The Pricing of Options with Default Risk

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ABSTRACT

This paper considers the pricing of options with default risk. The comparative statics of such options can differ from those of ordinary options, and early exercise of such American call options can be optimal. Several examples of options with default risk are considered.

TRADITIONALLY, IT HAS BEEN assumed that options have no default risk. Such an assumption is reasonable when one studies options traded on an organized exchange in the U.S. Default on such options would require, as Cox and Rubinstein [5] put it, "a very sudden and strong movement in stock prices, one considerably more extreme than any on record" (p. 71). However, many options and financial assets containing option-like payoffs are sold by firms that have limited assets. For such options, default is often a possibility that must be taken seriously.

In this paper, we study how options subject to default risk, which we hereinafter call vulnerable options, are priced. It turns out that many of the well-established results of the option-pricing literature (see, e.g., Smith [12] and references therein) do not hold for vulnerable options. In particular, the value of a vulnerable European option can fall with time to maturity, with the interest rate, and with the variance of the underlying asset. Furthermore, it may pay to exercise early a vulnerable American option on a non-dividend-paying asset.

There are many examples of options with default risk. Many options are privately written and are not guaranteed by a third party—currency options, options on precious metals, the cocoa options of the Coffee, Sugar, and Cocoa Exchange, real estate options, options granted by one firm to another, and so on. Insurance contracts can generally be viewed as vulnerable put options. (See Smith [13] for the pricing of insurance when default is ignored.) There have been defaults on some insurance contracts, and there is concern about the financial health of some insurance companies.

Many corporations market insurance contracts for bonds or mortgages, which are a form of vulnerable European put options. Some corporations have defaulted on mortgage insurance, and the bond ratings of several firms that market insurance contracts for bonds have been recently downgraded. Portfolio insurance (see Gatto et al. [6]) also involves vulnerable options. Many bonds issued by firms have option-like features on which these firms could default. Swaps often

incorporate option-like payoffs. Most project financings incorporate options of various kinds. The list could go on. While our paper cannot value every type of option with default risk, we apply our results to a number of examples, most of them related to various insurance contracts.

The paper is organized as follows. In Section I, we are able to show, without making distributional assumptions, how the comparative-static properties of vulnerable options differ from those of ordinary options, i.e., options without default risk. In Section II, we derive the value of a vulnerable option when the option writer offers a default-free bond or a fraction of the underlying asset as collateral. In Sections III and IV, we assume that the assets of the option writer follow a lognormal distribution. In Section III, the underlying asset follows a pure jump process, while, in Section IV, it follows a lognormal diffusion process. Section V provides concluding remarks.

I. Properties of Vulnerable Call Options

This section presents distribution-free results for the pricing of vulnerable call options and shows how such calls can be used to price other options.

A. Assumptions and Notation

The following notation is used throughout the paper. The underlying asset for the option has value $S(t)$ at time t. The option matures at time T, at which time it pays $\max(S(T) - X, 0)$ if the option writer is solvent. If the option writer cannot make the promised payment, the option holder receives the assets of the option writer, which have a total value of $V(t)$ at time t. The value of the European call is written $c(S(t), V(t), X, T)$.

Unless otherwise noted, the analysis rests on the following assumptions:

(A1) Markets for traded assets are perfect; i.e., there are no taxes, no transaction costs, and no limit on short sales, and all investors are price takers.

(A2) There exists unlimited borrowing at the constant interest rate R per unit of time.

(A3) $S(t)$ and $V(t)$ cannot take negative values. Furthermore, the joint distribution of the rates of return of $S(t)$ and $V(t)$ is not affected by a change in $S(t)$ or $V(t)$.

(A4) Markets are complete in the sense that it is possible to construct a portfolio that pays nothing until date T and $S(T)$ at date T and a portfolio that pays nothing until date T and $V(T)$ at date T.

Assumption (A4) will be strengthened later. This assumption is not excessively restrictive as the option is likely to be written on an asset and as the writer can be a firm with common stock that is publicly traded.

