TORSION IN HOMOLOGY OF RANDOM SIMPLICIAL COMPLEXES

DISSERTATION

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ABSTRACT

During the mid-twentieth century, Paul Erdős and Alfréd Rényi developed their now-standard random graph model. Beyond being practical in graph theory to nonconstructively prove the existence of graphs with certain interesting properties, the Erdős–Rényi model is also a model for generating random (one-dimensional) topological spaces. Within the last fifteen years, this model has been generalized to the higher-dimensional simplicial complex model of Nati Linial and Roy Meshulam. As in the case of the probabilistic method more generally, there are (at least) two reasons why one might apply random methods in topology: to understand what a "typical" topological space looks like and to give nonconstructive proofs of the existence of topological spaces with certain properties. Here we consider both of these applications of randomness in topology in considering the properties of torsion in homology of simplicial complexes. For the former, we discuss experimental results that strongly suggest torsion in homology of random Linial–Meshulam complexes is distributed according to Cohen–Lenstra heuristics. For the latter, we use the probabilistic method to give an upper bound on the number of vertices required to construct d-dimensional simplicial complexes with prescribed torsion in homology. This upper bound is optimal in the sense that it is a constant multiple of a known lower bound.

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PUBLICATIONS

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FIELDS OF STUDY

Major Field: Mathematics

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CHAPTER 1

INTRODUCTION TO TOPOLOGICAL COMBINATORICS

The purpose of this chapter is to provide an introduction, written for an audience of non-mathematicians, to the basics of topology. We will start with a broad overview of what topology is and follow by describing topology in the combinatorial setting to define the necessary terms assumed in later chapters. Along the way we will define *simplicial complexes*, topological spaces which are determined by discrete data, and *homology*, one of the main tools for studying topology. For the most part, in this chapter no mathematical knowledge at or beyond the calculus level is assumed of the reader. Everything in this chapter is essential to the study of topology and can be found in greater detail in a standard reference such as [22].

1.1 Basic overview of topology

Similar to geometry, topology is the study of shapes. However, unlike the study of geometry where lengths, volumes, and angles are often important properties of shapes, in topology the shapes are much less rigid. One should think of the shapes studied in topology, which are called *topological spaces*, as being made out of some infinitely elastic material that can be made arbitrarily large or small and can be deformed, but cannot be cut into pieces and glued back together. Thus to a topologist a cube and a sphere are the same. Indeed one could imagine "rounding out" the edges and corners of a cube to get a sphere. A central question of topology is to decide whether or not two spaces are the same in this sense. In the case of the cube and the sphere, our visualization and intuition are sufficient to decide, and we see that the cube may be continuously deformed to a sphere and vice versa. On the other hand, consider the example of the torus (the surface of a donut) and the sphere. Are these spaces the same topologically, that is, can one be deformed to the other and back? Intuitively, it doesn't seem possible, the torus has a hole in the middle and the sphere does not. Though how would one prove it? Perhaps it is incorrect to say that the torus has a hole in the middle because the hole could maybe be considered a property of the space around the torus rather than of the torus itself. Topology provides us with the tools to quantify the hole in the torus mathematically, to say that it *is* a property of the torus itself, and thus to rigorously show that the torus and the sphere are not the same space topologically.

In general topological spaces can be much more complicated than the examples given above. For example, maybe we are given two 9-dimensional topological spaces and asked to decide if they are the same or not. Here, we cannot rely on our intuition or visualization necessarily, but with the tools of topology we could count, say, the number of 5-dimensional holes in each one. Even though we cannot visualize a 9-dimensional space or a 5-dimensional hole, if the tools that we have tell us that one space has a 5-dimensional hole and the other space does not then, exactly like for the sphere and the torus, we can conclude that the two spaces are not the same.

1.2 Topological combinatorics

In the setting for this thesis, and more broadly in the research area of topological combinatorics, the spaces that are studied will often be *simplicial complexes*. While simplicial complexes can be quite complicated as topological spaces, there are convenient to work with because they can be described by data that can be succinctly written down or stored in a computer.

1.2.1 Simplicial complexes

A (finite) *simplicial complex* X on n vertices is a collection of subsets of the set $\{1, 2, 3, ..., n\}$ which is closed under taking subsets. For example,

$$X = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}\},$$

is a simplicial complex on 3 vertices as any set in X has the property that all of its subsets also belong to X. On the other hand,

$$Y = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 3\}, \{1, 2\}, \{1, 2, 3\}\},$$

is not a simplicial complex since $\{1, 2, 3\}$ belongs to Y, but $\{2, 3\}$ is a subset of it which does not belong to Y. The sets of a simplicial complex are called faces.

While the definition above is convenient for defining a simplical complex, in topological combinatorics they are most often studied geometrically or topologically. Given a simplicial complex X we let the *dimension* of a face refer to its size minus one, and we refer to the maximum dimension of a face in X as the dimension of X. We build a topological space from X called the *geometric realization of* X by representing faces of dimension zero (that is of size one) by vertices, faces of dimension one by edges, faces of dimension two by triangles, and faces of dimension three by tetrahedra. From this point we can no longer visualize the faces in 3-dimensions, but nevertheless we continue and geometrically interpret faces of dimension d by *d-simplicies* (higher-dimensional generalizations of triangles). As an example, Figure 1.1 shows the geometric realization of the simplicial complex

$$X = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{3, 4\}, \{1, 2, 3\}\}.$$

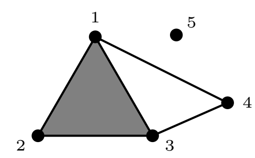

Figure 1.1: The geometric realization of X

1.2.2 Homology

One of the main tools for learning about the structure of a simplicial complex is homology. Roughly speaking, homology gives us a way to to count holes in a simplicial complexes as well as to quantify how twisted the complex is. Moreover, from the discrete nature of simplicial complexes, computer software can compute homology groups fairly easily. As an example consider the complex X shown in Figure 1.1. This complex has five vertices, five edges, and one triangle. We could give this list of eleven faces to standard homology software and it would output the following *homology groups of* X:

$$H_0(X) = \mathbb{Z}^2$$

$$H_1(X) = \mathbb{Z}$$

$$H_2(X) = 0$$

We will get into what the notation means below, but from this data one learns that X has two connected pieces, a single 1-dimensional hole, and no 2-dimensional holes. In the case of this particular complex we can see all of this from Figure 1.1. Indeed it is in two pieces, the single vertex labeled "5" and the rest of the complex, and has a 1-dimensional hole from the empty triangle on the vertices 1, 3, and 4.

In general, it will not be the case that we will be able to draw a given simplicial complex and see its structure exactly. Nevertheless, we can compute its homology to learn about its structure.

Basic overview

To be more precise than we were above, the homology of a simplicial complex is a sequence of *abelian groups* which encodes certain information about the space. There is much that could be said on the subject of abelian groups, but for the purpose of the topics in this thesis it is only necessary to give a basic overview of the relevant class of abelian groups, which are called *finitely-generated* abelian groups.

Informally speaking, an abelian group is a set of elements in which an addition operation is defined. The most basic example is the group of integers, denoted $\mathbb{Z}$, with addition defined in the normal way. A second class of examples are the finite cyclic groups denoted $\mathbb{Z}/n\mathbb{Z}$ for a natural number n. The elements of $\mathbb{Z}/n\mathbb{Z}$ are the numbers $\{0, 1, 2, 3, ..., n-1\}$ and the addition operation is addition modulo n; that is, we add in the usual way and then divide by n and keep the remainder. For example in $\mathbb{Z}/12\mathbb{Z}$, we have that $10 + 5 = 3$ since the remainder of 15 divided by 12 is 3.

For defining finitely-generated abelian groups, the only building blocks we need are the abelian groups described above, that is, $\mathbb{Z}$ and the finite cyclic groups $\mathbb{Z}/n\mathbb{Z}$. A finitely-generated abelian group is a finite direct sum of copies of $\mathbb{Z}$ and finite cyclic groups where the addition is defined component wise. For example the group $\mathbb{Z}^2 \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/10\mathbb{Z}$ has 4-tuples of the form (a, b, c, d) as its elements where a and b are integers, c is an element of $\mathbb{Z}/2\mathbb{Z}$, and d is an element of $\mathbb{Z}/10\mathbb{Z}$. To add two elements in this group we simply add in each coordinate according to the addition operation for that coordinate. Any finite abelian group A splits into a *free part* given by the direct sum of the copies of $\mathbb{Z}$ and a *torsion part* given by the direct sum of finite cyclic groups.

As mentioned above, homology is a way to assign a sequence of abelian groups (which in the present case of finite simplicial complexes are always finitely generated) to a simplicial complex. If X is a simplicial complex, then for each nonnegative integer i we define the ith homology group of X, which is denoted by $H_i(X)$. The number of copies of $\mathbb{Z}$ in the free part of $H_i(X)$ counts the number of i-dimensional holes. The torsion part of $H_i(X)$ roughly measures how "twisted" X is, and is best described with an example. Figure 1.2 shows a triangulation of the projective plane, a classical topological space. Note that this complex has six vertices; any two faces which are labeled by the same vertices are the same face. It is well known that the first homology group of this complex is $\mathbb{Z}/2\mathbb{Z}$. This is essentially because we have that the "boundary" of the complex is *twice* around the cycle on the vertices 1, 2, 3, and the homology group of order 2 encodes that fact. Think of this as an informal way to regard torsion in homology; it indicates some structure in X which is twisted around some smaller-dimensional structure some finite number of times.

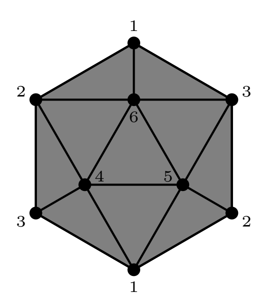

Figure 1.2: A triangulation of the projective plane

Formal definition

Above, we provided a basic overview to the non-mathematician to the idea of homology. In this section, we give a more formal definition. This section is written for the reader with more of a background in mathematics than the previous section. This section should familiarize someone who has studied math through linear algebra, but not necessarily topology, with the methods for computing homology of simplicial complexes.

Homology of a simplicial complex is defined in three steps. First we define chains, then we define boundary maps between chains to build a chain complex, lastly, we compute the homology from the chain complex in the natural way. To simplify the notation, we will assume that all simplicial complexes are finite.

