The Pricing of Stock Index Options in a General Equilibrium Model

Warren Bailey and René M. Stulz*

Abstract

This paper analyzes the pricing of stock index options in a simple general equilibrium model. In this model, the volatility of the stock index and the spot rate of interest are functions of a stochastic variable. The paper investigates the biases that arise when using the Black-Scholes model with the assumed volatility and interest rate dynamics. It is shown that the model can, in principle, explain the biases observed in empirical work on stock index options.

I. Introduction

Most asset allocation techniques require the direct purchase or the dynamic replication of stock index options. Since these options are so widely used in the management of large portfolios, small pricing errors in these options can lead to large dollar losses for investment strategies. Existing empirical research indicates that, when the same option formula is used to price index and stock options, it leads to different biases relative to the observed option prices. More precisely, existing empirical work seems to suggest that option pricing models generally overprice deep-in-the-money stock options and underprice deep-in-the-money index options, while the opposite biases often have been observed for out-of-the-money options.[1]

In this paper, we investigate the pricing of stock index options in a general equilibrium model. In general equilibrium, the volatility of the index is negatively related to the rate of interest. Consequently, it is not possible to consider the effect of stochastic volatility on index option prices without at the same time allowing the interest rate to vary stochastically. In this paper, we therefore price index options in a model that allows the interest rate and the volatility of the index to change randomly over time and to be related to each other. This model

* College of Business, The Ohio State University, Columbus, OH 43210. Part of this research was done while the second author was visiting the University of Chicago. The authors are grateful to Peter Carr, Herb Johnson, Francis Longstaff, *JFQA* Managing Editor Paul Malatesta, and to seminar participants at the University of California at Davis, the University of Maryland, and the University of Wisconsin for useful comments.

[1] See Evnine and Rudd (1985), Bailey (1986), Shastri and Tandon (1986), and Whaley (1986) for empirical evidence on the pricing of index spot and futures options, and see Geske and Trautman (1986) for a summary of the empirical evidence on the pricing of stock options.

can potentially explain the empirical evidence on stock index options. We find that, for one model of interest rate dynamics, the index option prices in our model are higher than Black-Scholes prices for deep-in-the-money options when the interest rates are negatively related to the level of the index and lower otherwise.

The paper is organized as follows. In Section II, we develop our model, assuming that the volatility of the index is constant. To stress the relevance of the link between the interest rate and the index volatility, we compare two economies that differ only in the volatility of the index and show that the economy with the highest index volatility can have lower index option prices. In Section III, we investigate analytically the case in which the volatility of the index changes stochastically, using the Cox, Ingersoll, and Ross (1985b) model. We are able to show that, in some cases, the sign of the Black-Scholes bias depends on whether the link between index volatility and interest rates is taken into account. In Section IV, we provide numerical simulations for the model of Section III as well as for an alternative model of interest rate dynamics. We compare the numerical values for both models to Black-Scholes values and to values that assume a stochastic volatility for the index but ignore the general equilibrium link between index volatility and the rate of interest.

II. Index Options in a Simple General Equilibrium Model

In this section, we study the pricing of index options in a simple production economy with perfect markets and continuous trading. Only one good is produced. The technology to produce that good is exogenously given and exhibits constant stochastic returns to scale. If $q(t)$ is invested in production at date t, its dynamics are given by

$$dq(t) = \mu_q q(t)dt + \sigma_q q(t)dz_q , \tag{1}$$

when input is always reinvested. μ_q and σ_q are assumed to be constant and dz_q is the increment of a standard Wiener process. We assume that production takes place through firms. Without loss of generality, we assume that firms finance production only through equity claims. Investors can buy and sell default-free bonds among themselves, so that risk-free borrowing and lending can take place. Because of the assumption of constant returns to scale, the value of a firm's common stock is equal to the quantity of the commodity used by that firm in production. If firms pay no dividends, the instantaneous rate of return on a firm's common stock is dq/q.[2] The value of the index at time t is written $I(t)$. With our assumptions, it is natural to define $I(t)$ as the number of units of the commodity owned by an investor who, at date t_0, $t_0 < t$, invested one unit of commodity in production and reinvested the proceeds continuously. Hence, by construction, $dI/I = dq/q$, so that the local variance and mean of the growth rate of the index are, respectively, σ_q^2 and μ_q. With this notation, a European index call option that matures at date t' and has exercise price equal to k, pays $\text{Max}(I(t')-k,0)$ at date t'.