B. Relation to Ordinary Options

Using the notation just introduced, the payoff at maturity of an ordinary European call is $\max(S(T) - X, 0)$, while the payoff of the vulnerable call is $\min[V(T), \max(S(T) - X, 0)]$.[1] One can construct an artificial asset with current

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[1] As the ordinary option pays at least as much as the vulnerable option, it cannot be worth less.

value $A(t)$ such that $\min[V(T), \max(S(T) - X, 0)] = \max(A(T) - X, 0)$, so that the vulnerable call can be viewed as an ordinary call on an asset that pays $A(T)$ at maturity. Let $A(T) = \min(X + V(T), S(T))$. It can be verified that

$$\max(A(T) - X, 0) = \max[\min(X + V(T), S(T)) - X, 0]$$
$$= \min[V(T), \max(S(T) - X, 0)]. \tag{1}$$

Hence, $c(S, V, X, T)$ is equal to the price of an ordinary option on A, which we write as $c^A(A, X, T)$, and we can focus the analysis on pricing an ordinary option on asset A. Interestingly, this ordinary option is an option on the minimum of two assets. However, because $A(t)$ is a function of the exercise price of the call, the distribution-free results of Stulz [14] do not apply directly.

C. Distribution-Free Comparative-Statics Results

To obtain comparative-statics results for calls, it is first necessary to derive comparative-statics results for the value of the artificial asset that pays $A(T)$ at date T. Let $e(x, y, T)$ denote the current value of a European option to exchange an asset with value y for an asset with value x at date T. (See Margrabe [9].) With this additional notation, it follows that

$$A(t) = S(t) - e[S(t), X\exp(-R(T-t)) + V(t), T] \equiv S - e^I$$
$$= X\exp(-R(T-t)) + V(t) - e[X\exp(-R(T-t)) + V(t), S(t), T]$$
$$\equiv X\exp(-R(T-t)) + V - e^{II}. \tag{2}$$

Using the fact that $e(x, y, T) = yc(x/y, 1, T)$ and using Merton's [10] distribution-free results, equation (2) implies that

$$\frac{\partial A}{\partial S} = -e_2^{II} > 0, \tag{3a}$$

$$\frac{\partial A}{\partial V} = -e_2^I > 0, \tag{3b}$$

$$\frac{\partial A}{\partial X} = -e_2^I \exp(-R(T-t)) > 0, \tag{3c}$$

$$\frac{\partial A}{\partial R} = (T-t)X \exp(-R(T-t))e_2^I < 0, \tag{3d}$$

and

$$\frac{\partial A}{\partial T} = RX \exp(-R(T-t))e_2^I - e_3^I < 0. \tag{3e}$$

It is not surprising that $A(t)$ increases with S, V, and X. The only reason that $A(t)$ falls with R is that the present value of X falls with R. Finally, $A(t)$ falls with T because of the discounting of X and because owning A involves selling an option.

It is now possible to derive the comparative statics of the call option written on asset A. Using Merton's [10] distribution-free results, it follows that

$$\frac{\partial c}{\partial S} = c_1^A \frac{\partial A}{\partial S} > 0, \tag{4a}$$

$$\frac{\partial c}{\partial V} = c_1^A \frac{\partial A}{\partial V} > 0, \tag{4b}$$

$$\frac{\partial c}{\partial X} = c_1^A \frac{\partial A}{\partial X} + c_2^A < 0, \tag{4c}$$

$$\frac{\partial c}{\partial R} = c_1^A \frac{\partial A}{\partial R} + \frac{\partial c^A}{\partial R} \gtreqless 0, \tag{4d}$$

and

$$\frac{\partial c}{\partial T} = c_1^A \frac{\partial A}{\partial T} + \frac{\partial c^A}{\partial T} \gtreqless 0. \tag{4e}$$

Not surprisingly, the value of the call option increases with the value of the underlying asset and with the value of the assets of the option writer. While the value of the call falls with the exercise price, this must be proved by arguments other than inspection of (4c) as the first term on the r.h.s. of (4c) is positive while the second is negative. The value of the call is a decreasing function of X because there is no state of the world in which an option with a larger exercise price pays off more than an option with a lower exercise price. More surprisingly, it is possible for the value of the call to be a decreasing function of the interest rate R and of the maturity date T. This happens because an increase in R or in T decreases the value of the underlying asset. If there is no default risk, the value of the underlying asset is $S(t)$ and its value depends neither on T nor on R, so that the value of the option increases with R and T.