Let X be a finite simplicial complex. For $i \geq 0$, let C_i denote the free abelian group generated by the i-dimensional faces of X. We refer to the elements of C_i as the *i-chains* of X. Formally, an i-chain of X is a linear combination of i-dimensional faces in X with coefficients in $\mathbb{Z}$. We may naturally regard each i-chain as an element of $\mathbb{Z}^{f_i(X)}$, where $f_i(X)$ denotes the number of i-dimensional faces of X, which records the coefficient on each i-dimensional face. Next, for each i, we define the i-dimensional *boundary map* of X to be a linear map $\partial_i : C_i \to C_{i-1}$. For a basis element of C_i, that is for a face, $[v_0, v_1, ..., v_i]$ we define $\partial_i([v_0, ..., v_i]) := \sum_{j=0}^{i}(-1)^i[v_0, ..., \hat{v_j}, ..., v_i]$ where $[v_0, ..., \hat{v_j}, ..., v_i]$ denotes the $(i-1)$-chain obtained by deleting the vertex v_j from the face $[v_0, ..., v_i]$. This gives us the following *chain complex.*

$$\cdots \xrightarrow{\partial_{i+1}} C_i \xrightarrow{\partial_i} C_{i-1} \xrightarrow{\partial_{i-1}} C_{i-2} \xrightarrow{\partial_{i-2}} \cdots \xrightarrow{\partial_1} C_0 \longrightarrow 0$$

Now for each i, denote by Z_i the kernel of ∂_i, and denote by B_i, the image of ∂_{i+1}. The elements of Z_i are called *i-cycles* and the elements of B_i are called *i-boundaries.* From the definition of the boundary maps, one can check that for every i, $\partial_{i-1} \circ \partial_i : C_i \to C_{i-2}$ is the zero map. It follows that for each i, we have $B_i \subseteq Z_i \subseteq C_i$. The ith homology group of X is the quotient group given by Z_i/B_i. From the natural way to associate an i-chain to an element in $\mathbb{Z}^{f_i(X)}$, we have a natural way to encode the boundary map ∂_i as a *boundary matrix* with coefficients in $\mathbb{Z}$ (and so that all of the coefficients of ∂_i are in $\{-1, 0, 1\}$). Using the standard machinery of linear algebra, one can compute Z_i and B_i for every boundary matrix, and thus the homology groups of X.

We point out that two spaces which are *homotopy equivalent* have the same homology groups in all dimensions. We mention this because using homology groups to enumerate non-homotopy-equivalent simplicial complexes on n vertices is an important part of Chapter 4 of this thesis, but we refer the reader to Chapter 0 of [22] for the precise definition of homotopy equivalence.

In some cases it is more convenient to compute homology with coefficients in a field rather than to compute the full *integer homology* of X (which is what we described above). Doing so is computationally much faster, but we lose information about the torsion in homology of X. Given a field $\mathbb{F}$, we can compute the homology of X with coefficients in $\mathbb{F}$, denoted $H_i(X; \mathbb{F})$ in the exact same way as described above except that we define the i-chains of X to be formal linear combinations of the i-dimensional faces of X with coefficients in $\mathbb{F}$ instead of in $\mathbb{Z}$. In this situation, $H_i(X; \mathbb{F})$ is not just an abelian group; it is a vector space over $\mathbb{F}$. It follows that $H_i(X; \mathbb{F})$ is completely described by its dimension as a vector space over $\mathbb{F}$. The dimension of $H_i(X; \mathbb{F})$ is denoted $\beta_i(X; \mathbb{F})$ and is called the ith Betti number of X with coefficients in $\mathbb{F}$.

The universal coefficient theorem, which may be found in full in Chapter 3 of [22] explains how homology with coefficients in a field $\mathbb{F}$ is related to homology with integer coefficients. Here we do not state the universal coefficient theorem in full detail, but we do review a few of the basic implications of it. Suppose X is a simplicial complex with homology groups $H_i(X) = \mathbb{Z}^{t_i} \oplus A_i$ where t_i is a nonnegative integer and A_i

is a finite abelian group for all $i \geq 0$. From the homology groups of X with integer coefficients, one can determine the homology groups of X with coefficients in $\mathbb{Q}$ or in $\mathbb{Z}/q\mathbb{Z}$ for q a prime. For $\mathbb{Q}$, one has that $H_i(X;\mathbb{Q}) = \mathbb{Q}^{t_i}$. Thus $\beta_i(X;\mathbb{Q})$ tells us the rank of the free part of $H_i(X)$, for this reason when we write $\beta_i(X)$ without specifying a field, we assume the field to be $\mathbb{Q}$. For $\mathbb{Z}/q\mathbb{Z}$, we let s_i denote the rank of the Sylow p-subgroup of A_i for each i, then $H_i(X;\mathbb{Z}/q\mathbb{Z}) = \mathbb{Z}/q\mathbb{Z}^{t_i+s_i+s_{i-1}}$. Observe that while homology with $\mathbb{Q}$ or $\mathbb{Z}/q\mathbb{Z}$ coefficients can be determined completely from homology with integer coefficients, the reverse is not true.

CHAPTER 2

RANDOM TOPOLOGY

There are at least two reasons to study random topology. The first is to use the probabilistic method to prove the existence of topological spaces with interesting properties. Classic examples of such an approach come from the study of Erdős–Rényi random graphs which are random 1-dimensional topological spaces. By using random methods, mathematicians of the twentieth century were able to prove the existence of graphs with certain properties, without actually constructing such graphs. The second reason to study random topology is to get an understanding of what a random space looks like. By understanding what a random space looks like one can understand when a space is far from being random. This provides a *topological null hypothesis* in the emerging field of topological data analysis. In ordinary statistics, one interprets data by comparing it to randomly-generated data. Analogously, one can interpret topological data by comparing it to random topological spaces.

In this thesis we will discuss research related to both of these approaches to random topology. In Chapter 3, we explore a model of random simplicial complex to understand typical topological spaces. In Chapter 4, we use the probabilistic method to prove the existence of topological spaces with interesting torsion in homology.

2.1 Notation

Starting in this chapter we will be using some standard notation from discrete mathematics, which we introduce here. If f and g are real-valued two functions (on $\mathbb{N}$), we have the following notations to compare f and g:

- We write $f(n) = O(g(n))$, which is read "f is big oh of g", if there exists a constant M so that for all n sufficiently large $f(n) \leq Mg(n)$.

- We write $f(n) = \Omega(g(n))$, which is read "f is big omega of g", if there exists a constant ϵ so that for all n sufficiently large $f(n) \geq \epsilon g(n)$.

- We write $f(n) = \Theta(g(n))$, which is read "f is big theta of g", if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

- We write $f(n) = o(g(n))$, which is read "f is little oh of g", if for every $\epsilon > 0$, $f(n) < \epsilon g(n)$ for all n sufficiently large.

- We write $f(n) = \omega(g(n))$, which is read "f is little omega of g", if for every $M > 0$, $f(n) > Mg(n)$ for all n sufficiently large.

Often we will use this notation in place of a function and write for example

$$\Pr(A(n)) = \frac{1}{2} + o(1)$$

which means that the probability of the event $A(n)$ is equal to $1/2$ plus some function that is $o(1)$, thus as n goes to infinity, the probability of $A(n)$ goes to $1/2$.

Regarding simplicial complexes, we will use the following notation for a simplicial complex X and $k \geq 0$.

- We write $\text{skel}_k(X)$ to denote the set of k-dimensional faces of X

- We write $f_k(X)$ to denote the number of k-dimensional faces of X, that is $f_k(X) = |\operatorname{skel}_k(X)|$.
- We write $X^{(k)}$ to denote the k-skeleton of X, the simplicial complex determined by all faces of X of dimension at most k.

Finally, if A is an abelian group (typically the homology group of a simplicial complex), we denote the torsion subgroup of A by A_T.

2.2 Erdős–Rényi random graphs

One of the first models of random topological spaces is the Erdős–Rényi random graph. First described in [18], $G(n,p)$, for $n \in \mathbb{N}$ and $p = p(n) \in [0,1]$, is the probability space of graphs on n vertices where $G \sim G(n,p)$ is sampled among all graphs on n vertices by including every possible edge of the complete graph on n vertices independently with probability p. Much of the research on random graphs has to do with finding *thresholds* for certain graph properties to hold *with high probability*. A graph property P is said to hold with high probability in $G(n,p)$ provided the probability that $G \sim G(n,p)$ has property P tends to 1 as n tends to infinity. Broadly speaking a threshold for a property P is a probability $p_0 = p_0(n)$ so that if $p > p_0$ then with high probability $G \sim G(n,p)$ has property P and if $p < p_0$ then with high probability G does not have property P.

To be more precise, within this thesis we will talk about three types of thresholds: sharp thresholds, coarse thresholds, and hitting-time thresholds. We discuss them here for graph properties, but they are the same for random complexes that we introduce in the next section. We say that graph property P has a sharp threshold at $p_0 = p_0(n)$ provided that for every $\epsilon > 0$ fixed, if $p < (1-\epsilon)p_0$ then $G \sim G(n,p)$ with high probability does not have property P while $p > (1+\epsilon)p_0$ implies that $G \sim G(n,p)$

has property P with high probability. We say that P has a coarse threshold at $p_0 = p_0(n)$ if $G \sim G(n,p)$ with high probability fails to have property P when $p = o(p_0)$ and has property P with high probability if $p = \omega(p_0)$. Hitting-time thresholds are a bit different and will be discussed below after giving an example. Here, we are mostly interested in sharp thresholds, so we when say "threshold" without qualification we mean a sharp threshold.