[2] Dividends are omitted for simplicity. As aggregate dividends equal consumption, the level of dividends would be endogenous in our model and would depend, for instance, on the volatility of the index.

To price the stock index option, we need to derive the interest rate dynamics. Let $r(t)$ be the rate of interest on a bond that matures instantaneously. We assume that the representative investor has a constant degree of relative risk tolerance, T, and a constant subjective rate of time preference, ρ.[3] There is no outside supply of default-free bonds. Therefore, the representative investor does not hold bonds in equilibrium and invests all his wealth $w(t)$ in production.

To solve for the dynamics of r, we assume that the representative investor conjectures that r is constant and show that the investor's conjecture is self-fulfilling. If investors can borrow and lend at the rate r, the fraction of the representative investor's wealth invested in production at time t, $n(t)$, satisfies[4]

$$n(t) = T\left[\frac{\mu_q - r}{\sigma_q^2}\right]. \tag{2}$$

In equilibrium, $r(t)$ must be such that $n(t) = 1$. This implies that

$$r(t) = \mu_q - \left(\frac{1}{T}\right)\sigma_q^2 . \tag{3}$$

By construction, the right-hand side of Equation (3) is constant, so that the investor's expectation of a constant rate of interest is rational. Note, however, that the rate of interest falls with the volatility of the index.[5] This is because an increase in the volatility of the index corresponds to an increase in the volatility of the representative investor's invested wealth. As a result, the investor would want to put some of his wealth in riskless bonds. The riskless bonds therefore must become a less attractive investment to induce the representative investor to keep no wealth invested in the risk-free asset.

As the rate of interest is constant, trading is continuous, and markets are perfect, the price of the index call option is given by the Black-Scholes formula. Let $c(I(t),x,t')$ be the price of the index option. Solving for $c(I(t),x,t')$ in terms of exogenous variables, we obtain

$$c(I(t),x,t') = I(t)N\left(d_1\right) - ke^{\left[\mu_q - \frac{1}{T}\sigma_q^2\right](t'-t)}N\left(d_2\right), \tag{4}$$

where

$$d_1 = \frac{\mathrm{Ln}(q(t)/k) + \left[\mu_q + \left(\frac{1}{2} - \frac{1}{T}\right)\sigma_q^2\right](t'-t)}{\sigma_q\sqrt{t'-t}},$$

$$d_2 = d_1 - \sigma_q\sqrt{t'-t} .$$

$N(.)$ is the cumulative standard normal distribution function. Inspection of Equation (4) immediately reveals that the value of the call option depends on the expected growth rate of the index and its volatility.

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[3] Bick (1987) also makes the assumption that the representative investor exhibits constant relative risk tolerance to construct an economy in which the Black-Scholes formula holds.

[4] See, for instance, Merton (1971).

[5] See Breeden (1986) for an extensive discussion of the relation between production risks and the rate of interest.

We now turn to the comparative statics of the index option. The interpretation of these comparative statics is that we are comparing option values in different economies, since we assume that in each economy the volatility of the index is constant. Not surprisingly, the value of the option increases with $q(t)$ and with time to maturity. It also decreases with the exercise price. More surprisingly, the value of the option increases with the rate of growth of the index because the rate of interest is an increasing function of the rate of growth of the index. The most surprising result, however, is that an increase in the volatility of the index has an ambiguous effect on the value of the option.

To understand why an increase in the volatility of the index does not necessarily increase the value of the option, it is useful to note first that, if the representative investor's relative risk tolerance is large, the value of the option increases with σ_q^2. This is because, when T becomes large, the effect of an increase in σ_q^2 on the rate of interest becomes negligible. In contrast, whenever T is not very large, an increase in σ_q^2 brings about a decrease in the rate of interest, which reduces the value of the option. In this case, the total effect of an increase in the volatility of the index on the value of the option is the sum of the positive volatility effect, for a fixed rate of interest, plus the negative interest rate effect caused by the negative relation between index volatility and the rate of interest. The sum of these effects is given by