The comparative statics of the vulnerable call option differ from the comparative statics of the ordinary call in another important dimension. It follows from Jagannathan [7] that adding nonpriced risk to the underlying asset necessarily increases the value of the ordinary option. One can easily construct examples that show that this is not necessarily the case for a vulnerable option. To see this, consider the case in which $S(T)$ and $V(T)$ are distributed in such a way that, if $S(T)$ exceeds its expected value, then $S(T) - X > V(T)$, while, if $S(T)$ comes short of its expected value, then $\max(S(T) - X, 0) < V(T)$. In this case, when $S(T)$ exceeds its expected value, the option holder receives not $S(T) - X$ but $V(T)$, so that the payoffs to the option holder in states of the world such that $S(T)$ exceeds its mean are unaffected if one adds nonpriced symmetric risk to the distribution of $S(T)$. However, when $S(T)$ is below its mean and $S(T) - X$ is positive, the option holder receives smaller payoffs as nonpriced risk is added to the distribution of $S(T)$. Therefore, it follows that, in this case, adding nonpriced risk to the underlying asset is likely to decrease the value of the vulnerable option. Notice, however, that, whenever the probability of default on the option is low, adding nonpriced risk to the underlying asset must increase the value of the option in the usual way.

D. A Parity Relation

A vulnerable European put with exercise price X pays at maturity:

$$\min[V(T), \max(X - S(T), 0)].$$

Let $p(S(t), V(t), X, T)$ be the value of the put at date t. As with the call, one can express the put as an ordinary put option on an artificial asset. Let $a(t)$ be the current value of an artificial asset that pays $\max(S, X - V)$ at time T. With this notation, a vulnerable put has the same value as an ordinary put written on the asset with value $a(t)$; i.e.,

$$p(S(t), V(t), X, T) = p^a(a(t), X, T). \tag{5}$$

The values of the put and of the call are related through a parity relation. Let $M(t)$ be the current value of an asset that pays $\min(a(T), X + V(T)) = \max(A(T), X - V(T))$ at date T. It can easily be verified that

$$M(t) = a(t) - e[a(t), X \exp(-R(T - t)) + V(t), T] \tag{6a}$$

and

$$M(t) = A(t) + e[X \exp(-R(T - t)) - V(t), A(t), T]. \tag{6b}$$

The parity relation between vulnerable puts and calls can now be stated in the following way:

$$p^a(a(t), X, T) = c^A(A(t), X, T) - M(t) + X \exp(-R(T - t)). \tag{7}$$

The proof is simple. Since the put will be exercised only if $a(T) < X$, there is no harm in replacing $a(t)$ with $M(t)$ in (7). Furthermore, since the call will be exercised only if $A(T) > X$, there is no harm in replacing $A(t)$ with $M(t)$ in (7). Thus, (7) is simply put-call parity for options on M with exercise price X.

E. American Puts and Calls

To discuss the properties of vulnerable American puts and calls, we have to replace assumption (A4) with assumption (A4′):

(A4′) $S(t)$ and $V(t)$ are the prices of non-dividend-paying traded assets that pay $S(T)$ and $V(T)$ at date T.

