Normal numbers - a primer

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Plan

- **Normal number - definitions and preliminaries**

- Some basic facts that we will need

- What will we learn?

What is a normal number?

- $b \in \{2, 3, \dots\}$: called a **base**
- Every $x \in [0, 1)$ admits a representation:

$$x = 0.x_1 x_2 \cdots = \sum_{k=1}^{\infty} x_k b^{-k}, \qquad x_k \in \{0, 1, \dots, b-1\}.$$

- $\{x_k : k \geq 1\}$: **digits** of x in base b

Normality in base b

Say x is ***b*-normal** if $\forall k \in \mathbb{N}$ and $(d_1, \dots, d_k) \in \{0, 1, \dots, b-1\}^k$,

$$\lim_{N \to \infty} \frac{1}{N} \# \{0 \leq j \leq N-1 : (x_{j+1}, \dots, x_{j+k}) = (d_1, \dots, d_k)\} = \frac{1}{b^k}.$$

▶ É. Borel, Les probabilités dénombrables et leurs applications arithmétiques, Rend. Circ. Math. Palermo 27 (1909), 247-271.

▶ I. Niven & H. S. Zuckerman, On the definition of normal numbers, Pacific J. Math. 1 (1951), 103-109.

Examples, non-examples, knowns and unknowns

- **Exercise:** Show that rational numbers are not b-normal for any b:

$$Hint: \quad \frac{1}{7} = 0.142857142857\cdots \quad \text{(decimal expansion)}$$

- The decimal expansion obtained by concatenating all positive integers

$$\xi_c := 0.1234567891011121314\cdots$$

  is normal in base 10.

  - D. G. Champernowne, The construction of decimals normal in the scale of ten, J. London Math. Soc. 8 (1933), 254–260.

- Borel (1909) **Lebesgue almost every $x \in [0,1]$ is b-normal $\forall b \geq 2$.**
- Not known whether $\sqrt{2}, \pi, e$ are normal in *any* base.

Why study normal numbers?

Connections to many problems in

- Number theory
- Geometric measure theory
- Harmonic analysis
- Ergodic theory
- Computer Science
- ⋮

some of which will appear in this mini-course.

- Y. Bugeaud, Distribution modulo one and Diophantine approximation, Cambridge Tracts in Mathematics, 193. Cambridge University Press, Cambridge, 2012.

What is this course about? Take 1

- **Sets of partly normal numbers:** For $\mathscr{B}, \mathscr{B}' \subseteq \mathbb{N} \setminus \{1\}$,

$$\mathscr{N}(\mathscr{B}, \mathscr{B}') := \left\{ x \in [0,1) \,\middle|\, \begin{array}{l} x \text{ is } b\text{-normal } \forall b \in \mathscr{B}, \\ x \text{ is not } b'\text{-normal } \forall b' \in \mathscr{B}' \end{array} \right\}.$$

- Borel's theorem $\implies$ $\mathscr{N}(\mathscr{B}, \mathscr{B}')$ is Lebesgue-null if $\mathscr{B}' \neq \emptyset$.
- Preliminary question: When is $\mathscr{N}(\mathscr{B}, \mathscr{B}') \neq \emptyset$? (next slide)

A general question about numbers normal in some, but not all, bases

What can we say about finer

- analytic,
- measure-theoretic

properties of nontrivial $\mathscr{N}(\mathscr{B}, \mathscr{B}')$?

Plan

- Normal number - definitions and preliminaries
- **Some basic facts that we will need**
  - Size of sets of partly normal numbers
  - Normality and equidistribution
  - Normality almost everywhere
  - Sets of uniqueness and their connection to normality
- What will we learn?

How large are sets of partly normal numbers?

Multiplicative independence

- Say $r, s \in \mathbb{N} \setminus \{1\}$ are **multiplicatively independent** & write $r \sim s$ if

$$\frac{\log r}{\log s} \in \mathbb{Q}, \quad \text{i.e.} \quad \exists m, n \in \mathbb{N} \text{ such that } r^m = s^n.$$

- For $\mathscr{B} \subseteq \mathbb{N} \setminus \{1\}$, the **multiplicatively dependent closure** of $\mathscr{B}$ is

$$\overline{\mathscr{B}} := \{c \in \mathbb{N} \setminus \{1\} : \exists b \in \mathscr{B} \ni c \sim b\}.$$

- For $r \sim s$, a number is r-normal $\iff$ it is s-normal.

$$\mathscr{N}(\mathscr{B}, \mathscr{B}') = \begin{cases} \emptyset & \text{if } \overline{\mathscr{B}} \cap \overline{\mathscr{B}'} \neq \emptyset, \\ \text{uncountable} & \text{otherwise.} \end{cases}$$

  - W. Schmidt, On normal numbers, Pacific J. Math. 10 (1960) 661-672.

$$\dim_{\mathbb{H}}(\mathscr{N}(\mathscr{B}, \mathscr{B}') = 1 \quad \text{if } \overline{\mathscr{B}} \cap \overline{\mathscr{B}'} = \emptyset.$$

  - A. Pollington, The Hausdorff dimension ... numbers, Pacific J. Math 10 (1981) 193-204.