$$\text{(5)} \qquad \frac{dc}{d\sigma_q^2} = -\frac{1}{T}\tau k e^{-r\tau} N\left(d_2\right) + k e^{-r\tau} N'\left(d_2\right)\frac{\sqrt{\tau}}{\sigma_q^2},$$

where $\tau = t' - t$ and $N'(d_2)$ is the first derivative of $N(d_2)$ with respect to d_2. The right-hand side of Equation (5) is the sum of a negative and of a positive term. The positive term goes to zero as d_2 becomes large, while the negative term goes to $-(1/T)\tau k\exp(-r\tau) < 0$. Consequently, the right-hand side of Equation (5) is negative for large values of d_2. This means that the index option is more likely to be a decreasing function of the volatility of the index if it is deep in the money. As time to maturity becomes large, however, the right-hand side of Equation (5) goes to zero. This is because the value of an option with infinite time to maturity is equal to the value of the underlying asset and, hence, does not depend on the volatility of that asset.

At this point, one might be tempted to believe that our surprising results depend excessively on the strong assumptions we made to derive our model. However, the key result that underlies our analysis is that the interest rate depends positively on the expected rate of change of invested wealth and negatively on its volatility. This result is fairly robust and is not specific to our model, as evidenced by the analysis of Breeden (1986).

III. Stochastic Volatility and Index Option Prices in General Equilibrium

In this section, we explore analytically the case in which the representative investor has a logarithmic utility function, and the index follows

$$\text{(6)} \qquad \frac{dI(t)}{I(t)} = \mu_I x(t)dt + \sigma_I\sqrt{x(t)}dz_I(t).$$

These dynamics imply that both the local mean and the local variance of the growth rate of the index are proportional to $x(t)$. Like Cox, Ingersoll, and Ross (1985b), who use Equation (6) for the dynamics of aggregate wealth, we assume that $x(t)$ satisfies

$$dx(t) = \alpha(\beta - x(t))dt + \sigma_x \sqrt{x(t)}\, dz_x(t) . \tag{7}$$

We assume that $\alpha \geqslant 0$, $\beta \geqslant 0$, and $\sigma_x > 0$. With these dynamics, $x(t)$ drifts back to β with a speed given by α.

Since the representative household has a logarithmic utility function, it does not wish to hedge against unanticipated changes in interest rates and, consequently, the interest rate is given by Equation (3) with the coefficient of relative risk tolerance set equal to one,

$$r(t) = \mu_I x(t) - \sigma_I^2 x(t) . \tag{8}$$

Differentiating Equation (8) yields the interest rate dynamics[6]

$$dr(t) = \mu_I dx(t) - \sigma_I^2 dx(t) . \tag{9}$$

Equations (7)–(9) imply that the interest rate cannot be negative. It is important to note that the index volatility dynamics cannot be specified arbitrarily, as they may yield implausible interest rate dynamics. For instance, if the local mean growth rate of the index is assumed to be constant, the interest rate is negative when $x(t)$ exceeds μ_I/σ_I^2. In the next section, we investigate numerically a case in which the local mean growth rate of the index is constant and the interest rate is never negative.

If we let $\lambda = 1/(\mu_I - \sigma_I^2)$, it follows that $x(t) = \lambda r(t)$. Consequently, the stock index and interest rate dynamics can be written as functions of $\lambda r(t)$ instead of $x(t)$. Given Equations (6) and (7), the state of the world is completely specified if one knows $I(t)$ and $r(t)$, so that the index option depends on only two stochastic variables and time. Cox, Ingersoll, and Ross (1985b) solve for the price of default-free discount bonds when the rate of interest follows Equation (9) and show that the returns on bonds are perfectly negatively correlated with changes in the interest rate. This means that the index option can be hedged by taking appropriate positions in the index and a discount bond. The option therefore can be valued by discounting its payoff at maturity at the risk-free rate, using risk-neutral price dynamics for the index. If $C(I,k,r,t)$ is the value of the index option, it must satisfy

$$C(I,k,r,t) = E_{r(t),I'(t)} \left\{ e^{-\int_t^{t+\tau} r(s)ds} \max\, [I'(t+\tau) - k, 0] \right\} , \tag{10}$$

where

$$\frac{dI'(t)}{I'(t)} = r(t)dt + \lambda^{1/2} \sigma_I \sqrt{r(t)}\, dz_I . \tag{11}$$

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[6] Ramaswamy and Sundaresan (1986) study the pricing of options on index futures using the interest rate process of Equation (9), but keep the volatility of the index constant.