As American options can be exercised before maturity, it is simpler to assume that they are written on traded financial assets unless otherwise stated. Let $C(S(t), V(t), X, T)$ and $P(S(t), V(t), X, T)$ be the values of a vulnerable American call and put, respectively. Merton's [10] results hold, so that

$$C(S(t), V(t), X, T) \geq c(S(t), V(t), X, T) \tag{8a}$$

$$P(S(t), V(t), X, T) \geq p(S(t), V(t), X, T). \tag{8b}$$

It is well known that an ordinary American put on a stock that pays no dividends before T may be exercised before T. As an ordinary option is a special case of a vulnerable option, vulnerable American puts may be exercised before maturity. More interestingly, however, vulnerable American calls on non-dividend-paying stocks may also be exercised before maturity, unlike ordinary American calls. This result follows directly from the fact that the price of a vulnerable European call may fall with time to maturity. This is clearly not possible for an American call, as the option holder can always behave as if the maturity of the option were

at an earlier date, say $T' < T$, and exercise at that date. Because a European call that matures at T' may be more valuable than one that matures at date T, it may pay to exercise early an American call that matures at date T. Note that we cannot have $C \geqslant V$, for then the asset with price V would dominate the vulnerable call. However, if $C < V$, then we must have $S - X < V$ since $C \geqslant S - X$. Hence, it follows that a vulnerable American call should be exercised at (or possibly before) the first instant that $S - X \geqslant V$. Note that, if prices move continuously, then the holder of a vulnerable American call option never collects less than the payment promised by the option writer, which means that the option writer never defaults.

II. Two Simple Cases

In this section, we examine two types of vulnerable options that have simple closed-form solutions when $S(t)$ follows

$$\frac{dS}{S} = \mu dt + \sigma dz, \tag{9}$$

where σ, the standard deviation of the rate of return on S, is constant, dz is the increment of a standard Wiener process, and μ is the expected return on S. First, we compute the value of the covered option. Next, we provide a formula for the value of an option guaranteed with a fixed margin.

A. Covered Options

We consider the value of a European call written by an investor whose only asset is α shares of the underlying asset. In this case, using our previous notation, the value of the call is $c(S(t), \alpha S(t), X, T)$. As the call depends on only one stochastic variable, i.e., the price of the underlying asset, it must follow the same partial differential equation as the one followed by an ordinary call. At maturity, the value of the call must satisfy

$$S(T) - X \quad \text{if} \quad \alpha S(T) \geqslant S(T) - X \geqslant 0, \tag{10a}$$

$$\alpha S(T) \quad \text{if} \quad S(T) - X > \alpha S(T), \tag{10b}$$

and zero otherwise. Now, (10a) implies that, if the option writer is solvent at maturity, the payoff of the option is the same as the payoff of an ordinary option, while (10b) states that the option holder receives the assets of the option writer if the writer is not solvent and the option is in the money at maturity. The payoff given by (10a) has the same probability as $X/(1 - \alpha) \geqslant S(T) \geqslant X$, while the payoff stated in (10b) has the same probability as $S(T) \geqslant X/(1 - \alpha)$. The value of the call can be computed by taking the expectation of its payoffs under the assumption that $\mu = R$ and discounting it back to date t at the rate R. It immediately follows that, for $\alpha < 1$,

$$c(S(t), \alpha S(t), X, T))$$

$$= c(S(t), X, T) - (1 - \alpha)c(S(t), X/(1 - \alpha), T), \tag{11}$$

where the prices of the two calls on the r.h.s. of (11) are given by the Black and Scholes [2] formula. Another way to see this is to note that the payoff at maturity can be written as $S(T) - X$ if $S(T) > X$ minus $(1 - \alpha)(S(T) - X/(1 - \alpha))$ if $S(T) > X/(1 - \alpha)$. The first part of the payoff produces the first term on the r.h.s., and the second part produces the second term. By inspection, equation (11) yields the Black and Scholes formula as $\alpha \to 1$. Whenever $\alpha > 1$, the Black and Scholes [2] formula holds because the probability of default on the option is zero. Because the value of the vulnerable option is a function of the difference of two ordinary calls, it is not surprising that one can get comparative-static results of opposite signs for some variables from those one obtains for the Black and Scholes formula. One can show that the value of the vulnerable option given in equation (11) may fall with increases in T, R, and σ^2.