Normality and uniform distribution mod 1

- A sequence $\{x_n : n \geq 1\} \subseteq \mathbb{R}$ is **uniformly distributed mod 1** if

$$\lim_{N\to\infty} \frac{1}{N} \# \{1 \leq n \leq N : \{x_n\} \in [\alpha, \beta)\} = \beta - \alpha \quad \forall 0 \leq \alpha < \beta < 1.$$

▶ $\{x\} = x - \lfloor x \rfloor$ fractional part of x.

Weyl's criterion for uniform distribution (1916)

$\{x_n\}$ is uniformly distributed mod 1 if and only if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} e^{-2\pi i h x_n} = 0 \; \forall h \in \mathbb{Z} \setminus \{0\}.$$

▶ H. Weyl, Über die Gleichverteilung von Zahlen mod. Eins., Math. Ann. 77 (1916), 313-352.

Wall's characterization of normality (1949)

$x \in [0, 1)$ is b-normal $\iff (b^n x)_{n \geq 1}$ is uniformly distributed mod 1.

▶ D. D. Wall, Normal numbers. PhD thesis, University of California, Berkeley, CA, 1949.

Measures and normality almost everywhere

- μ : Borel probability measure on $[0,1)$, $b \in \mathbb{N} \setminus \{1\}$.

$$\widehat{\mu}(n) = \int_0^1 e^{-2\pi i n x} d\mu(x) = n^{\text{th}} \text{Fourier coefficient of } \mu, \quad n \in \mathbb{Z}.$$

A criterion of Davenport, Erdős and LeVeque (1963)

If
$$\sum_{N=1}^{\infty} \frac{1}{N} \int_0^1 \left| \frac{1}{N} \sum_{n=0}^{N-1} e^{-2\pi i h b^n x} \right|^2 \mathrm{d}\mu(x)$$
$$= \sum_{N=1}^{\infty} \frac{1}{N^3} \sum_{m,n=0}^{N-1} \widehat{\mu}(h(b^n - b^m)) < \infty \quad \forall h \in \mathbb{Z} \setminus \{0\},$$

then μ-a.e. $x \in [0,1]$ is b-normal.

▶H. Davenport, P. Erdős and W. J. LeVeque, On Weyl's criterion for uniform distribution, Michigan Math. J. 10 (1963), 311–314.

A few warm-up exercises

- **Exercise:** Replicate DEL's Lebesgue-measure proof for a general μ.
- **Exercise:** Use DEL's criterion to prove Borel's theorem on absolutely normal numbers.
- **Exercise:** Use DEL's summability criterion to show that for $\kappa > 0$,

$$\limsup_{n\to\infty} \frac{\left|\widehat{\mu}(n)\right|}{|\log\log n|^{1+\kappa}} < \infty \implies \mu \text{ a.e. } x \text{ is } b\text{-normal } \forall b \geq 2.$$

Hint: Try out the simpler case of measures with power decay en route to the general case.

What is this mini-course about? Take 2

A question of Kahane and Salem (1964)

"Is the set of non-normal numbers a set U^*?"

In our notation, if $\mathscr{N}(\mathscr{B}, \mathscr{B}') \neq \emptyset$, does $\exists$ a probability measure $\mu \ni$

- μ-a.e. $x \in \mathscr{N}(\mathscr{B}, \mathscr{B}')$,
- μ is Rajchman, i.e., $\left|\widehat{\mu}(n)\right| \to 0$ as $|n| \to \infty$

- Kahane & Salem's question was posed initially for dyadic expansions.

Lyons (1986)

$\exists$ a Rajchman measure μ_{L} for which a.e. $x \in \mathscr{N}(\cdot; \{2\})$.

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μ_{L}-a.e. x is odd-normal but not even-normal.

- No natural generalizations of μ_{L} for arbitrary $(\mathscr{B}, \mathscr{B}')$.

Motivation: Normality and Fourier analysis

Sets of uniqueness and of multiplicity

A Lebesgue measurable set $E \subseteq [0, 1)$ is a **set of uniqueness** if

$$\sum_{n\in\mathbb{Z}} c_n e^{2\pi inx} = 0 \ \ \forall x \in [0, 1) \setminus E \implies c_n = 0 \ \ \forall n \in \mathbb{Z}.$$

Otherwise, it is a **set of multiplicity**.

- Rich history originating in the work of Riemann, Heine, Cantor ...
- **Exercise:**

$$\text{Countable sets} \subset \text{sets of uniqueness} \subset \text{Lebesgue-null sets.}$$

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So ... what is this course really about? Take 3

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- Non-empty partly normal sets $\mathscr{N}(\mathscr{B}, \mathscr{B}')$ are sets of multiplicity.
- If $b \not\sim b'$ for all $(b, b') \in \mathscr{B} \times \mathscr{B}'$, then $\exists$ a Rajchman measure $\mu \ni$

$$\mu - \text{a.e. } x \in \mathscr{N}(\mathscr{B}, \mathscr{B}').$$

We will discuss some ideas behind the proofs:

- **Lecture 1: What are skewed measures?**
- **Lecture 2: Randomness in multiplicative independent bases**
- **Lecture 3: (Non)-normality & Rajchman property**