$E_{r(t),I'(t)}\{\cdot\}$ is the expectation conditional on $r(t)$ and $I'(t)$. The assumption that $r(s)$ is independent from $I(j)$, $j \in [0,s]$, for all s, greatly simplifies Equation (10) and is used in the remainder of this section. With this assumption, the distribution of Ln$I(j)$ conditional on $\int_t^{t+\tau} r(s)ds$ is normal.[7] Define $v(t+\tau) = \int_t^{t+\tau} r(s)ds$. With this new notation, Equation (10) becomes

$$(12) \qquad C(I,k,r,t) = E_{r(t),I'(t)}\left\{E_{v(t+\tau),I'(t)}\left\{e^{-v(t+\tau)} \max\ [I'(t+\tau)-k,0]\right\}\right\}.$$

The expression in curly brackets on the right-hand side of Equation (12) corresponds to the expected payoff of the option discounted at the risk-free rate for a given value of $v(t+\tau)$. Hence, to compute the value of this expression, we can use the Black-Scholes formula. Designating this expression by $C(I,k,r,t \mid v(t+\tau))$, we have

$$(13a) \qquad C(I,k,r,t \mid v(t+\tau)) = IN\left(d_1\right) - ke^{-v(t+\tau)}N\left(d_2\right),$$

where

$$(13b) \qquad d_1 = \frac{\text{Ln}(I/k) + \left(1+(1/2)\sigma_I^2\lambda\right)v(t+\tau)}{\sigma_I\sqrt{\lambda v(t+\tau)}},$$

$$(13c) \qquad d_2 = d_1 - \sigma_I\sqrt{\lambda v(t+\tau)}.$$

It follows from this that the value of the index option is the expected value of the option price calculated for a given $v(t+\tau)$. $v(t+\tau)$ has the interpretation of τ times the average interest rate.

We now turn to the comparative statics of the index option price. Since the comparative statics of options that exhibit stochastic volatilities have been studied previously,[8] we compare the comparative statics of $C(I,k,r,t)$ with those of a call option on I when I follows Equation (6), but the rate of interest is assumed constant. We denote the value of the option that ignores the relation between index volatility and the rate of interest by $C(I,k,x,t)$ and $\int_t^{t+\tau} x(s)ds$ by $h(t+\tau)$. Let $C(I,k,t \mid z)$ be the Black-Scholes value of a call option on I evaluated at the mean value of z, where z is either $v(t+\tau)$ or $h(t+\tau)$. Let $\bar{v}$ and $\bar{h}$ be the mean values of $v(t+\tau)$ and $h(t+\tau)$. Hull and White (1987) use Jensen's inequality to show that

$$(14) \qquad \text{sign}\{C(I,k,x,t) - C(I,k,t \mid h)\} = \text{sign}\left(\frac{\partial^2 C}{\partial \bar{h}^2}\right).$$

[7] See Hull and White (1987) for this argument.

[8] See Hull and White (1987), Johnson and Shanno (1987), Wiggins (1987), and Scott (1987). Note that these authors assume different dynamics for the volatility. For instance, Hull and White (1987) assume that the volatility of the rate of change of the index is constant, while here it falls as the state variable increases to conform with the interest rate dynamics of Cox, Ingersoll, and Ross (1985b).

This relation follows since $C(I,k,x,t)$ is the expectation of $C(I,k,t \mid h)$. Evaluating $\partial^2 C/\partial \bar{h}^2$, Hull and White (1987) find that the Black-Scholes formula, computed for the mean volatility, overprices at-the-money options and underprices deep in- or out-of-the-money options. To compare $C(I,k,r,t)$ to $C(I,k,x,t)$, we note first that, if the interest rate in $C(I,k,t \mid h)$ is set at $\bar{v}/\tau$, then $C(I,k,t \mid h) = C(I,k,t \mid v)$. We therefore can use Jensen's inequality, so that

$$\text{sign}\{C(I,k,r,t) - C(I,k,x,t)\} = \text{sign}\left\{\frac{\partial^2 C}{\partial \bar{v}^2} - \frac{\partial^2 C}{\partial \bar{h}^2}\right\}. \tag{15}$$

