his own proofs. Loomis succeeded at this admirably, giving a very simple proof with perhaps one defect: it uses the Maximum Theorem, which both calls into question how “elementary” it is, and means it is valid only for $\mathbb{R}$. Nonetheless, it is the simplest proof of the original theorem I’ve seen, subject to the constraint of using no advanced theorems other than the Maximum Theorem. Certainly, it is far more elegant than von Neumann’s original proof. So, I’ve included an adaptation of it here, in Section 4, which I believe has two advantages for the modern student of game theory over the original Loomis paper: I have stripped it of some extra generality, which simplifies the notation, and I have rewritten it in the notation of game theory, in a way which emphasizes strategic intuition. The resulting argument is simpler than my first proof, but with the relative weakness of its appeal to compactness.

2 Definitions and Proof Outline

All of our games will be finite two-player zero-sum games, allowing a streamlining of notation. Such a game will be a triple $G = (A_1, A_2, u)$ where $A_1, A_2$ are finite sets and $u$ is a function
from $A_1 \times A_2$ to $\mathbb{F}$ for some ordered field $\mathbb{F}$. The function $u$ is the utility of Player 1; Player 2’s utility is $-u$. The game is symmetric if $A_1 = A_2$ and for all $a_1$ and $a_2$, $u(a_1, a_2) = -u(a_2, a_1)$.

For each player $i$, a mixed strategy is, formally, a non-negative-valued function from $A_i$ to $\mathbb{F}$ which sums to 1, and $\Sigma$ is the set of mixed strategies. In a slight overloading of notation, any $a_i \in A_i$ will sometimes be considered to be the mixed strategy assigning probability 1 to $a_i$. We extend the function $u$ to $\Sigma_1 \times \Sigma_2$ by linearity as usual. For any $\sigma_i^1, \sigma_i^2 \in \Sigma_i$ and $\alpha \in [0, 1]$, the mixture $\alpha \sigma_i^1 + (1 - \alpha) \sigma_i^2 \in \Sigma_i$ has the expected meaning, assigning probability $\alpha \sigma_i^1(a_i) + (1 - \alpha) \sigma_i^2(a_i)$ to each $a_i$.

The minimax theorem can then be stated as follows:

**Theorem 1 (Minimax Theorem)** For any finite two-player zero-sum game $G$,

$$\max_{\sigma_1 \in \Sigma_1} \min_{\sigma_2 \in \Sigma_2} u(\sigma_1, \sigma_2) = \min_{\sigma_2 \in \Sigma_2} \max_{\sigma_1 \in \Sigma_1} u(\sigma_1, \sigma_2)$$ (1)

Note that when we work in an arbitrary $\mathbb{F}$, there is no immediate reason that either side of (1) must be well-defined. It is part of the content of the theorem that they are. The common value in (1) is called the value of $G$. It is straightforward to see that the theorem is equivalent to the existence of a Nash equilibrium $(\sigma^*_1, \sigma^*_2)$. Indeed, $\sigma^*_1$ would optimize the left-hand-side, and $\sigma^*_2$ the right-hand-side, with the common value $u(\sigma^*_1, \sigma^*_2)$.

We call a mixed strategy bulletproof if it guarantees non-negative utility. Our main lemma will be the following:

**Lemma 1 (Bulletproof Lemma)** In any two-player zero-sum game, at least one player has a bulletproof strategy.

From the Bulletproof Lemma, the Minimax Theorem follows in two simple and intuitive steps. First, the lemma easily implies the theorem for symmetric games. Indeed, the lemma along with symmetry tells us that both players have bulletproof strategies, and then both sides of (1) equal 0.