For an overview of many known thresholds in $G(n,p)$, we refer the reader to $[1, 7, 25]$. Here we focus on two important thresholds in $G(n,p)$ as they will relate to high-dimensional models of random spaces. The first of these is the *connectivity threshold*, first described in [19]. In [19], Erdős and Rényi show that $p = \dfrac{\log n}{n}$ is the sharp threshold for connectivity of a random graph. Explicitly, this means that for any $\epsilon > 0$, one has that $p > (1+\epsilon)\log n/n$ implies

$$\lim_{n\to\infty} \Pr(G \sim G(n,p) \text{ is connected.}) = 1,$$

and that $p < (1-\epsilon)\log n/n$ implies

$$\lim_{n\to\infty} \Pr(G \sim G(n,p) \text{ is connected.}) = 0.$$

In fact, see for example $[7, 19]$, a hitting-time threshold is known for connectivity of the random graph. Specifically, it is known that if one considers the process time version of the Erdős–Rényi random graph in which one begins with the empty graph on n vertices and adds one edge at each step uniformly at random from among all edges not already in the graph then this evolving graph will become connected exactly when the last isolated vertex is covered by an edge. This is a canonical example of a result which establishes a *hitting-time threshold.* For hitting-time thresholds we consider some stochastic process, here it is the process of starting with the empty graph on n vertices and adding the edges one at a time in random order, and two events A and B and show that with high probability the two events occur at the same time. In the present example, A is the event that the graph is connected and B is the event that it has no isolated vertices. Thus the hitting-time threshold for A is when B occurs.

The second threshold for $G(n, p)$ that we highlight here is the *Erdős–Rényi phase transition* at $p = 1/n$ first described in [19]. The Erdős–Rényi phase transition is a threshold for two interesting properties: the emergence of unique giant component and the emergence of cycles. Regarding the former, one has that for any $\epsilon > 0$ if $p < (1 - \epsilon)/n$, then with high probability $G \sim G(n, p)$ consists of many connected components, each of which has size $O(\log n)$, and each component has at most one cycle. On the other hand for $p > (1 + \epsilon)/n$, one has that with high probability $G \sim G(n, p)$ has a unique giant component on a constant fraction of the vertices while the rest of the components have size $O(\log n)$. The threshold for cycles to emerge is also $1/n$ and is a one-sided sharp threshold. That is, for $p = (1 - \epsilon)/n$ (with $\epsilon > 0$, fixed) one has a positive probability that $G \sim G(n, p)$ has cycles and a positive probability that it does not, but for $p = (1 + \epsilon)/n$, $G \sim G(n, p)$ contains cycles with high probability.

We discuss the connectivity-threshold and the Erdős–Rényi phase transition here as in the next section we discuss analogues of these thresholds in higher dimensions.

2.3 Linial–Meshulam random simplicial complexes

In the 2006 publication [32], Linial and Meshulam introduce their higher-dimensional analogue of the Erdős–Rényi random graph. For a fixed dimension d, the d-dimensional Linial–Meshulam model of random simplicial complex $Y_d(n, p)$ is sampled by considering the $(d-1)$-skeleton of the simplex on n vertices and including each d-dimensional face independently with probability p. Observe that $Y_1(n, p)$ is exactly the Erdős–Rényi random graph model. As with $G(n, p)$, much of the research on $Y_d(n, p)$ has been to discover threshold for various topological properties. For an overview of many of these properties we refer the reader to [26]; here we discuss the analogues of the two graph properties described above.

2.3.1 Homological connectivity

The property that a graph is connected is a homological property. Indeed a graph G is connected exactly if its zeroth reduced[1] homology group $\tilde{H}_0(G)$ is trivial. Thus the connectivity threshold for $G(n,p)$ is a threshold for this particular topological property. For $d \geq 2$, we see that $Y \sim Y_d(n,p)$ is always connected in the ordinary sense regardless of the value of p since Y has complete 1-skeleton. Thus to generalize connectivity to d-dimensions, one considers *homological connectivity.* We say that a d-dimensional simplicial complex Y is homologically connected if $\tilde{H}_i(Y) = 0$ for all $i < d$. Therefore to generalize the connectivity threshold to higher dimensions, [32] and [40] study the vanishing threshold for the $(d-1)$st homology group of $Y_d(n,p)$. In the paper which introduces the Linial–Meshulam model, [32], it is shown that $2\log n/n$ is the sharp threshold for the first homology group of $Y \sim Y_2(n,p)$ *with coefficients in* $\mathbb{Z}/2\mathbb{Z}$ to vanish. This is extended in [40] to show that $d\log n/n$ is the threshold for the codimension-1 homology group of $Y \sim Y_d(n,p)$ with coefficients in any fixed finite abelian group R to vanish.

We point out here that the coefficient ring is critically important when $d \geq 2$ due to the possibility of torsion in homology. It is well known that graph cannot have torsion in homology. Thus the homology groups (with integer coefficients) of a graph can be fully computed from homology with, say, $\mathbb{Z}/2\mathbb{Z}$ coefficients; the same cannot

[1]For any topological space X, $H_0(X)$ is isomorphic to the free abelian group generated by the connected components of X, thus $\beta_0(X)$ counts the number of connected components. A precise definition of reduced homology groups may be found in [22], for our purposes here however we just point out that for $i > 0$, the reduced homology group $\tilde{H}_i(X)$ is the same as the ordinary homology group $H_i(X)$, while $H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}$

be said about high-dimensional simplicial complexes due to the possibility of torsion. Of course by the universal coefficient theorem, if one knows that for *every* prime q, $H_i(X; \mathbb{Z}/q\mathbb{Z}) = 0$, then $H_i(X) = 0$. However, the result of [40] does not imply that $H_{d-1}(Y_d(n,p); \mathbb{Z}/q\mathbb{Z}) = 0$ simultaneously for all primes q provided that $p > d \log n/n$. Thus, [40] alone does not find the vanishing threshold for homology with integer coefficients. Indeed the sharp threshold for integer homology of $Y_d(n,p)$ to vanish is conjectured to be $d \log n/n$ but this remains open in all dimensions greater than 2.

For $d = 2$, a 2016 paper of Łuczak and Peled [37] proves a hitting-time result for the first homology group of $Y_d(n,p)$ to vanish. The result of [37] is that the in the discrete-time process version of the 2-dimensional Linial–Meshulam model, the first homology group vanishes exactly when the last isolated edge is covered by a 2-dimensional face, and so in particular $2 \log n/n$ is the sharp threshold for the first homology group of a random 2-complex to vanish. For $d \geq 3$, the best-known result is the result of Hoffman, Kahle, and Paquette from [24] that the vanishing threshold for the $(d-1)$st homology group is at most $80d \log n/n$. In particular, this shows that $\log n/n$ is a coarse threshold for homological connectivity of $Y_d(n,p)$.

2.3.2 Emergence of top homology and noncollapsibility

As discussed above, the other phase transition from $G(n,p)$ that we will compare with higher dimensional complexes is the Erdős–Rényi phase transition at $p = 1/n$. Recall from Section 2.2 that the Erdős–Rényi phase transition is both the (one-sided sharp) threshold for the emergence of cycles in $G(n,p)$ and the threshold for the emergence of a unique giant component. Here we discuss results related to higher-dimensional generalizations of both of these properties.

As in the case of generalizing the usual notion of connectivity of a random graph to homological connectivity of a random simplicial complex, we can view emergence of cycles in $Y_d(n,p)$ as a topological property. Indeed, the first homology group of a graph is generated by its cycles and a graph G is acyclic if and only if $H_1(G) = 0$. Therefore, a natural way to generalize the question of whether or not a graph has cycles to d-dimensional simplicial complexes is to ask whether or not a given d-dimensional simplicial complex has nontrivial dth homology group. This generalization in the context of $Y_d(n,p)$ is considered in [3] and [35].

Regarding the threshold for nontrivial top homology to emerge in $Y_d(n,p)$ there is first an easily obtained upper bound of $p = \dfrac{d+1}{n}$. Indeed if $c > d+1$ and $p = c/n$, then the expected number of d-dimensional faces in $Y \sim Y_d(n,p)$ is given by

$$\mathbb{E}[f_d(Y)] = \frac{c}{n}\binom{n}{d+1} = \frac{c}{d+1}\binom{n-1}{d}.$$

For n large enough this expectation is larger than $\binom{n}{d}$, that is the expected number of d-dimensional faces is larger than the number of $(d-1)$-dimensional faces. Thus by Markov's inequality on has that with high probability $f_d(Y) > f_{d-1}(Y)$. Moreover, if $f_d(Y) > f_{d-1}(Y)$, then the dth boundary matrix of Y has more columns than rows and so it has a nontrivial kernel, and therefore $H_d(Y) \neq 0$. It turns out, however that this upper bound can be improved. In [3], Aronshtam and Linial compute an explicit constant $c_d < d+1$ for $d \geq 2$ so that if $c > c_d$ and $p = c/n$, then with high probability $H_d(Y) \neq 0$ for $Y \sim Y_d(n,p)$. We omit the details for the computation of c_d here, referring the reader to [3], however an approximation for c_2 is given by 2.754, an approximation for c_3 by 3.907, and as d tends to infinity c_d approaches $d+1$, but is always strictly smaller.

Continuing to examine this question of emergence of top homology, [35] shows that $p = c_d/n$ is indeed the threshold for top homology *with real coefficients* to emerge in $Y_d(n,p)$ in proving the following theorem

Theorem 2.1 (Theorem 1.1 of [35]). *For $c < c_d$, with high probability $H_d(Y;\mathbb{R})$ is* *generated by a bounded number of copies of the boundary of the* $(d+1)$*-simplex for* $Y \sim Y_d(n, c/n)$.

Note that the boundary of a $(d+1)$-simplex is a generator for a homology class of $H_d(Y)$, and that if c is any positive constant then there is a positive probability that $Y_d(n, c/n)$ contains the boundary of a $(d+1)$-simplex. Indeed we can easily compute the expected number of $(d+1)$-simplex boundaries in $Y_d(n, c/n)$. A $(d+1)$-simplex boundary contains $d+2$ vertices and has $d+2$ d-dimensional faces each of which is included in $Y \sim Y_d(n, p)$ independently with probability p. Thus if $p = c/n$ and X counts the number of embedded copies of the $(d+1)$-simplex boundary in $Y \sim Y_d(n, p)$, we have the following expectation:

$$\mathbb{E}(X) = \binom{n}{d+2}\left(\frac{c}{n}\right)^{d+2} \approx \frac{n^{d+2}}{(d+2)!}\frac{c^{d+2}}{n^{d+2}} = \frac{c^{d+2}}{(d+2)!}.$$

If c is a constant, then through routine methods one can show that the distribution of X approaches a Poisson distribution with mean $\dfrac{c^{d+2}}{(d+2)!}$. Thus for $0 < c < c_d$, the result of [35] tells us that there is a positive probability that $Y_d(n, c/n)$ has nontrivial top homology group and a positive probability that it does not. Thus for $d \geq 2$, c_d/n is a one-sided sharp phase transition for the emergence of nontrivial top homology group, generalizing the same type of phase transition for the emergence of cycles in $G(n, p)$.