Evaluating the term in curly brackets on the right-hand side of Equation (15), we obtain

$$\begin{aligned}\frac{\partial^2 C}{\partial \bar{v}^2} - \frac{\partial^2 C}{\partial \bar{h}^2} = -ke^{-v(t+\tau)}\Bigg\{&\frac{1}{2}N_1(d_2)d_2\sigma_I\lambda^{1/2}v^{1/2} \\ &+ \frac{1}{2}N_1(d_2)\left(\frac{\text{Ln}I/k + v - \frac{3}{2}\sigma_I^2\lambda v}{\sigma_I\lambda^{1/2}v^{3/2}}\right) + N(d_2)\Bigg\}.\end{aligned} \tag{16}$$

The right-hand side of Equation (16) is negative for large values of LnI/k and is negligible for small values. If the rate of interest exceeds ³⁄₂ times the mean volatility of the growth rate of the index, then the right-hand side of (16) is negative for all positive values of LnI/k, while otherwise it is negative for all sufficiently large values of LnI/k. Hence, Equation (16) suggests that the theoretical Black-Scholes formula underpricing of deep-in-the-money options is lessened when one takes into account the relation between index volatility and the interest rate. This result follows because, for deep-in-the-money options, the Black-Scholes formula is a concave function of the interest rate and a convex function of the volatility. For an option that is sufficiently deep in the money, it is possible for the interest rate effect to dominate the volatility effect, so that the option value falls as the volatility increases and the Black-Scholes formula overprices the option. For this case, ignoring the link between the interest rate and the index volatility leads one to accept the nongeneral equilibrium comparative statics (Equation (14)) and to mistakenly believe that the Black-Scholes formula underprices the option. Numerical simulations have to be used to obtain further insights into the comparative statics of index options when the index follows the dynamics assumed here. Such simulations are presented in the next section, along with simulations for an alternative model of index dynamics.

IV. Numerical Comparisons of Index Option Pricing Models

In Section III, we consider a model of index price dynamics with the property that both the drift and the volatility of the index are proportional to the state variable $x(t)$. The advantages of using these index dynamics are that (1) bond prices satisfy a known formula derived by Cox, Ingersoll, and Ross (1985b), and (2) the interest rate is always positive. In this section, we numerically investigate option values for an alternative model of index dynamics with constant drift. This model makes it possible to focus the analysis squarely on the implications of stochastic volatility. However, since the drift is constant, one must restrict the range of the volatility if one wants to insure that the rate of interest is always positive. The option values obtained for this model of index dynamics are compared to Black-Scholes values, to the values that obtain for the model used in Section III, and to stochastic volatility values that assume a constant interest rate.

Assuming that the drift of the index is constant, we have the following price dynamics for the index

$$\frac{dI(t)}{I(t)} = \mu_I dt + \sigma_I \sqrt{x(t)}\, dz_I(t) . \tag{17}$$

Equation (17) implies that the local variance of the index is proportional to $x(t)$, which is also the case for the model of volatility dynamics assumed in Section III. The dynamics for $x(t)$ are the same as those assumed in Section III, that is

$$dx(t) = \alpha(\beta - x(t))dt + \sigma_x \sqrt{x(t)}\, dz_x(t) , \tag{18}$$

where α, β, and σ_x are assumed to be positive.

As in Section III, we assume that the representative investor has a logarithmic utility function. Consequently, with the assumed index dynamics, the interest rate equals

$$r(t) = \mu_I - \sigma_I^2 x(t) . \tag{19}$$

Equation (19) implies that the interest rate is always nonnegative only if $x(t)$ cannot exceed μ_I/σ_I^2. β must be chosen so that, when $x(t)$ reaches its upper limit, it is pulled back down toward β.

