Small simplicial complexes with large torsion in homology

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Facets of Complexity Monday Lecture at TU Berlin

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Background

Question

For $n, d \geq 2$, what is the maximum size of the torsion subgroup of $H_{d-1}(X)$ among all d-dimensional simplicial complexes X on n vertices?

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$$\mathbb{Z}/3\mathbb{Z} \quad \oplus \quad \mathbb{Z}/33422490829392704640180589006211126262726635946771649216509246377615037674383589816431016547860503398\mathbb{Z}.$$

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d	n	Approximate size of $H_{d-1}(X)_T$
2	50	2.728×10^{4}
2	75	2.203×10^{27}
2	100	9.236×10^{58}
3	20	2.748×10^{6}
3	30	8.503×10^{82}
3	40	7.832×10^{294}
4	15	4.464×10^{3}
4	20	3.172×10^{94}
5	14	3.516×10^{7}
5	17	7.521×10^{82}

Table: Examples of randomly-generated simplicial complexes X with huge torsion in homology

Background

The complexes in the table are generated from the *torsion burst* of the Linial–Meshulam model of random simplicial complexes. See:

Kahle, Lutz, N., Parsons. *Cohen–Lenstra heuristics for torsion in homology of random complexes.* To appear in Experimental Mathematics. 2018. arXiv:1710.05683.

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Theorem (Kalai, 1983)

For every $d \geq 2$, there exists a constant $k_d > 0$ so that for every n large enough there exists a d-dimensional simplicial complex X on n vertices which has $|H_{d-1}(X)_T| \geq \exp(k_d n^d)$.

Complexes with prescribed torsion in homology

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Given a finite abelian group G and a dimension d, how small can n be so that there exists a simplicial complex X on n vertices with $H_{d-1}(X)_T \cong G$?

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Notation

For a finite abelian group G and a dimension d, let $T_d(G)$ denote the answer to the above question.

A lower bound on $T_d(G)$

Theorem (Kalai, 1983)

For every $d \geq 2$, if X is a simplicial complex on n vertices then $|H_{d-1}(X)_T| \leq \sqrt{d+1}^{\binom{n-1}{d}}$.

Thus there exists a constant $c_d = \frac{d}{e}\sqrt[d]{\frac{2}{\log(d+1)}}$, so that

$$T_d(G) \geq c_d \sqrt[d]{\log |G|}$$

An upper bound on $T_d(G)$

Theorem (N., 2018)

For every $d \geq 2$, there exist a constant C_d so that for any finite abelian group G,

$$T_d(G) \leq C_d \sqrt[d]{\log |G|}.$$

Thus for fixed d, $T_d(G) = \Theta_d(\sqrt[d]{\log |G|})$.

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1. Given G, build a sparse d-complex X on $O(\log|G|)$ vertices and $H_{d-1}(X)_T \cong G$
2. Use the probabilistic method to take a random quotient of X in a way that the number of vertices will be $O(\sqrt[d]{\log|G|})$ and the torsion in homology will not change.

The initial construction

Goal: For every dimension $d \geq 2$ we want to find a constant $K = K(d)$ so that for any finite abelian group G there is a simplicial complex X with:

- $\Delta(X^{(1)}) \leq K$.
- $n(X) \leq K \log_2(|G|)$.
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It suffices to prove this claim for cyclic groups. Let $G = \mathbb{Z}/m\mathbb{Z}$.

The initial construction

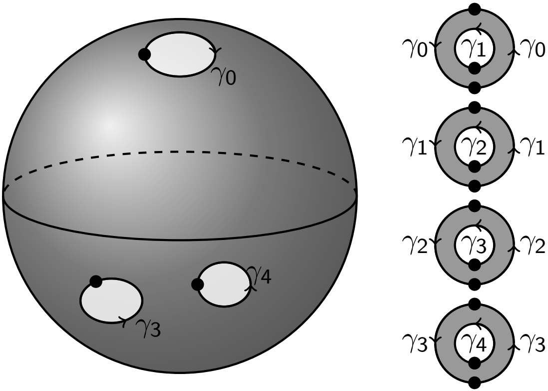

Figure: The topological space which we triangulate in our construction for $m = 25$ and $d = 2$.