While generalizing cycles in a graph to nontrivial d-dimensional homology classes in a d-dimensional simplicial complex is natural, it is not the only possible generalization. Another generalization comes from *d-collapsibility*. This generalization is considered in [2, 4]. If Y is a d-dimensional simplicial complex then we say that a $(d-1)$-dimensional face of Y is *free* provided it is contained in exactly one d-dimensional face. An elementary collapse of Y at the free face τ is the deletion of τ and the unique d-dimensional face σ that contains it. Observe that an elementary collapse is a homotopy equivalence so in particular an elementary collapse of Y

preserves all of its homology groups. Following the convention of [4] we say that a d-dimensional simplicial complex Y is d-collapsible provided there is a sequence of elementary collapses of Y that eliminate all d-dimensional faces.

For graphs, the property of being 1-collapsible and the property of having no cycles are equivalent. In higher dimensions, however, d-collapsibility of a d-complex Y implies that $H_d(Y) = 0$, but they are not equivalent properties. Indeed, any triangulation of the projective plane is not 2-collapsible, but has trivial top homology group. Examining the question of d-collapsibility, [4] show that for $d \geq 2$, there is a constant $\gamma_d > 0$ so that for $c < \gamma_d$, the probability that $Y_d(n, c/n)$ is d-collapsible given that it does not contain the boundary of a $(d+1)$-simplex tends to one. Following up on this lower bound, [2] show that for $c > \gamma_d$, $Y \sim Y_d(n, c/n)$ is with high probability not d-collapsible. This result is improved in [36] where it is shown that for $c > \gamma_d$, $Y \sim Y_d(n, c/n)$ is far from being d-collapsible in the sense that a constant proportion of the d-dimensional faces must be deleted to arrive at a complex which is d-collapsible. As with the constant c_d in the top-homology threshold, we omit the computation of γ_d here, but an approximation in the $d = 2$ case is 2.455 and in the $d = 3$ case is 3.089. Asymptotically γ_d is $\Theta(\log d)$. In particular, these two generalizations of acyclicity do not have the same thresholds for $d \geq 2$.

While we have two notions for generalizing the property that $G \sim G(n, p)$ contains cycles, the question of generalizing the emergence of a giant component appears more subtle. However, [35] discuss one possible and convincing generalization: the *shadow* of a simplicial complex. For a d-dimensional simplicial complex Y on n vertices with complete $(d-1)$-skeleton, the shadow of Y over a field $\mathbb{F}$, denoted $SH_{\mathbb{F}}(Y)$, is defined, first by [34] as

$$SH_{\mathbb{F}}(Y) := \{\sigma \in \Delta_{n-1}^{(d)} \setminus \text{skel}_d(Y) \mid H_d(Y; \mathbb{F}) \subsetneq H_d(Y \cap \{\sigma\}; \mathbb{F})\}$$

Recall from the Erdős–Rényi phase transition that if $c < 1$ and $p = c/n$ then with high probability all components of $G \sim G(n,p)$ are on $O(\log n)$ vertices. Now, the shadow of a graph G on n vertices (over any field) is the collection of edges of $K_n \setminus E(G)$ that when added to G complete a cycle, that is, $SH(G)$ is the collection of non-edges of G between vertices in the same component. If all components of G have size at most $C \log n$ for some constant C then the size of the shadow is at most $n\binom{C\log n}{2} = O(n \log^2 n) = o(n^2)$. On the other hand when $p = c/n$ for $c > 1$, the giant component of G contributes $\Omega(n^2)$ edges to $SH(G)$. Making this connection between the giant component and the size of the shadow, Linial and Peled prove the following where c_d is the constant from the top homology threshold.

Theorem 2.2 (Theorem 1.4 from [35])**.** *Let $Y \sim Y_d(n, c/n)$ for $d \geq 2$ and $c > 0$.*

1. *If $c < c_d$, then with high probability,*

$$|\operatorname{SH}_{\mathbb{R}}(Y)| = \Theta(n).$$

2. *If $c > c_d$, then with high probability,*

$$|\operatorname{SH}_{\mathbb{R}}(Y)| = (\delta + o(1))\binom{n}{d+1}$$

where δ is an explicit positive constant depending on c and d.

Moreover, [35] shows that the shadow of a random d-complex undergoes a discontinuous first-order phase transition at $c = c_d$. That is, the size of the shadow jumps at c_d, in other words for $\delta = \delta(c,d)$ as above one has that for $d \geq 2$,

$$\lim_{c \to c_d^+} \delta(c,d) > 0.$$

Linial and Peled [35] contrast this with the Erdős–Rényi random graph where the density of the shadow (and consequently the density of the largest component) undergoes a continuous phase transition in $p = c/n$ at $c = 1$.

2.3.3 The torsion burst

While [3] and [35] provide a sharp phase transition for top-dimensional homology to emerge in $Y_d(n,p)$ and show that it is also the point where a giant shadow appears, computational experiments point to a perhaps even more remarkable property which occurs around the same point. Namely, torsion appears in the $(d-1)$st homology group. This torsion was first mentioned in the literature in 2016 in [36, 37], and in Chapter 3 we will discuss some experiments examining the torsion burst.

In order to see this torsion in the $(d-1)$st homology group one must consider the (discrete-time) stochastic-process version of $Y_d(n,p)$. Borrowing notation from [37], let

$$\mathcal{Y}_d(n) = \left\{ Y_d(n,m) \mid 0 \leq m \leq \binom{n}{d+1} \right\}$$

denote the stochastic Linial–Meshulam process. That is $\mathcal{Y}_d(n)$ is a Markov chain with states $Y_d(n,m)$ as m ranges from 0 to $\binom{n}{d+1}$ where $Y_d(n,0)$ is the complete $(d-1)$-dimensional simplicial complex on n vertices and $Y_d(n,m)$ is generated from $Y_d(n,m-1)$ by adding a d-dimensional face uniformly at random from among the d-dimensional faces on the vertex set $\{1,2,...,n\}$ which are not contained in $Y_d(n,m-1)$. While there is not yet an established hitting-time threshold for emergence of the first nontrivial cycle in the top-homology group, computational experiments indicate torsion in homology around where one would expect the hitting time to be based on [3] and [35]. We refer to this phenomenon, first in [27], as the *torsion burst* because we see torsion appear, get very large, and then vanish within a small interval in $\mathcal{Y}_d(n)$. A typical example of the torsion burst can be found in Table 2.1

Table 2.1: The torsion burst for an single instance of $\mathcal{Y}_2(65)$

m	H_2	H_1
1839	$\mathbb{Z}$	$\mathbb{Z}^{178}$
1840	$\mathbb{Z}$	$\mathbb{Z}^{177} \times \mathbb{Z}/9\mathbb{Z}$
1841	$\mathbb{Z}$	$\mathbb{Z}^{176} \times \mathbb{Z}/9\mathbb{Z}$
1842	$\mathbb{Z}$	$\mathbb{Z}^{175} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/1701881199302081202\mathbb{Z}$
1843	$\mathbb{Z}^2$	$\mathbb{Z}^{175} \times \mathbb{Z}/9\mathbb{Z}$
1844	$\mathbb{Z}^3$	$\mathbb{Z}^{175} \times \mathbb{Z}/3\mathbb{Z}$
1845	$\mathbb{Z}^4$	$\mathbb{Z}^{175}$

In Chapter 3, we will explore the torsion burst experimentally. In particular we provide data and make conjectures on *Cohen–Lenstra heuristics* within the torsion burst as well as other properties. In Chapter 4, we give a method for constructing complexes which have large torsion in homology and in Chapter 5 we discuss other models of random abelian group which apparently exhibit a torsion burst.

A proof to explain why we see the torsion burst is unknown. It is also unknown why there does not appear to be torsion anywhere else. Indeed, [37] make the following conjecture regarding the absence of torsion away from the torsion burst.

Conjecture 2.3 (Łuczak and Peled [37])**.** *For every $d \geq 2$ and $p = p(n)$ such that $|np - c_d|$ is bounded away from 0, $H_{d-1}(Y_d(n, p); \mathbb{Z})$ is torsion-free with high probability.*

In Chapters 3 and 5 we discuss some partial and conditional results towards a proof of Conjecture 2.3.

2.4 $\mathbb{Q}$-acyclic complexes

In order to motivate some of our results and conjectures on torsion in homology of simplicial complexes we begin with a discussion of $\mathbb{Q}$-acyclic complexes, a family of complexes with large torsion in homology. The study of $\mathbb{Q}$-acyclic complexes began with the classic paper of Kalai, [28], in which he proves a higher-dimensional generalization of Cayley's formula for enumerating spanning trees in the complete graph. Recall that a tree on n vertices may be defined as a graph T on n vertices which satisfies the following three conditions.

1. T has $n-1$ edges.

2. T is connected.

3. T has no cycles.

Cayley's formula states that the number of labeled spanning tress on the complete graph on n vertices is n^{n-2}. Kalai extends this to higher dimensions by first defining $\mathbb{Q}$-acyclic complexes. Observe that (2) and (3) above are really topological statements. We could replace (2) with the statement that $\tilde{H}_0(T) = 0$ and (3) with the statement that $H_1(T) = 0$. This is noted in [28] where Kalai defines a d-dimensional $\mathbb{Q}$-acyclic complex on n vertices to be a d-dimensional simplicial complex X on n vertices which has complete $(d-1)$-skeleton and satisfies the following three conditions:

1. X has $\binom{n-1}{d}$ d-dimensional faces.

2. $\tilde{H}_{d-1}(X)$ is a finite group.

3. $H_d(X) = 0$.

Note that torsion is a possibility in $H_{d-1}(X)$ when $d > 1$, that is these $\mathbb{Q}$-acyclic complexes are not necessarily $\mathbb{Z}$-acyclic. This torsion plays a major role in the main theorem of [28] given below.