Using Equation (19), the state variable x can be eliminated from Equation (17),

$$\frac{dI(t)}{I(t)} = \mu_I dt + \sqrt{(\mu - r(t))}\, dz_I . \tag{20}$$

We also can write the interest rate dynamics as a function of $r(t)$,

$$dr(t) = \alpha\left[\left(\mu - \sigma_I^2 \beta\right) - r(t)\right]dt - \sigma_I \sigma_x \sqrt{(\mu - r(t))}\, dz_x . \tag{21}$$

In Section III, the index option could be priced by arbitrage since we knew from Cox, Ingersoll, and Ross (1985b) that the rate of return of default-free bonds is perfectly correlated with the rate of change of the state variable. This approach cannot be used in this section because we do not have a formula for bond prices with the state variable dynamics assumed here. However, since this economy is the same as the one in Cox, Ingersoll, and Ross (1985a), the expected rate of return on any asset is equal to the covariance of the rate of return of that asset with the rate of growth of the index. Consequently, the index option with value $C(I,k,r,t)$ must satisfy the following partial differential equation

$$\begin{aligned} & C_I Ir + C_r\left\{\alpha\left[\left(\mu - \sigma_I^2\beta\right) - r\right] + \rho\sigma_I\sigma_x(\mu - r)\right\} \\ & + \frac{1}{2}\left[C_{II}I^2(\mu - r) + C_{rr}\sigma_I^2\sigma_x^2(\mu - r) - 2C_{Ir}I\rho\sigma_I\sigma_x(\mu - r)\right] \\ & + C_t - rC = 0 . \end{aligned} \tag{22}$$

Parameter ρ is the correlation between dz_I and dz_x. Table 1 provides option values computed by solving Equation (22) numerically for C after imposing the appropriate boundary conditions.[9] The table also reports numerically computed values for a comparable nongeneral equilibrium stochastic variance model and for the stochastic drift and variance model of Section III. The option values are computed with $X(0)$ and β set at one and $r(0)$ set at 5 percent. We assume that the state variable is at its steady-state value so that the departures of our option prices from Black-Scholes prices are not simply brought about by transitory departures of the state variable from its steady-state value. The volatility parameters, correlation, and initial index value are varied to produce comparative statics. The table also includes Black-Scholes values computed using the variance parameter σ_I^2, and the constant T-bill yield consistent with the life of the option and Equation (21).

In Table 1, option prices always increase with the volatility of the index. However, the price increase is greatest for the Black-Scholes model when the correlation between the index and the state variable is negative and is greatest for the stochastic volatility model otherwise. The stochastic volatility call option values increase as the correlation between the index and the state variable increases. Finally, the effect of an increase in the volatility of the state variable also depends on that correlation. If the correlation between the index and the state variable is zero, a change in the volatility of x has no discernible effect on the option values. If ρ is greater than zero, an increase in the volatility of x decreases option values, while it has the opposite effect if ρ is smaller than zero.

Table 1 illustrates the importance of the correlation coefficient between the state variable and the index. Since, for the state variable dynamics assumed in this section, the interest rate is a decreasing linear function of the state variable, it follows that, if the correlation coefficient between the state variable and the index is positive, the correlation coefficient between the index and the rate of interest is negative. Since a negative relation between stock returns and interest rates has been often documented,[10] the model developed in this section leads one to expect index option prices to have higher values than predicted by the Black-Scholes model. For the simulations reproduced in Table 1, the magnitude of the mistake made when using the Black-Scholes formula instead of the correct stochastic volatility formula does not exceed 10 percent for the most plausible parameter values. However, if the index volatility is high, the bias can become extremely substantial for deep-in-the-money options, as evidenced by a bias of a third when the index standard deviation is 0.4, the volatility of the state variable is 0.3, and the index is 120 percent of the exercise price.

It is important to recognize that the dynamics for the index volatility assumed here differ from those of earlier papers that consider the pricing of options on assets with stochastic volatilities. Consequently, our stochastic volatility option values assuming a constant interest rate differ from those that one would obtain using the assumptions made in earlier papers. Hull and White (1987), for instance, assume that the volatility follows a lognormal diffusion process,

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[9] Numerical values are computed with the mixed explicit/implicit hopscotch method of Gourlay and McKee (1977).

[10] See Fama and Schwert (1977), for instance.