B. *Options Guaranteed by a Fixed Margin*

We consider the valuation of a European call on an asset with price dynamics that follow equation (9) that is guaranteed by a fixed margin deposit of M. In this case, the option pays $\min[\max(S(T) - X, 0), M]$ at maturity. Hence, as M is fixed, the option holder receives only M when $S - X$ exceeds M. It immediately follows that the value of the call option is

$$c(S(t), M, X, T) = c(S(t), X, T) - c(S(t), M, T). \tag{12}$$

Hence, the call is isomorphic to subordinated debt. It is known from Black and Cox [1] that the value of subordinated debt can fall with increases in T, R, and σ^2. Note that, if the margin is deposited in riskless securities so that its value grows at the rate R, M on the r.h.s. of (12) is replaced by $M \exp(-R(T - t))$.

III. When the Underlying Asset Follows a Pure Jump Process

In this section, we consider the valuation of vulnerable options written on an asset with a current value that follows a pure jump process and discuss examples of such options. In this case, $S(t)$ follows

$$dS = \mu(S)Sdt + \begin{cases} \overset{\lambda(S)}{\diagup} & (\tilde{k}(S) - 1)S \\ \underset{1 - \lambda(S)}{\diagdown} & 0, \end{cases} \tag{13}$$

where $\lambda(S)$ is the probability of a jump taking place. If a jump takes place, the value of the asset jumps by $(k(S) - 1)S$, where $k(S)$ is a realization of the random variable $\tilde{k}(S)$. We assume that the value of the assets of the option writer, $V(t)$, follows

$$dV = \mu_V Vdt + \sigma_V Vdz_V, \tag{14}$$

where dz_V is the increment of a standard Wiener process. Both μ_V and σ_V are assumed to be constant.

Cox and Ross [4] provide closed-form solutions for ordinary options written on an asset that follows a pure jump process. Their approach relies on the fact that, if at any time the asset price can move to only one of two values, the payoff

of the option can be replicated by holding a portfolio with positions in the underlying asset and the safe asset. Unfortunately, the Cox and Ross [4] approach does not generalize to the case of vulnerable options. To understand this, consider a European call with exercise price X and time to maturity T written on an asset that pays F $(F \geq X)$ at maturity with probability π and zero with probability $(1 - \pi)$. If the option is an ordinary call, investing $-(F - X)/F$ in the risky asset provides a perfect hedge for the option. However, if the option is a vulnerable call, it pays $\min[V^*, F - X]$, where V^* is the value of V at maturity, with probability π and zero with probability $(1 - \pi)$. The only way to hedge the payoff of the option if the underlying asset is worth F at maturity is to short V and buy a call on V with exercise price $F - X$. In this case, however, one is left with $-\min[V^*, F - X]$ if the underlying asset is worthless at maturity. Hence, the option cannot be priced by arbitrage. Merton [11] faces a similar problem when pricing an option on a stock with dynamics that are a mixture of jump and diffusion processes. To price the option, Merton [11] assumes that the jump risk is nonpriced risk. In the following, we make this assumption and can therefore use Merton's [11] solution technique. The idea is that a perfect hedge can be formed for the diffusion risk, while the other risk is diversifiable, so that none of the risk of a portfolio containing the option hedged against the diffusion risk is priced. Therefore, the portfolio must earn the risk-free rate, and the Cox and Ross [4] method of discounting the expected payoff of a contingent claim using risk-neutral price dynamics for the underlying assets can be used to price the vulnerable option.

A. Vulnerable European Puts and Calls

We now consider the case of jumps driven by a Poisson process, which means that $\lambda(S)$ is a constant, λ. In this case, the price of a vulnerable call on asset S with time to maturity T and exercise price X is

$$c(S, V, X, T) = \sum_{n=0}^{\infty} \frac{e^{-\lambda T}(\lambda T)^n}{n!} [E_n[e^{-RT}\min(V^*, \max(S_n - X, 0))]], \quad (15)$$

where n is the number of jumps, E_n is the expectation operator with the expectation conditional on n jumps having taken place, and S_n is the value of the underlying asset if n jumps have taken place. Unless the size of the jumps is nonstochastic, S_n will be a random variable. Because of the put-call parity theorem of Section I, equation (15) makes it possible to derive the value of a vulnerable put option. While equation (15) provides a useful way to calculate the value of a vulnerable option, it is useless unless the distribution of the jump size is known. In the following, we present applications of equation (15) for which the distribution of the jump sizes is known.