Theorem 2.4 (Kalai [28]). *For $d, n \in \mathbb{N}$, let $\mathcal{T}_d(n)$ denote the family of d-dimensional $\mathbb{Q}$-acyclic complexes on n vertices then*

$$\sum_{X \in \mathcal{T}_d(n)} |\tilde{H}_{d-1}(X)|^2 = n^{\binom{n-2}{d}}.$$

Of course, graphs cannot have torsion in the zeroth homology group, so in the 1-dimensional case this is exactly Cayley's formula for enumerating spanning trees. In higher dimensions however $\mathbb{Q}$-acyclic complexes can have (extremely large) torsion in homology. Indeed as a corollary to Theorem 2.4, Kalai proves the following:

Theorem 2.5 (Kalai [28]). *For every dimension $d \geq 2$ there is a positive constant k_d so that,*

$$\mathbb{E}(|H_{d-1}(X)|^2) \geq \exp(k_d n^d)$$

where X is chosen uniformly from $\mathcal{T}_d(n)$.

Moreover Kalai shows that if X is a d-dimensional simplicial complex on n then $|H_{d-1}(X)_T| \leq \sqrt{d+1}^{\binom{n-2}{d}}$. Thus one can say that a typical d-dimensional $\mathbb{Q}$-acyclic complex on n vertices has torsion in homology of size $\exp(\Theta(n^d))$.

While there might be hope that $\mathbb{Q}$-acyclic complexes would provide a way to generate examples of small simplicial complexes with large torsion in homology, so far examples of $\mathbb{Q}$-acyclic complexes have been difficult to find. One particular family of specific examples of $\mathbb{Q}$-acyclic complexes is the class of sum complexes first described in [33] and discussed again in [39]. Given n, and $A \subseteq \mathbb{Z}/n\mathbb{Z}$, Linial, Meshulam, and Rosenthal, [33], define the sum complex X_A to be the simplicial complex of dimension $d = |A| - 1$ with vertex set $\mathbb{Z}/n\mathbb{Z}$ and complete $(d-1)$-skeleton where a d-dimensional face is included if and only if the sum of the vertices that define it are in A. The main result of [33] is that if n is a prime then for any d and any set $A \subseteq \mathbb{Z}/n\mathbb{Z}$ with $|A| = d + 1$, X_A is a $\mathbb{Q}$-acyclic complex. This family of complexes does provide

Table 2.2: Torsion in sum complexes

n	A	$H_{d-1}(X_A)_T$
7	$\{1,2,4\}$	$\mathbb{Z}/2\mathbb{Z}$
8	$\{1,2,3,5\}$	$\mathbb{Z}/4\mathbb{Z}$
9	$\{1,2,3,5\}$	$\mathbb{Z}/19\mathbb{Z}$
10	$\{1,2,3,5,8\}$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/1364\mathbb{Z}$
11	$\{1,2,3,5,8\}$	$\mathbb{Z}/3\mathbb{Z} \times (\mathbb{Z}/30\mathbb{Z})^5$
12	$\{1,3,4,6,7,8\}$	$\mathbb{Z}/20\mathbb{Z} \times \mathbb{Z}/40\mathbb{Z} \times \mathbb{Z}/1481480\mathbb{Z} \times \mathbb{Z}/201481280\mathbb{Z}$

some examples of small simplicial complexes with large torsion in homology, but other classes of examples remain elusive. In Table 2.2, we give some examples found by an exhaustive search of sum complexes on small numbers of vertices to find the one with the largest torsion group in codimension-1 homology. In Section 4.6, we discuss a method for constructing $\mathbb{Q}$-acyclic complexes with extremely large torsion in homology.

CHAPTER 3

EXAMINING THE TORSION BURST

In this chapter we summarize experimental results examining the torsion burst in the Linial–Meshulam model, particularly in the 2-dimensional case. Much of this appears in my collaboration with Matt Kahle, Frank Lutz, and Kyle Parsons [27], however when possible this chapter will add to and supplement what is discussed in [27]. The experiments conducted in [27] were carried out in order to especially examine Cohen–Lenstra heuristics for the torsion burst. We discuss this in detail in Section 3.4. In this chapter we discuss the design of these experiments, that is how we efficiently generate examples of the torsion burst, the statistics on the size of the torsion burst and its duration, and the empirical distribution of the groups within the torsion burst. We also develop a hitting-time conjecture that makes a connection between the torsion burst and the giant component in the Erdős–Rényi random graph.

3.1 Experimental design

From very early experiments as well as the comments of Łuczak and Peled in [37], it appears necessary to consider the stochastic process version of $Y_d(n, p)$ in order to find the torsion burst. That is one does not typically see torsion in $Y \sim Y_d(n, p)$ for any fixed $p = p(n)$. Thus to sample random complexes at the torsion burst it becomes necessary to not only have an efficient way to compute integer homology, but to have an efficient way to find where the torsion burst occurs. Fortunately, discrete Morse theory, which we describe in the setting of simplicial complexes, provides us with a way to do both in the setting of $\mathcal{Y}_d(n)$.

3.1.1 Discrete Morse theory

Discrete Morse theory is a combinatorical analogue of standard Morse theory for smooth manifolds. Discrete Morse theory was developed by Robin Forman in [20] which is the standard reference for the subject. Forman's development of discrete Morse theory is quite general and works for CW complexes, however for the present situation of simplicial complexes we can simplify much of the the theory and rather than describing Morse functions in full generality, we'll work in the setting of *discrete gradient vector fields* described for simplicial complexes in Sections 3-6 of [20].

Given a simplicial complex X, we consider the Hasse diagram of X as a directed graph. That is we construct a graph H from X where the vertices are the faces of X and given two faces τ and σ in X there is a directed edge from σ to τ provided τ is a codimension-1 face of σ. A discrete vector field of X is a partial matching of this directed Hasse diagram X. We say that this discrete vector field of X is a discrete gradient vector field if reversing the direction of every edge in the partial matching of the directed Hasse diagram (and not reversing the edges outside the matching) results in a directed graph with no directed cycles. The relevant theorem due to Forman is the following

Theorem 3.1 (Theorem 6.3 of [20])**.** *Suppose X is a simplicial complex with a discrete gradient vector field V, and let g_i denote number of i-dimensional faces of X which are unsaturated by V. Then X is homotopy equivalent to a CW-complex with exactly g_i faces of dimension i for all $i \geq 0$.*

Moreover, using standard methods (see Section 7 of [20]) one can construct the boundary matrices of the CW-complex resulting from applying Theorem 3.1 to a simplicial complex X with discrete gradient vector field V. Thus, given a simplicial complex if one can find a matching of its Hasse diagram which saturates many of the faces, then one can use the resulting discrete gradient vector field to find a much smaller CW-complex from which the homology groups of X may be computed. In the present situation of wanting to examine the torsion burst in a reasonable amount of time, it is useful to have a way to find a discrete gradient vector field in order to compute homology. In [27], we did so using a slightly modified implementation of an algorithm of Bruno Benedetti and Frank Lutz first described in [6].

3.1.2 Benedetti–Lutz algorithm

Given a finite simplicial complex X, the Benedetti–Lutz algorithm finds a discrete gradient vector field on X via iterative elementary collapses. As previously discussed in Section 2.3.2 an elementary collapse of X is modifying X be removing a free face τ and the unique face σ which contains it; moreover an elementary collapse is a homotopy equivalence. Given a simplicial complex X of dimension d, the Benedetti–Lutz algorithm starts by performing all possible elementary collapses until no free $(d-1)$-dimensional faces remain. At that point a d-dimensional face is selected uniformly at random from among all remaining d-dimensional face, this becomes the first critical face of dimension d. We then repeat the process of elementary collapse until no elementary collapses are possible and delete a second critical face. We continue in this way until all faces of dimension d are gone and then begin the process again at dimension $(d-1)$ with the complex that remains. Eventually the entire complex will be deleted. The discrete gradient vector field is given by pairs $\{\tau, \sigma\}$ so that σ was deleted as part of an elementary collapse at τ. The critical faces are those faces which had to be (chosen randomly and) deleted in order to continue the elementary collapsing process.

Now for running this process on a Linial–Meshulam random complex, we can stop when all top-dimensional faces are deleted. As the codimension-1 skeleton of a Linial–Meshulam random simplicial complex is complete, all homology groups in codimension greater than 1 are trivial. It follows that the top-dimensional boundary matrix of the CW-complex determined by the discrete gradient vector field together with the Euler characteristic give all the information that we need to compute the top two homology groups. Additionally, immediately before deleting the first critical faces, we may remove the isolated codimension-1 faces since each elementary collapse is a homotopy equivalence and the pure part of a simplicial complex determines the torsion part of the codimension-1 homology group and the top Betti number. This only applies before removal of the first critical face though. After that face is deleted we cannot delete codimension-1 faces except by elementary collapses. We also modify the algorithm to favor the most-recently-added faces as critical faces instead of selecting them randomly. To make this precise we provide the algorithm for finding a discrete Morse matching for a portion of the stochastic Linial–Meshulam process below.

Algorithm Modified Benedetti–Lutz for stochastic Linial–Meshluam process

Input: An instance $\mathcal{Y}_d(n)$ of the stochastic Linial–Meshulam process and two integers k and K in $\left\{0, 1, ..., \binom{n}{d+1}\right\}$.

Output: A discrete gradient vector field on $Y_d(n, K) \in \mathcal{Y}_d(n)$ in which the critical faces of dimension d are ordered so that the critical d-dimensional faces added to $\mathcal{Y}_d(n)$ in steps $k+1$ to K are first in the ordering.

1. Let $X_0 = Y_d(n, K)$ in $\mathcal{Y}_d(n)$ and for each $i \in \{1, ..., K\}$ let σ_i denote the d-dimensional face added to obtain $Y_d(n, i)$ from $Y_d(n, i-1)$. Let

$$F_1 := \{\sigma_{k+1}, \sigma_{k+2}, ..., \sigma_K\}.$$

2. Perform a sequence of elementary collapses to arrive at a simplicial complex $X_1 \subseteq X_0$ which has no free $(d-1)$-faces. Record the pairing of $(d-1)$-dimensional and d-dimensional faces given by this sequence of collapses, and let X_2 be the pure part of X_1.

3. Let

$$F_2 = F_1 \cap \mathrm{skel}_d(X_2).$$

   Delete the faces of F_2 from X_2 and record them as critical faces. Call the resulting complex X_3.

4. Perform the standard Benedetti–Lutz algorithm on X_3, and return the resulting discrete gradient vector field on $Y_d(n, K)$.