TABLE 1
Simulated Call Option Values from Four Models[a]

I/K[b]	σ_I[c]	σ_x[d]	ρ[e]	General Equilibrium[f]	Ordinary Stochastic Variance[g]	General Equilibrium Stochastic Drift and Variance[h]	Black-Scholes[i]
0.8	0.3	0.2	−0.5	0.044	0.044	0.045	0.046
0.9	0.3	0.2	−0.5	0.084	0.084	0.085	0.087
1.0	0.3	0.2	−0.5	0.138	0.137	0.140	0.142
1.1	0.3	0.2	−0.5	0.202	0.201	0.205	0.211
1.2	0.3	0.2	−0.5	0.275	0.274	0.280	0.289
0.8	0.3	0.2	0.0	0.045	0.045	0.045	0.045
0.9	0.3	0.2	0.0	0.086	0.086	0.086	0.087
1.0	0.3	0.2	0.0	0.142	0.142	0.142	0.142
1.1	0.3	0.2	0.0	0.211	0.211	0.211	0.211
1.2	0.3	0.2	0.0	0.289	0.289	0.289	0.289
0.8	0.3	0.2	0.5	0.046	0.046	0.046	0.046
0.9	0.3	0.2	0.5	0.089	0.089	0.088	0.087
1.0	0.3	0.2	0.5	0.146	0.147	0.145	0.142
1.1	0.3	0.2	0.5	0.221	0.223	0.218	0.211
1.2	0.3	0.2	0.5	0.306	0.308	0.302	0.289
0.8	0.3	0.3	−0.5	0.044	0.044	0.045	0.045
0.9	0.3	0.3	−0.5	0.083	0.083	0.084	0.087
1.0	0.3	0.3	−0.5	0.136	0.135	0.138	0.142
1.1	0.3	0.3	−0.5	0.198	0.197	0.202	0.211
1.2	0.3	0.3	−0.5	0.269	0.268	0.276	0.289
0.8	0.3	0.3	0.0	0.045	0.045	0.045	0.045
0.9	0.3	0.3	0.0	0.086	0.086	0.086	0.087
1.0	0.3	0.3	0.0	0.141	0.141	0.142	0.142
1.1	0.3	0.3	0.0	0.211	0.211	0.210	0.211
1.2	0.3	0.3	0.0	0.289	0.289	0.289	0.289
0.8	0.3	0.3	0.5	0.047	0.047	0.046	0.045
0.9	0.3	0.3	0.5	0.091	0.091	0.089	0.087
1.0	0.3	0.3	0.5	0.148	0.150	0.147	0.142
1.1	0.3	0.3	0.5	0.227	0.230	0.224	0.211
1.2	0.3	0.3	0.5	0.315	0.320	0.313	0.289
0.8	0.3	0.4	−0.5	0.043	0.043	0.045	0.045
0.9	0.3	0.4	−0.5	0.082	0.081	0.083	0.087
1.0	0.3	0.4	−0.5	0.133	0.133	0.137	0.142
1.1	0.3	0.4	−0.5	0.194	0.194	0.199	0.210
1.2	0.3	0.4	−0.5	0.265	0.264	0.272	0.289
0.8	0.3	0.4	0.0	0.045	0.045	0.045	0.045
0.9	0.3	0.4	0.0	0.086	0.086	0.086	0.087
1.0	0.3	0.4	0.0	0.140	0.141	0.141	0.142
1.1	0.3	0.4	0.0	0.211	0.211	0.210	0.211
1.2	0.3	0.4	0.0	0.289	0.290	0.288	0.289
0.8	0.3	0.4	0.5	0.047	0.048	0.046	0.046
0.9	0.3	0.4	0.5	0.092	0.093	0.091	0.087
1.0	0.3	0.4	0.5	0.152	0.153	0.151	0.142
1.1	0.3	0.4	0.5	0.234	0.238	0.235	0.211
1.2	0.3	0.4	0.5	0.325	0.332	0.335	0.289
0.8	0.2	0.3	−0.5	0.018	0.018	0.019	0.019
0.9	0.2	0.3	−0.5	0.050	0.050	0.050	0.051
1.0	0.2	0.3	−0.5	0.103	0.103	0.102	0.104
1.1	0.2	0.3	−0.5	0.173	0.173	0.172	0.177
1.2	0.2	0.3	−0.5	0.255	0.254	0.254	0.262
0.8	0.2	0.3	0.0	0.018	0.018	0.018	0.019
0.9	0.2	0.3	0.0	0.051	0.051	0.051	0.051
1.0	0.2	0.3	0.0	0.104	0.104	0.104	0.104
1.1	0.2	0.3	0.0	0.177	0.177	0.176	0.177
1.2	0.2	0.3	0.0	0.262	0.262	0.262	0.262

TABLE 1 (Cont.)