Next, we show that the symmetric case implies the general case. Given $G = (A_1, A_2, u)$, we create a symmetric version where both players simultaneously play both roles and we add the payoffs. That is, we let $\tilde{G} = (A_1 \times A_2, A_1 \times A_2, \tilde{u})$ where $\tilde{u}((a_1, a_2), (a_1', a_2')) = u(a_1, a_2') - u(a_1', a_2)$. It is a fairly easy exercise to see that a minimax strategy in $\tilde{G}$ must have marginal distributions on each coordinate which solve the min-max and max-min problem of $G$ and establish (1).\(^3\)

So, the bulk of the work is in the proof of the lemma. It is here that I believe my approach is somewhat novel. The details are in the next section; here is the idea. We proceed by induction, reducing the players’ options in a way which preserves the existence (or non-existence) of a bulletproof strategy for each player. This reduction does not necessarily preserve the value of the game; it does preserve its sign.

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1 Ordered fields include the real numbers $\mathbb{R}$ and subsets of the reals which are closed under the field operations, such as the rationals $\mathbb{Q}$. There are also ordered fields which cannot be embedded in $\mathbb{R}$, called non-Archimedean; such a field includes infinite and infinitesimal elements. One useful introduction to the topic is Propp (2013).

2 The interval $[0,1]$ should be interpreted as a subset of $\mathbb{F}$, not a subset of $\mathbb{R}$ as per usual.

3 This reduction of the general case to the symmetric case appears in Kuhn & Tucker (1950), p. 81. The authors credit it to von Neumann. It is equivalent to flipping a coin to decide who plays what role. That article also uses a version of the Bulletproof Lemma on the way to the main result.
3 Proof of the Bulletproof Lemma

We will induct on \(|A_1|\). Let \(G = (A_1, A_2, u)\) be any two-player zero-sum game. If \(|A_1| = 1\) the result is easy, so let \(|A_1| \geq 2\). Pick any action \(a_1^* \in A_1\); in order to apply the inductive hypothesis, we will construct a new game \(G'\) where Player 1 has action set \(A'_1 = A_1 - \{a_1^*\}\). Player 2’s new action set \(A'_2\) will be a finite subset of \(\Sigma_2\). To construct it, we first partition \(A_2\) as follows:

\[
A'^b_2 = \{a_2 \in A_2 : u(a_1^*, a_2) \leq 0\}, \quad A^b_2 = \{a_2 \in A_2 : u(a_1^*, a_2) > 0\}
\]

That is, actions in \(A'^b_2\), the “good” actions, are those that defeat or tie the eliminated action \(a_1^*\), and \(A^b_2\), for “bad”, lose to action \(a_1^*\). If \(A'^b_2 = \emptyset\), then \(a_1^*\) is bulletproof, so assume not. For each pair \((g, b) \in A'^b_2 \times A^b_2\), there is a unique mixture which has expected payoff 0 against \(a_1^*\), which we denote \(\langle g, b \rangle\), i.e.

\[
\langle g, b \rangle := \frac{u(a_1^*, b)}{u(a_1^*, b) - u(a_1^*, g)} g + \frac{-u(a_1^*, g)}{u(a_1^*, b) - u(a_1^*, g)} b
\]

Then we set

\[
A'_2 = A'^b_2 \cup \{\langle g, b \rangle : g \in A'^b_2, b \in A^b_2\} \cong A^b_2 \cup \emptyset
\]

Notice that both players’ action sets are subsets of their mixed strategy sets from \(G\), so that payoffs in \(G'\) can be simply inherited from those in \(G\). We now can apply the inductive hypothesis to \(G' = (A'_1, A'_2, u)\).

The case that Player 2 has a bulletproof strategy \(\sigma_2\) in \(G'\) is very simple. This very same strategy is bulletproof in \(G\), because all actions in \(A'_2\) were chosen to have non-negative payoff against the missing action \(a_1^*\).

If Player 1 has a bulletproof strategy \(\sigma_1\) in \(G'\), there is a bit more work. We plan to find an \(\alpha \in [0, 1]\) such that \(\sigma_1^\alpha := \alpha a_1^* + (1 - \alpha)\sigma_1\) is bulletproof in \(G\). Note that by definition of “bad”, \(a_1^*\) has positive payoff against bad actions, and since good actions are in \(A'_2\), \(\sigma_1\) has non-negative payoff against good actions. Therefore, a sufficiently large \(\alpha\) works against all bad actions, and a sufficiently small one against all good actions. We now construct an \(\alpha\) which works against both.