A.1. Pensions

It is common for firms to offer compensation contracts that promise some fixed payment at some age or after a given period of employment. For instance, a firm might promise a large lump-sum payment at retirement age if the employee is still alive. We assume that the probability of survival to any date is given by an exponential distribution. We also assume that only the employee, and not his

or her heirs, is entitled to the pension. Let $S(t)$ be the current value of an artificial asset that pays F at date $t + T$ if the employee is alive and zero otherwise. Suppose the firm puts aside funds that, at time $t + T$, will be worth $X < F$, so that the employee will get X for sure. It immediately follows that the current value of the unfunded part of the lump-sum payment is the value of a vulnerable call option on the asset with value $S(t)$ with exercise price X:

$$c(S, V, X, T) = e^{-\lambda T}[V - c(V, F - X, T)], \tag{16}$$

where $e^{-\lambda T}$ is the probability that the employee will still be alive at maturity of the contract and V is the value, per pension, of the firm net of the funds set aside to pay X. Interestingly, the value of the option falls with time to maturity and with the variance of the rate of return of the value of the firm.

A.2. Bond Insurance

Consider a bond issued by a foreign country where there is some probability that a change in political regime will occur that would make the debt worthless. The probability that the current political regime is still in place at any date is given by an exponential distribution. The value of bond insurance in this case is the value of a vulnerable put that pays the face value of the bond at maturity if default has occurred. The value of the insurance is

$$p(S, V, X, T) = (V - c(V, X, T))(1 - e^{-\lambda T}), \tag{17}$$

where S is the current value of the uninsured debt, V is the value of the assets of the insurer, $e^{-\lambda T}$ is the probability that the government will not have been overthrown during a time interval of T, and X is the face value of the debt. Here, an increase in time to maturity has an ambiguous effect. As time to maturity increases, the present value of the payment promised by the insurer falls. However, the probability of default increases with time to maturity. Note also that the value of the put falls with the variance of the rate of change of the value of the insurer's assets.

B. American Puts and Calls

In this section, we extend the analysis to American puts and calls. If s is the time at which the call option is exercised, the value of the call is given by

$$C(V, S, X, T) = \text{E}^*[e^{-Rs}\min[V(s), \max(S(s) - X, 0)]], \tag{18}$$

where E^* denotes the expectation taken with respect to risk-neutral price dynamics for $V(s)$. The time of exercise is not known beforehand. However, at time s, it must be true that

$$C(V, S, X, T - s) = \min(V(s), S(s) - X). \tag{19}$$

With the dynamics assumed in this section, there is no simple characterization of the exercise policy. This is due to the fact that the vulnerable option may be exercised when $S - X < V$; i.e., an American option on a non-dividend-paying asset may be exercised before maturity even when the option writer can fulfill his promise to pay $S - X$ when assigned. An ordinary call is not exercised early because of the time value of money and the volatility effect. However, if the

interest rate is zero, there is no time value, and, if $dV = -dS$, then the volatility effect is negative for a vulnerable call sufficiently in the money. Hence, there are cases where early exercise is optimal. In the following, we discuss two examples in which the exercise policy is easily characterized.