I/k[b]	σ_I[c]	σ_x[d]	ρ[e]	General Equilibrium[f]	Ordinary Stochastic Variance[g]	General Equilibrium Stochastic Drift and Variance[h]	Black-Scholes[i]
0 8	0 2	0 3	0 5	0 018	0 018	0 018	0 019
0 9	0 2	0 3	0 5	0 051	0 051	0 051	0 051
1 0	0 2	0 3	0 5	0 106	0 106	0 106	0 105
1 1	0 2	0.3	0 5	0 180	0 180	0.182	0 177
1 2	0 2	0 3	0 5	0 268	0 269	0 271	0 262
0 8	0 4	0 3	−0 5	0 070	0 069	0 074	0 076
0 9	0 4	0 3	−0 5	0 111	0 110	0 118	0 122
1 0	0 4	0 3	−0.5	0 177	0 176	0 178	0 180
1 1	0 4	0 3	−0.5	0 220	0.218	0.234	0 247
1 2	0 4	0 3	−0 5	0 285	0 283	0.303	0 322
0 8	0 4	0 3	0.0	0.076	0 076	0 076	0 076
0 9	0 4	0 3	0.0	0 123	0 123	0 122	0 123
1 0	0 4	0 3	0 0	0 179	0 179	0 179	0 180
1 1	0 4	0 3	0 0	0 248	0 248	0 247	0 247
1 2	0 4	0 3	0 0	0 323	0 324	0 322	0 322
0 8	0 4	0 3	0 5	0 088	0 090	0 082	0 076
0 9	0 4	0 3	0 5	0 146	0 151	0 138	0 123
1 0	0 4	0 3	0 5	0 206	0 215	0 204	0 180
1 1	0 4	0 3	0 5	0.311	0 326	0 307	0 248
1 2	0 4	0 3	0 5	0 413	0 437	0 422	0 322

[a] The initial interest rate is 5 percent, the initial value and long-run mean of the state variable, x, are 1, the speed of adjustment is 0 5, and the time to expiration is one year

[b] Index level, I, divided by exercise price, k

[c] Volatility parameter for index dynamics

[d] Volatility parameter for stochastic volatility dynamics

[e] Correlation between noise processes of index and volatility dynamics

[f] General equilibrium stochastic variance model with constant drift and index volatility proportional to a state variable, x, that follows a square-root mean reverting process

[g] Comparable nongeneral equilibrium stochastic variance model (i e , model with stochastic index volatility but constant interest rate)

[h] General equilibrium model with index drift and variance proportional to a state variable, x, that follows a square-root mean reverting process

[i] The Black-Scholes values are computed using a steady-state index volatility and a default-free interest rate given by the model that yields Column b option values

whereas here the volatility of the state variable follows a square-root process. The difference in the assumptions about the volatility dynamics plays a nontrivial role in the results. With the assumption made by Hull and White (1987), the percentage bias decreases as the option becomes more in the money, while here, for in-the-money options, the bias increases when the state variable is positively correlated with the index value. An important lesson from this paper is, therefore, that the economic relevance of stochastic volatility option models depends nontrivially on the assumed volatility dynamics.

V. Conclusion

This paper discusses the valuation of index options in a simple general equilibrium model. While the analysis was conducted for two different types of index dynamics, it can be extended to other dynamics. Our analysis indicates that the approach used here helps to explain the biases observed by empirical researchers. Since these biases may have alternative explanations, it would be useful to test additional restrictions on the data implied by our model. For instance, the assumed model for index volatility could be tested directly.

The general equilibrium considerations emphasized in this paper have a role to play in the valuation of other options. For instance, the value of currency options depends crucially on the volatility of the exchange rate and on the difference between the domestic and the foreign interest rates.[11] However, the interest rate differential is itself generally a function of both the mean and the volatility of the exchange rate. When valuing currency options, one wants, therefore, to take into account the relation between exchange rate volatility and the interest rate differential.

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[11] See Garman and Kohlhagen (1983) or Stulz (1983).