The algorithm above outputs a discrete gradient vector field in which the d-dimensional faces are ordered according to the order in which they were deleted from the complex. By standard methods this matching gives us a boundary matrix which may be used to compute the top two homology groups of $Y_d(n, i)$ for $i \in \{k, ..., K\}$. Indeed the algorithm above finds a discrete gradient vector field for $Y_d(n, K)$ from which we get a boundary matrix M. The columns of M are indexed by the critical faces which are ordered. For any $i \in \{k, ..., K\}$ if we remove from M those columns corresponding to $\{\sigma_i, \sigma_{i+1}, \sigma_{i+2}, ..., \sigma_K\} \cap F_2$ we have that the resulting submatrix is the boundary matrix for the CW-complex associated to the discrete gradient vector field restricted to $Y_d(n, i)$. The algorithm therefore works to our advantage since we may find one discrete gradient vector field and one corresponding boundary matrix from which we can compute $H_{d-1}(Y_d(n,i))$ for all i in a given list $\{k, ..., K\}$ rather than having to compute a new discrete gradient vector field and boundary matrix at each step.

3.1.3 Generating complexes at the torsion burst

To conduct the experiments described in this chapter we needed a way to generate many examples of random simplicial complexes at the torsion burst. Here we describe exactly how we used the Benedetti–Lutz algorithm to generate these complexes and compute their homology. For some experiments we only needed to find and compute the largest torsion group that appears within the torsion burst, for other experiments we want to capture the entire torsion burst.

We first consider the algorithm for finding and computing the largest torsion group within a single instance of $\mathcal{Y}_d(n)$, which we denote $LT(n,d)$. In theory this algorithm would run the entire $\binom{n}{d+1}$ process $\mathcal{Y}_d(n)$ and compute integer homology at each step, however in practice we make some reduction to vastly improve the speed of this process. The first reduction is to just consider the portion of the $\binom{n}{d+1}$ process where torsion is apparently likely to appear. As described in Section 2.3.3, we expect torsion to appear around the threshold for top homology to emerge. Thus we set $m^* = \left\lfloor \frac{c_d}{n}\binom{n}{d+1} \right\rfloor$ where c_d is the constant described by [3, 35] so that c_d/n is the sharp threshold for $Y \sim Y_d(n,p)$ to have nontrivial cycles in $H_d(Y)$. Now $LT(n,d)$ will start by searching some window around m^* to search for torsion in homology. The exact window varied between different rounds of experiments, but based on our most recent trials searching the window from $m_1 = m^* - 100$ to $m_2 = m^* + 100$ appears to almost always be sufficient to find the torsion burst.

Considering this window, we use the Benedetti–Lutz algorithm described above with $k = m_1$ and $K = m_2$, this gives us a discrete gradient vector field on $Y_d(n, m_2) \in$

$\mathcal{Y}_d(n)$ and an associated boundary matrix where the leftmost columns are associated to F_2, the critical faces added during the window from m_1 to m_2, Starting with the boundary matrix with the columns associated to F_2 deleted we can compute the homology of $Y_d(n, m_1)$, and by revealing the columns associated to F_2 in the order that the faces appear we can compute the homology of $\mathcal{Y}_d(n)$ at every state from m_1 to m_2.

Furthermore we are aided by the fact that $LT(n, d)$ searches for the largest torsion group rather than, say, the first torsion group. If β_i denotes the ith Betti number over $\mathbb{Q}$, we observe that the size of the torsion group can increase only when a face is added which decreases β_{d-1}, and can decrease only when β_d increases (this is because $H_{d-1}(Y_d(n, m+1))$ is a quotient of $H_{d-1}(Y_d(n, m))$ by the Mayer–Vietoris sequence). Thus, if we compute β_d with rational coefficients, at each stage in the window that we search we only need to compute integer homology of $Y_d(n, m)$ if $\beta_d(Y_d(n, m-1)) = \beta_d(Y_d(n, m)) < \beta_d(Y_d(n, m+1))$. This seems to be a large reduction in the number of integer homology computations. Before the phase transition, it is rare that β_d will increase and after the phase transition it is rare that it will not. This is made precise by considering the homological shadow of $Y_d(n, p)$ and its phase transition, due to [35], given above as Theorem 2.2. Furthermore since $\beta_d(Y_d(n, m))$ is a nondecreasing sequence of m, we may use a binary search, rather than a linear search, to find all values of m at which we will compute integer homology.

Computing β_{d-1} or β_d with rational coefficients is certainly faster than computing full integer homology, however, the process of computing the rank of the relevant boundary matrix typically results in the entries growing arbitrarily large, so the process is still slow. To save time, we instead pick a large prime q_0 and compute β_d with $\mathbb{Z}/q_0\mathbb{Z}$ coefficients instead of computing β_d with rational coefficients, and then we compute integer homology for all values of m for which $\beta_d(Y_d(n, m-1); \mathbb{Z}/q_0\mathbb{Z}) =$

$\beta_d(Y_d(n,m);\mathbb{Z}/q_0\mathbb{Z}) < \beta_d(Y_d(n,m+1);\mathbb{Z}/q_0\mathbb{Z})$. We keep the largest torsion group found in the codimension-1 homology as $LT(n,d)$. If there is a tie between two non-isomorphic largest torsion groups of the same size, we keep the first one as $LT(n,d)$. In practice, ties like this never occurred.

Example 1. We run our implementation of $LT(60,2)$ with $q_0 = 10007$. First, the predicted place for torsion to appear is computed as $m^* = \left\lfloor \frac{c_2}{60}\binom{60}{3} \right\rfloor$. This value is 1570; so we set the window to be from $m_1 = 1470$ to $m_2 = 1670$. Now we generate a random 2-complex on 60 vertices with 1670 2-dimensional faces which are randomly ordered. This random ordering gives us the first 1670 steps in an instance of $\mathcal{Y}_2(60)$. Now we do a binary search to find all values of m between 1471 and 1669 so that $\beta_2(Y_2(60,m-1);\mathbb{Z}/10007\mathbb{Z}) = \beta_2(Y_2(60,m);\mathbb{Z}/10007\mathbb{Z}) < \beta_2(Y_2(60,m+1),\mathbb{Z}/10007\mathbb{Z})$. In this particular instance there happened to be two such values of m, 1543 and 1545. Finally we compute integer homology of $Y_2(60,1543)$ and $Y_2(60,1545)$ and compare the torsion parts which are $\mathbb{Z}/66911823408\mathbb{Z}$ and $\mathbb{Z}/4\mathbb{Z}$, respectively. The output therefore is $\mathbb{Z}/66911823408\mathbb{Z}$.

In experiments where we needed to get the whole torsion burst, we simply found $LT(n,d)$ as above and then from the associated m with $H_{d-1}(Y_d(n,m))_T$ realizing this largest torsion group, we recorded $H_{d-1}(Y_d(n,m-1))_T, H_{d-1}(Y_d(n,m-2))_T, \ldots$ until we reached a state where the codimension-1 homology group was torsion free, and then did the same in the other direction recording $H_{d-1}(Y_d(n,m+1))_T, H_{d-1}(Y_d(n,m+2))_T, \ldots$ until there was no torsion.

3.2 Statistics of the torsion burst

Likely the first property of the groups within the torsion burst that one notices is that the largest group within the torsion burst tends to be extremely large. In this section we examine evidence to support the notion that the size of the peak group within the torsion burst grows like $\exp(n^d)$ for d-dimensional complexes on n vertices. Recall from Theorem 2.5 [28] that for every dimension d there exists a constant k_d so that the largest torsion group which may appear in the $(d-1)$st homology group for a simplicial complex X on n vertices has size at least $\exp(k_d n^d)$, so the existence of simplicial complexes with extremely large torsion is not surprising, but explicit examples in a random setting are perhaps unexpected.

We consider the $d = 2$ case. Primarily for the purposes of searching for Cohen–Lenstra heuristics, we generated 10,000 instances of $LT(n, 2)$ each for $n = 50$, 60, 75, 100, and 125. Table 3.1 summarizes the average value of $\log(|LT(n, 2)|)$ determined experimentally from these trials. In this table and all similar tables reporting averages, the standard deviation is also indicated with the "$\pm$" symbol

Table 3.1: Order of $\log(|LT(n, 2)|)$ for 10,000 trials for each value of n.

n	$\log(\lvert LT(n, 2)\rvert)$
50	12.4683 ± 4.2591
60	28.5229 ± 5.1663
75	64.8400 ± 6.1673
100	156.6246 ± 15.5111
125	291.0269 ± 11.8643

Using this data we approximated a best-fit quadratic function of $0.0328109n^2 - 2.0328n + 32.2885$ which models the size of $\log(|LT(n, 2)|)$ from the data set. In order to test this model, we also generated a few examples on more vertices and compared with the value predicted by the model.

Table 3.2: Predicted and empirical size of $\log(|LT(n)|)$.

n	Number of Trials	Predicted Size of $\log(\|LT(n,2)\|)$	Empirical Size of $\log(\|LT(n,2)\|)$
150	100	465.614	464.743
175	25	681.382	685.413
200	2	938.164	948.902
250	1	1574.770	1590.538
260	1	1721.780	1742.036
270	1	1875.350	1889.057

For $d \geq 3$, it takes longer to generate complexes at the torsion burst, but we do have a few examples of the torsion burst for larger values of d given in Tables 3.3, 3.4, and 3.5 to give a sense a how large these groups can be relative to the number of vertices.