B.1. An Insurance Policy

Consider an insurance policy that pays up to F if some damage takes place and has a deductible of X. If one is willing to assume that the probability of the damage taking place does not depend on the actions of the insurance holders, the damage can be modeled as exponentially distributed. In reality, the damage is not a traded asset, but we assume for simplicity that it is. We assume that the damage can take place only once and that its value stays fixed once it has occurred; hence, the optimal exercise policy is to exercise when the damage occurs. We ignore the fact that insurers have incentives to try to delay payment. The value of the insurance policy is equal to the value of a vulnerable call on the present value of the damage S with exercise price X minus the value of a vulnerable call on the present value of S with exercise price F. If $I(V, S, F, X, T)$ is the value of the insurance policy, we have

$$I(V, S, F, X, T) = C(V, S, X, T) - C(V, S, F, T). \qquad (20)$$

Note now that

$$C(V, S, X, T) = E^*[e^{-Rs}\min[V(s), \max(S(s) - X, 0)]], \qquad (21)$$

where s is the time of the damage. The same result applies for $C(V, S, F, T)$.

Suppose now that the damage is fixed and takes value F. In this case, in the absence of a deductible, we have

$$C(V, S, F, T) = \int_0^T \lambda e^{-\lambda s}[Fe^{-\int_0^s R dv} - P(V, F, s)] \, ds, \qquad (22)$$

where $P(V, F, s)$ is the price of an ordinary put on the firm's assets, maturing at time s, with exercise price F. Not surprisingly, the value of the insurance policy falls with the variance of the growth rate of the assets of the insurance company and increases with the value of these assets. The value of the insurance policy always increases with time to maturity.

B.2. A Warranty

Consider a warranty that pays X if the product breaks down, provided that the breakdown occurs less than T from now. Here, X is a promised payment by the manufacturer, so that, if the product breaks down at time s, the owner of the product receives

$$\min(X, V(s)),$$

where $V(s)$ is the value at time s of the assets of the manufacturer per warranty outstanding. If the breakdown is modeled as exponentially distributed and if the breakdown occurs for all products at the same time, we have

$$W(S, V, X, T) = \int_0^T e^{-(\lambda+R)s} \lambda \mathrm{E}^*[\min(V(s), X)]\, ds$$

$$= \int_0^T e^{-(\lambda+R)s} \lambda [X - P(V, X, s)]\, ds, \qquad (23)$$

where $W(S, V, X, T)$ is the value of the warranty. Not surprisingly, the value of the warranty falls as the variance of the rate of change of the value of the firm increases and increases as the value of the firm increases. Note that the value of a warranty when the products sold by the firm break down at random but different times could be modeled in the same way. However, in this case, V would follow a mixed diffusion-jump process.

IV. When the Underlying Asset Follows a Lognormal Diffusion Process

In this section, we assume that the value of the underlying asset follows

$$dS = \mu_S S dt + \sigma_S S dz_S, \qquad (24)$$

where σ_S is assumed to be constant and dz_S is the increment of a standard Wiener process. When both S and V follow lognormal diffusion processes, it is possible to construct a perfect hedge for a vulnerable option. The value x of any vulnerable option must satisfy the following partial differential equation:

$$-\frac{\partial x}{\partial T} = Rx - RV \frac{\partial x}{\partial V} - RS \frac{\partial x}{\partial S} - \frac{1}{2} \sigma_S^2 S^2 \frac{\partial^2 x}{\partial S^2}$$

$$-\frac{1}{2} \sigma_V^2 V^2 \frac{\partial^2 x}{\partial V^2} - \rho \sigma_S \sigma_V VS \frac{\partial^2 x}{\partial S \partial V}, \qquad (25)$$

where ρ is the instantaneous correlation coefficient between the rates of return on assets S and V. The boundary conditions for equation (25) depend on the contractual provisions of the vulnerable option under study. We examine European vulnerable options when S and V follow a bivariate normal distribution.