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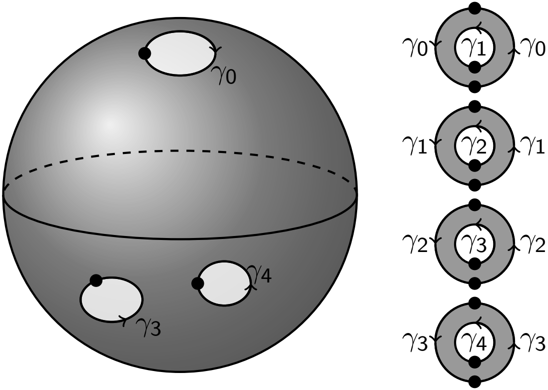

$$H_1(X)_T \cong \langle \gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4 \mid 2\gamma_0 = \gamma_1, \ldots, 2\gamma_3 = \gamma_4, \gamma_0 + \gamma_3 + \gamma_4 = 0 \rangle$$
$$\cong \langle \gamma_0 \mid (2^0 + 2^3 + 2^4)\gamma_0 = 0 \rangle \cong \langle \gamma_0 \mid 25\gamma_0 = 0 \rangle.$$

The initial constuction

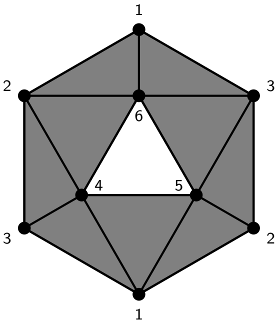

Figure: The building block for the telescope

The initial construction

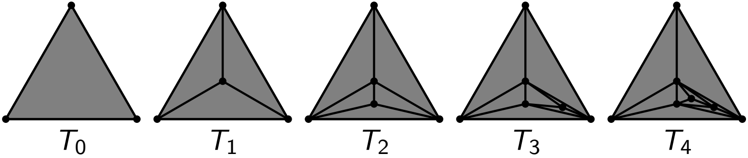

Figure: Iterative stacking to triangulate the sphere

Taking a quotient of the complex

Definition

If X is a simplicial complex and c is a proper coloring of its vertices then c induces a coloring (a pattern) of every face of X, and we define (X, c) as the quotient space of X with respect to this coloring.

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Lemma

If X is a d-dimensional simplicial complex with a proper coloring c of its vertices so that no two $(d-1)$-dimensional faces receive the same pattern, then
$H_{d-1}(X)_T \cong H_{d-1}((X, c))_T$.

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- Good news: A random approach might work.
- Bad news: We have dependency between intersecting faces and the birthday problem working against us.
- Great news: Lovász Local Lemma!

Lovász Local Lemma

Theorem (Erdős–Lovász, 1975)

Let $A_1, A_2, ..., A_k$ be events in an arbitrary probability space. Suppose that each event A_i is mutually independent of a set of all the other events A_j but at most t, and that $\Pr[A_i] \leq p$ for all $1 \leq i \leq k$. If $ep(t+1) \leq 1$ then $\Pr[\bigwedge_{i=1}^{k} \overline{A_i}] > 0$.

Coloring the vertices

Lemma

Let X be a d-dimensional simplicial complex, for $d \geq 2$, on n vertices with $\Delta(X^{(1)}) \leq K - 1$ for some integer $K \geq 5$, then there exists a proper coloring c of $V(X)$ having at most

$$18K^{5d+3}d^6\sqrt[d]{n} = K(3d^5K^{5d})(6K^2d\sqrt[d]{n}) \text{ colors}$$

so that no two $(d-1)$-dimensional faces of X receive the same pattern by c.

Proper coloring

No intersecting $(d-1)$-faces have the same pattern.

No disjoint $(d-1)$-faces have the same pattern.

Reformulation of the main theorem

Theorem

For every $d \geq 2$, there exists $\delta_d > 0$ so that for all n, one has that every abelian group G of size at most $\exp(\delta_d n^d)$ appears as the torsion subgroup of $H_{d-1}(X)$ for some simplicial complex $X = X(G)$ on n vertices.

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Corollary

Let $h(n)$ denote the number of homotopy types of simplicial complexes on n vertices. Then $\log(h(n))$ grows faster than any polynomial in n.

An application to counting simplicial complexes

A refinement of the prescribed-torsion theorem:

Theorem (N., 2018+)

For d large enough and for every finite abelian group G,

$$T_d(G) \leq 100d \sqrt[d]{\log |G|}.$$

Therefore $T_d(G) = \Theta(d \sqrt[d]{\log |G|})$.