If \(\sigma_1\) has non-negative payoff against all bad actions, we are done (with \(\alpha = 0\), so assume not. Then, for each bad action \(b\), let \(\alpha_b\) be the smallest \(\alpha\) that works against \(b\), i.e. it is chosen so \(u(\sigma_1^{\alpha_b}, b) = 0\) (or \(\alpha_b = 0\) if \(u(\sigma_1, b) \geq 0\)).\(^4\) Let \(\alpha = \max_{b \in A'^b_2} \alpha_b = \alpha_b\) for appropriate \(b\). By construction, \(\sigma_1^\alpha\) is bulletproof against all bad actions. Also, for any good action \(g\), recall that \(u(a_1^*, (g, b)) = 0\), and also, because \(\sigma_1\) is bulletproof in \(G'\), \(u(\sigma_1, (g, b)) \geq 0\), which together imply \(u(\sigma_1^\alpha, (g, b)) \geq 0\). But also \(u(\sigma_1^\alpha, b) = 0\), so we must have \(u(\sigma_1^\alpha, g) \geq 0\), and the proof is complete. \(\square\)

\(^4\)It might appear that we use continuity here, but actually \(\alpha_b\) can be calculated by field operations; it is the solution to a linear equation. Indeed, for \textit{linear} functions, the Intermediate Value Theorem holds in any ordered field, which is the core reason that the Minimax Theorem doesn’t require the use of \(\mathbb{R}\).
The proof of Loomis (1946) for $F = \mathbb{R}$

If the theorem fails, let $G = (A_1, A_2, u)$ be a counterexample for which $|A_1| + |A_2|$ is smallest. Because $\Sigma_1, \Sigma_2$ are compact, the expressions in (1) are well-defined. Let $\sigma_1^*$ optimize the left-hand-side (the max-min problem) and $\sigma_2^*$ the right (the min-max problem), and call the optimal values $\underline{v}$ and $\bar{v}$. It is immediate that

$$\underline{v} \leq u(\sigma_1^*, \sigma_2^*) \leq \bar{v}$$

and by assumption, at least one inequality is strict; say the first. Then there is some $a_2^b$ (a “bad” strategy for Player 2) with $\underline{v} < u(\sigma_1^*, a_2^b)$. Form $G'$ from $G$ by eliminating $a_2^b$. By inductive hypothesis, (1) holds for $G'$, say with common value $v$. The min-max can only go up when we eliminate one of Player 2’s strategies, so $\bar{v} \leq v$ and thus $\underline{v} < v$. Let $\sigma_1^{**}$ solve the max-min problem in $G'$. It is then easy to check that for small enough $\varepsilon > 0$, the strategy $\sigma_1^\varepsilon := (1 - \varepsilon)\sigma_1^* + \varepsilon \sigma_1^{**}$ improves on the alleged solution $\sigma_1^*$ to the max-min problem of $G$. Indeed, the strict inequality $\underline{v} < u(\sigma_1^*, a_2^b)$ holds for $\varepsilon = 0$, hence also in a neighborhood of 0, and for $a_2 \neq a_2^b$ we simply have, for any $\varepsilon \in (0, 1)$,

$$\underline{v} < (1 - \varepsilon)\underline{v} + \varepsilon v \leq (1 - \varepsilon)u(\sigma_1^*, a_2) + \varepsilon u(\sigma_1^{**}, a_2) = u(\sigma_1^\varepsilon, a_2)$$

The contradiction proves the desired result. □

This proof requires that the underlying field be $\mathbb{R}$ in the first paragraph, in order to apply compactness and conclude that the max-min and min-max exist. The implicit use of continuity in the final paragraph is only for convenience and could be replaced by field operations.

References