A. Vulnerable European Call Options

The value of a vulnerable European call option can be obtained by taking the expectation of the payoff of the option under the assumption of risk-neutral price dynamics. The value of the option can therefore be written as (see Johnson [8])

$$c(S, V, X, T) = \frac{e^{-RT}}{2\pi(1 - \rho^2)^{1/2} \sigma_V \sigma_S T}$$

$$\times \left[\int_X^\infty \frac{dS^*}{S^*} \int_0^{S^*-X} dV^* e^{-D} + \int_0^\infty \frac{dV^*}{V^*} \int_X^{V^*+X} \frac{dS^*}{S^*} e^{-D}(S^* - X) \right], \qquad (26a)$$

where

$$D = \frac{A^2 - 2\rho AB + B^2}{2(1 - \rho^2)} \qquad (26b)$$

$$A = \frac{\log(S/S^*) + (R - \sigma_S^2/2)T}{\sigma_S\sqrt{T}} \tag{26c}$$

$$B = \frac{\log(V/V^*) + (R - \sigma_V^2/2)T}{\sigma_V\sqrt{T}}. \tag{26d}$$

Asterisks denote asset values at maturity of the vulnerable call. In equation (26a), the first double integral corresponds to the expected payoff if the option writer is bankrupt at maturity, while the second integral corresponds to the expected payoff in the other case.

Table I gives some numerical values for the vulnerable European calls. Notice that the value for the vulnerable call, given in the next-to-last column, is always less than the corresponding Black and Scholes value, given in the last column; moreover, in some cases the vulnerability severely reduces the value of the option. Note that the vulnerable call price is less sensitive to the five parameters that enter the Black and Scholes equation than is the price given by the Black and Scholes equation itself. While the values in the table were obtained by doing the integrals numerically, other approaches can be used, e.g., the lattice approach of Boyle [3].

The results of this section apply to any privately written call option. It is interesting to note than an option written by a hedger is more valuable than an option written by a speculator as one would expect ρ to be positive for a hedger.

B. Debt Guarantees

Risky-debt guarantees are used in a variety of situations. In project financing, it is often the case that a third party guarantees a loan. While the third party is

Table I

Numerical Values for Vulnerable Call Options

σ_S	σ_V	ρ	R	T	X	S	V	$c(S, V, X, T)$	c_{BS}
.2								1.77	2.17
.3								1.96	3.07
.4								2.04	3.98
	.2							1.92	3.07
	.4							2.00	3.07
		−.5						1.69	3.07
		0						1.82	3.07
			.0833					1.33	1.46
			.5833		30			2.18	4.19
					50			4.50	10.58
								.34	.44
						30		.13	.15
						50		4.24	10.64
							3	1.34	3.07
							10	2.77	3.07

Except when otherwise stated, the values for σ_S, σ_V, ρ, R, T, X, S, and V are .3, .3, .5, .0488, .3333, 40, 40, and 5, respectively. Values for the vulnerable calls are given in the next-to-last column on the right, and Black and Scholes values, c_{BS}, are given in the last column for comparison.

sometimes a government agency, it can also be a private firm. Mortgage insurance is a risky-debt guarantee, and so are other types of bond insurance. When the debt guaranteed corresponds to a risky discount bond, the value of the debt guarantee is the value of a vulnerable European put option on the debt. If the value of the risky debt and of the guarantor follow jointly a lognormal distribution, one can use the results of Section IV.A together with the put-call parity results of Section I to value the debt guarantee. Alternatively, the value of the debt guarantee can be computed directly by evaluating the following integrals:

$$p(S, V, X, T) = \frac{e^{-RT}}{2\pi(1 - \rho^2)^{1/2}\sigma_V\sigma_S T}$$

$$\times \left[\int_0^X \frac{dS^*}{S^*} \int_0^{X-S^*} dV^* \, e^{-D} + \int_0^\infty \frac{dV^*}{V^*} \int_{X-V^*}^X \frac{dS^*}{S^*} e^{-D}(X - S^*) \right]. \quad (27)$$

Note that these results also apply to the valuation of privately written puts on assets other than bonds. It is again the case that options written by hedgers are more valuable, but here hedgers are investors who, for example, are short the underlying asset.

V. Conclusions

This paper has considered the pricing of vulnerable options, i.e., options for which the probability of default is not zero. These options can have unusual properties. For example, it is possible for the value of a vulnerable European option to fall with time to maturity, with the interest rate, and with the variance of the underlying asset. Furthermore, it may pay to exercise early a vulnerable American call on a non-dividend-paying asset.

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