Table 3.3: Homology groups of one instance of $\mathcal{Y}_3(25)$

Faces	H_3	H_2
1949	$\mathbb{Z}^4$	$\mathbb{Z}^{79}$
1950	$\mathbb{Z}^4$	$\mathbb{Z}^{78} \times \mathbb{Z}/6\mathbb{Z}$
1951	$\mathbb{Z}^4$	$\mathbb{Z}^{77} \times \mathbb{Z}/778016791830702358378590352176 0\mathbb{Z}$
1952	$\mathbb{Z}^5$	$\mathbb{Z}^{77} \times \mathbb{Z}/5\mathbb{Z}$
1953	$\mathbb{Z}^6$	$\mathbb{Z}^{77}$

Table 3.4: Homology groups of one instance of $\mathcal{Y}_4(17)$

Faces	H_4	H_3
1787	$\mathbb{Z}^{10}$	$\mathbb{Z}^{43}$
1788	$\mathbb{Z}^{10}$	$\mathbb{Z}^{42} \times \mathbb{Z}/2\mathbb{Z}$
1789	$\mathbb{Z}^{10}$	$\mathbb{Z}^{41} \times \mathbb{Z}/2\mathbb{Z}$
1790	$\mathbb{Z}^{10}$	$\mathbb{Z}^{40} \times \mathbb{Z}/2\mathbb{Z}$
1791	$\mathbb{Z}^{10}$	$\mathbb{Z}^{39} \times \mathbb{Z}/4923498678446918889877 4\mathbb{Z}$
1792	$\mathbb{Z}^{11}$	$\mathbb{Z}^{39}$

Table 3.5: Homology groups of one instance of $\mathcal{Y}_5(16)$

Faces	H_5	H_4
2972	$\mathbb{Z}^6$	$\mathbb{Z}^{37}$
2973	$\mathbb{Z}^6$	$\mathbb{Z}^{36} \times \mathbb{Z}/1147712621067945810235354141226409657574376675\mathbb{Z}$
2974	$\mathbb{Z}^7$	$\mathbb{Z}^{36}$

3.3 A hitting-time conjecture and a connection to the Erdős–Rényi giant component

In Chapter 2, we discussed the work of Linial and Peled [35] to generalize the Erdős–Rényi giant component to a giant shadow. Moreover, [35] show that the density of the shadow of $Y_d(n, c/n)$ undergoes a discontinuous phase transition at $c = c_d$ while the density of the largest component of $G(n, c/n)$ undergoes a continuous phase transition at $c = 1$. While this alone does not provide a hitting-time version for the emergence of the giant shadow it does suggest the possibility that the emergence of the giant shadow has hitting-time threshold.

This is best understood by a comparison to the random graph. Suppose we have a single instance of $\mathcal{Y}_1(n)$, that is the discrete-time stochastic process random graph, and are asked how many edges are there when the giant component appears. Since a giant component is defined to be a component that is on a constant fraction of the vertices, the exact hitting-time in this instance of $\mathcal{Y}_1(n)$ may be difficult to even define. Is a component on one percent of the vertices a giant component? On 10%, 50%, or 99%? Indeed the fact that the density of the largest component of $G(n, c/n)$

has a continuous phase transition tells us that when $c > 1$, $G(n, c/n)$ will always have a giant component on δn vertices where δ depends on c, but δ will be going to zero as c approaches one from above. Therefore the largest component will slowly grow larger and larger as we add more and more edges, but it will not be clear to define the exact moment that it becomes a giant component.

On the other hand, for $d \geq 2$ one could consider an instance of $\mathcal{Y}_d(n)$ and ask when it has a giant shadow over $\mathbb{R}$. As [35] show, the density of the shadow of $Y_d(n, c/n)$ undergoes a discontinuous phase transition at $c = c_d$, and so it is plausible that at a single step in the process $\mathcal{Y}_d(n)$ the size of the shadow will experience a huge jump. In the follow-up to [35], [36] conjecture that there is such a jump and that it coincides with the the giant torsion group appearing in $H_{d-1}(Y)$. We conjecture the same here and in [27] but also add a third event that occurs at the same moment.

Conjecture 3.2. *Let $\mathcal{Y}_d(n)$ be the stochastic Linial–Meshulam process in d dimensions for $d \geq 2$. Then there exists a constant δ depending on d so that with high probability there exists $m_0 \in \left\{1, \ldots, \binom{n}{d+1}\right\}$ so that the following three events occur.*

1. *The torsion part of $H_{d-1}(Y_d(n, m_0))$ is isomorphic to $LT(n, d)$.*
2. *$|SH_{\mathbb{R}}(Y_d(n, m_0))| \geq \delta n^{d+1}$, but $|SH_{\mathbb{R}}(Y_d(n, m_0 - 1))| \leq O(n)$.*
3. *$Y_d(n, m_0)$ contains a core Y' which spans the entire vertex set and has $H_{d-1}(Y') \cong LT(n, d)$, but $Y_d(n, m_0 - 1)$ does not contain such a core.*

Following the definition in [4], a d-dimensional core is a d-dimensional simplicial complex in which every $(d-1)$-dimensional face is contained in at least two d-dimensional faces. Thus cores are obstructions to d-collapsibility. Our conjecture allows us to define a higher-dimensional analogue of the giant component in terms of the faces of the complex rather than in terms of the shadow. Given a d-dimensional simplicial complex Y on n vertices we say that Y contains a *homological giant* Y' if Y' is a core of Y, $H_{d-1}(Y')$ is finite, and $V(Y) = V(Y')$. The conjecture says the hitting time for the emergence of a homological giant is the same as the hitting time for the emergence of the giant shadow. If this is true, then with high probability the homological giant has its top-dimensional homology generated by a bounded number of $(d+1)$-simplex boundaries at the moment it first appears. This means that after removal of one face from each of these $(d+1)$-simplex boundaries we have that the remaining part of the homological giant is a d-tree, i.e., a complex with $\beta_{d-1} = \beta_d = 0$. This is a bit different than a $\mathbb{Q}$-acyclic complex in the sense of Kalai in that the $(d-1)$-skeleton is not complete, but nevertheless it is reasonable to say that this complex is a higher-dimensional analogue of a tree. Indeed, it is a homologically connected maximal spanning forest of itself in the sense of Duval, Klivans, and Martin [15].

3.4 Cohen–Lenstra heuristics

Cohen–Lenstra heuristics refer to a family of probability distributions on finite abelian groups, first appearing in [11] in the context of class groups of number fields, in which the probability of each group G in the support of the distribution is inversely proportional to $|G|^\alpha |\operatorname{Aut}(G)|^\beta$, for fixed $\alpha, \beta \geq 0$. Since Cohen–Lenstra heuristics were first introduced they have appeared in number theory as well as in various models of random integral matrix cokernels (see for example [9, 11, 16, 21, 30, 38, 51–53]). The variety of settings in which Cohen–Lenstra heuristics appear suggest they are a natural family of distributions on finite abelian groups.

One of the most common distributions within the family of Cohen–Lenstra distributions is the case where $\alpha = 0$ and $\beta = 1$. However, it is well-known that this does not give a distribution on the set of *all* finite abelian groups. That is there is no distribution on the set of all finite abelian groups so that for every group G,

$$\Pr(G) \propto \frac{1}{|\operatorname{Aut}(G)|}.$$

(A simple proof of this fact appears in, for example, Chapter 5 of [31].) However, if one restricts to the family of q-groups $\mathcal{G}_q$ for any fixed prime q then there is such a distribution so that the probability assigned to any q-group is inversely proportional to the number of automorphisms of that group. Indeed, Cohen and Lenstra prove in their original paper [11] that for a fixed prime q,

$$\sum_{G \in \mathcal{G}_p} \frac{1}{|\operatorname{Aut}(G)|} = \frac{1}{\prod_{i=1}^{\infty}(1-q^{-k})}.$$

This family of distributions on q-groups is the one we consider most often here. However, in Section 3.4.2 we also discuss families of distributions with $\beta = 1$, but α equal to a positive integer.

3.4.1 Peak Cohen–Lenstra distribution

In this section we summarize data in support of the following conjecture:

Conjecture 3.3. *For a fixed prime q and fixed dimension $d \geq 2$, the Sylow q-subgroup of $LT(n, d)$ is asymptotically distributed according to the Cohen–Lenstra distribution which assigns probability $\dfrac{\prod_{k=1}^{\infty}(1-q^{-k})}{|\operatorname{Aut}(G)|}$ to any q-group G.*

To establish evidence for this conjecture, we ran $LT(n, 2)$ enough times to generate 10,000 nontrivial torsion groups each for n = 50, 60, 75, 100, and 125. We do not think omitting trials where $LT(n, 2)$ returns the trivial group changes our data too much, and consider such a situation an error state for our experiments. There are three reasons why $LT(n, 2)$ might return the trivial group:

1. There is no torsion burst.

2. There is a torsion burst, but it occurs before or after the window that we search, so we miss it.

3. There is q_0-torsion which affects our search process since in this case the Betti number $\beta(Y_2(n,m); \mathbb{Z}/q_0\mathbb{Z})$ is not always equal to $\beta_2(Y_2(n,m); \mathbb{Q})$.

Since we do not do anything to distinguish between these three possibilities we just omit and replace trials with $LT(n,2) = 0$. The simplest reason this is unlikely to significantly impact our data is that it is rare that $LT(n,2)$ returns the trivial group. In the trials on 50 vertices, only about 5% returned the trivial group (that is about 500 of the initial 10,000 trials returned the trivial group and were replaced with an equal number of new trials all of which returned nontrivial groups), and that proportion dropped significantly as n increased all the way to less than 0.1% for trials on 125 vertices. Moreover, for our experiments here we used $q_0 = 10007$, and so if our conjecture is true, we only expect to miss the torsion burst due to the presence of q_0-torsion in about 0.01% of trials. Finally, the conjectured distribution is a distribution on the q-part of $LT(n,d)$ for a fixed prime q, but this distribution may not be extended to a distribution on all finite abelian groups, as discussed above. Therefore, while the distribution may be extended to assign a positive probability to the event that there is no q-torsion for a fixed finite set of primes q (by taking the product measure over the given finite set of primes), it cannot be extended to a distribution which assigns a nonzero probability to the trivial group. That is, it cannot be extended to assign a positive probability to the event that there is no q-torsion simultaneously for *every* prime q.[1] Thus, based on our conjecture and the

[1] Lengler [31] does develop a theory of global Cohen–Lenstra heuristics compatible with the present distribution on Sylow q-subgroups in which some sets of groups are measurable and some are not. Within in this global Cohen–Lenstra theory it is generally impossible to assign any probability to a singleton set, however it does make since to assign probability zero to the trivial group.

evidence in favor of it, we expect that asymptotically the event that $\mathcal{Y}_2(n)$ has no torsion burst is exceptionally rare.