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Corollary

For n large enough $h(n) \geq \exp(\exp(0.003n))$.

Proof of the corollary

Let n be large and set $d = \lfloor n/(100e) \rfloor$. Every cyclic group of size up to $\exp(\exp(d))$ is realizable as $H_{d-1}(X)_T$ for some simplicial complex X on n vertices.

Number of simplicial complexes up to homotopy type

- From the corollary and a trivial upper bound on $h(n)$ we have:
$$2^{2^{0.004n}} < h(n) < 2^{2^n}$$

Number of simplicial complexes up to homotopy type

- From the corollary and a trivial upper bound on $h(n)$ we have:

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- This approach gives a better bound than counting possible sequences of Betti numbers (upper bounded by 2^{n^2}), or possible fundamental groups (upper bounded by 2^{n^3}).

Sketch of the proof of the theorem

- Up to homeomorphism the initial construction X is exactly the same.
  - The triangulation of the telescope is unchanged.
  - The triangulation of the sphere is changed in a way that requires more vertices but a smaller bound on the maximum degree of a vertex.
- Keep track of bounds on $\Delta_{0,d-1}(X)$ and $n(X)$ as this is all that is relevant to use the Local Lemma.
- Color in two steps rather than three.
  - Explicitly give a proper coloring of the *square* of the 1-skeleton of X (in the graph-power sense) using at most $10(d+1)$ colors.
  - Remember the colors from the first-round to color the vertices in the second round with fewer colors.

Sketch of the proof of the theorem

Given G and d, the initial construction $X = X(G)$ satifies the following

- $n(X) \leq 182d^4 \log |G|$
- $\Delta_{0,d-1}(X) \leq 2(3^d d) + 3d^3$
- The square of $X^{(1)}$ may be properly colored using at most $10(d+1)$ colors.

Color X a second time using at most $\sqrt[d]{3e\Delta^2 n}$ colors so that in the product of the two colorings, no two $(d-1)$-dimensional faces receive the same pattern. Therefore,

$$T_d(G) \leq (10d + 10)\sqrt[d]{546(2(3^d d) + 3d^3)^2 ed^4}\sqrt[d]{\log |G|}$$

Improving the constant

For every d, there exist three constants δ_d, k_d, h_d so that

- Every abelian group of size at most $\exp(\delta_d \binom{n}{d})$ is realizable as $H_{d-1}(X)_T$ for X on n vertices.
- At least one abelian group of size at least $\exp(k_d \binom{n}{d})$ is realizable as $H_{d-1}(X)_T$ for X on n vertices.
- No abelian group of size at least $\exp(h_d \binom{n}{d})$ is realizable as $H_{d-1}(X)_T$ for X on n vertices.

With currently known bounds as $d \to \infty$, $\delta_d \to 0$, but $k_d, h_d \to \infty$.

Toward an algorithmic version

Is it possible to give a fully constructive proof of the prescribed torsion result?

m	$d = 2$		$d = 4$	
	$n(X)$	$n((X, c))$	$n(X)$	$n((X, c))$
10^{25}	273	69	434	52
10^{100}	1106	142	1722	70
10^{400}	4465	280	6888	95
10^{1600}	17773	541	27496	130

Table: Greedy-coloring approach to bound $T_d(\mathbb{Z}/m\mathbb{Z})$. For each d, X denotes the initial construction and (X, c) denotes the final construction.

Toward an algorithmic version

An algorithmic version might help to understand what happens on a small number of vertices, rather than only asymptotics.

n	Every cyclic group	Random example	Upper bound
50	1.342×10^{8}	2.728×10^{4}	3.526×10^{280}
100	2.076×10^{34}	9.236×10^{58}	1.809×10^{1157}
150	9.485×10^{80}	6.691×10^{205}	2.340×10^{2630}
200	4.651×10^{136}	3.102×10^{406}	7.6366×10^{4699}

Table: Bounds on torsion groups for $d = 2$.

Concluding Remarks

- Main result: For every $d \geq 2$ and every finite abelian group G,

$$(1-o_d(1))\frac{1}{e}d\sqrt[d]{\log|G|} \leq T_d(G) \leq (1+o_d(1))90d\sqrt[d]{\log|G|}$$

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  - How much can the upper bound be improved? Can it be improved to find a δ_d which does not tend to zero?

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Thank you!