Now Conjecture 3.3 in particular implies that the probability of a particular q-group G is proportional to $|\operatorname{Aut}(G)|$. Accordingly, Tables 3.6, 3.7, and 3.8 show for any given group G the observed ratio between the number of instances where the Sylow q-subgroup of $LT(n,2)$ is trivial (remember, this is not saying $LT(n,2)$ is trivial, only that it has no q-torsion) and the number of instances where the Sylow q-subgroup of $LT(n,2)$ is isomorphic to G. If our conjecture is true then for every group G this ratio should approach the value given in the "Cohen–Lenstra Ratio" column. By the definition of the distribution in Conjecture 3.3, this Cohen–Lenstra ratio is exactly $|\operatorname{Aut}(G)|$. For example $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ has six automorphims, so in the limiting distribution $LT(n,2)$ should have trivial 2-part six times as often as it has $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ as its 2-part. From Table 3.6, 3.7, and 3.8 we see that there is strong evidence to support Conjecture 3.3.

Table 3.6: The empirical ratio of the probability that $LT(n, 2)$ has trivial 2-part to the probability of the given 2-group, compared to the predicted ratio from the Cohen–Lenstra distribution (10,000 trials for each n)

2-Subgroup	$n = 50$	$n = 60$	$n = 75$	$n = 100$	$n = 125$	Cohen–Lenstra Ratio
Trivial Group	1	1	1	1	1	1
$\mathbb{Z}/2\mathbb{Z}$	0.9045	1.0364	0.9752	0.9965	1.0076	1
$\mathbb{Z}/4\mathbb{Z}$	1.8822	1.9795	2.0063	2.0314	2.0423	2
$\mathbb{Z}/8\mathbb{Z}$	3.8530	4.3975	3.8548	4.2812	4.2525	4
$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	5.3591	5.6712	6.0705	5.8846	5.9958	6
$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$	6.4772	7.6464	7.4177	8.2820	7.9560	8
$\mathbb{Z}/16\mathbb{Z}$	7.2026	7.6666	7.1561	7.3040	7.3316	8
$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$	13.1756	14.6363	14.9526	17.5120	13.7251	16
$\mathbb{Z}/32\mathbb{Z}$	15.3466	16.8488	16.2342	19.9109	14.6262	16

Table 3.7: The empirical ratio of the probability that $LT(n, 2)$ has trivial 3-part to the probability of the given 3-group, compared to the predicted ratio from the Cohen–Lenstra distribution (10,000 trials for each n)

3-Subgroup	$n = 50$	$n = 60$	$n = 75$	$n = 100$	$n = 125$	Cohen–Lenstra Ratio
Trivial Group	1	1	1	1	1	1
$\mathbb{Z}/3\mathbb{Z}$	1.9622	1.9277	2.0334	1.9381	2.0482	2
$\mathbb{Z}/9\mathbb{Z}$	6.0997	5.8486	5.9510	6.0010	5.9337	6
$\mathbb{Z}/27\mathbb{Z}$	17.4937	17.8290	17.0212	17.8295	17.8012	18
$\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$	44.1507	44.9349	47.8632	42.0075	47.8220	48
$\mathbb{Z}/81\mathbb{Z}$	47.1440	54.7227	54.3689	56.5816	64.8620	54

Table 3.8: The empirical ratio of the probability that $LT(n, 2)$ has trivial 5-part to the probability of the given 5-group, compared to the predicted ratio from the Cohen–Lenstra distribution (10,000 trials for each n)

5-Subgroup	$n = 50$	$n = 60$	$n = 75$	$n = 100$	$n = 125$	Cohen–Lenstra Ratio
Trivial Group	1	1	1	1	1	1
$\mathbb{Z}/5\mathbb{Z}$	4.1215	4.1488	4.1628	4.0488	3.9284	4
$\mathbb{Z}/25\mathbb{Z}$	20.4866	20.7777	20.1259	19.5384	20.2080	20

3.4.2 Universal Cohen–Lenstra heuristics

While the peak torsion group which appears within the torsion burst is perhaps the most surprising, it is worth examining the smaller groups that appear. In this section we discuss experimental results and establish conjectures on the smaller torsion groups that appear within the torsion burst. Toward being able to describe these groups we need some definitions. We first begin with a formal definition of the torsion burst that until now has not been necessary.

Definition. Given an instance of $\mathcal{Y}_d(n)$ let $LT(n,d)$ refer to the largest torsion group which appears in $H_{d-1}(Y_d(n,m))$ over all $m \in \left\{0, 1, ..., \binom{n}{d+1}\right\}$, where ties between nonisomorphic torsion groups of maximum size are broken by the group which appears first in the stochastic process. Let m_0 be the first time $LT(n,d)$ appears in the codimension-1 homology group over $\mathcal{Y}_d(n)$. The *torsion burst* of $\mathcal{Y}_d(n)$ is the unique maximal consecutive set of states B in $\mathcal{Y}_d(n)$ so that $Y(n, m_0) \in B$ and every state in B realizes torsion in codimension-1 homology. The *duration* of the torsion burst is $|B|$.

Note that this definition defines the torsion burst as a nested sequence of simplicial complexes $X_1 \subseteq X_2 \subseteq X_3 \subseteq \cdots \subseteq X_{|B|-1} \subseteq X_{|B|}$, where $H_{d-1}(X_j)_T = LT(n,d)$ for some $j \in 1, ..., |B|$. Moreover, while it is *a priori* possible for torsion to appear, vanish, and reappear in the process, we did not see that occur in our experiments at all[2] and here we ignore that possibility and only consider (at most) one particular, well-defined portion of $\mathcal{Y}_d(n)$ which has torsion in homology.

Now we give names to the groups within a torsion burst.

[2]That is, this was observed to never occur in our early experiments. Once we started using the approach described in Section 3.1.3 we were not able to detect if torsion appeared, disappeared, and reappeard again.

Definition. Let $X_1 \subseteq X_2 \subseteq \cdots \subseteq X_{|B|}$ be the torsion burst in an instance of $\mathcal{Y}_d(n)$. By applying $H_{d-1}(\cdot)_T$ to this sequence of complexes we obtains a sequence of groups. Let $G_0 = LT(n,d)$ be (any instance of) the largest torsion group within the torsion burst. Now beginning at G_0 go back in the sequence and find the first group in the sequence that is not equal to G_0, we denote this by G_{-1} and call it the *first subcritical torsion group.* Now if the ith subcritical torsion group G_{-i} is nontrivial, go back until a finite group different from G_{-i} is found. We call it the $(i+1)$*st subcritical torsion group*, and denote it by $G_{-(i+1)}$. If G_{-i} is trivial, we stop and do not define the jth subcritical torsion group for $j > i$. Similarly, we may start at G_0 and work our way forward to define the *first, second, third, etc. supercritical torsion groups* denoted $G_1, G_2, \ldots$. We refer to the number of i so that G_i is defined and nontrivial as the *number of phases* of the torsion burst of $\mathcal{Y}_d(n)$.

Example 2. In the instance of $\mathcal{Y}_2(65)$ shown in Table 2.1, the largest torsion group is $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/1701881199302081202\mathbb{Z}$, the first subcritical torsion group is $\mathbb{Z}/9\mathbb{Z}$, the second subcritical torsion group is the trivial group, and for every $k \geq 3$, the kth subcritical group is undefined. On the other side, the first supercritical torsion group is $\mathbb{Z}/9\mathbb{Z}$, the second supercritical torsion group is $\mathbb{Z}/3\mathbb{Z}$, the third supercritical torsion group is trivial, and all higher supercritical torsion groups are undefined. The number of phases in this instance of $\mathcal{Y}_2(65)$ is four. We observe that the number of phases is different than the number of distinct groups since $\mathbb{Z}/9\mathbb{Z}$ is counted twice as both G_{-1} and as G_1. The number of phases is also different than the duration of the torsion burst. In this particular case, the duration of the torsion burst is five.

We make the following conjectures about the subcritical and supercritical torsion subgroups.

Conjecture 3.4. *For $k \in \mathbb{N}$, let λ_k denote the probability distribution on the set* *of all abelian groups which assigns probability proportional to* $\dfrac{1}{|G|^k|\operatorname{Aut}(G)|}$ *to each finite abelian group* G*. Then for each* k *and each* $d \geq 2$*, the conditional distribution on the* k*th subcritical torsion group of* $\mathcal{Y}_d(n)$ *given that it exists converges to* λ_k*.*

Conjecture 3.5. *Let* λ_k *be as above. For each* k *and each* $d \geq 2$*, the conditional distribution on the* k*th supercritical torsion group of* $\mathcal{Y}_d(n)$ *given that it exists converges to* λ_k*.*

Now there is an obvious symmetry to these conjectures, namely for each $k > 0$ we expect G_{-k} and G_k to have the same distribution. In the (apparently likely) case of a unimodal torsion burst, one has that the subcritical torsion groups are iterative subgroups of $LT(n, d)$ and the supercritical torsion groups are iterative quotients of $LT(n, d)$. Despite this difference in paradigm on either side of $LT(n, d)$, our evidence is strong in support of both conjectures.

Tables 3.9, 3.10, and 3.11 summarize the results of 10,000 trials at $n = 60$ and compares them to the conjectured Cohen–Lenstra distributions by providing the ratios of the number of instances a particular sub- or supercritical group was trivial to the number of instances where it was a particular group, for common groups, as in Tables 3.6, 3.7, and 3.8. For each k, we condition on the event that the kth sub- or the kth supercritical group exists; the data in the tables reflects this. Now, the differences between the observed and conjectured ratios are larger in the Tables 3.9, 3.10, and 3.11, especially in Tables 3.10 and 3.11, than in the Tables 3.6, 3.7, and 3.8. This can be explained, however, by the sample size and the high concentration of instances of the trivial group. As further evidence for our conjecture, we found the total variation distance between the empirical distribution and the conjectured distribution for the subcritical and supercritical groups to be less than 0.06 in all cases.

Table 3.9: Observed ratios for the first subcritical torsion group and the first supercritical torsion group for 10,000 instances of $\mathcal{Y}_2(60)$, and a comparison to the conjectured limiting ratios

Group	1st Subcritical Group	1st Supercritical Group	Cohen–Lenstra Ratio
Trivial Group	1	1	1
$\mathbb{Z}/2\mathbb{Z}$	1.9482	2.0793	2
$\mathbb{Z}/3\mathbb{Z}$	5.7533	6.4367	6
$\mathbb{Z}/4\mathbb{Z}$	7.7613	8.0608	8
$\mathbb{Z}/6\mathbb{Z}$	11.9222	11.3984	12
$\mathbb{Z}/5\mathbb{Z}$	18.1864	21.1449	20
$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	23.3261	21.9950	24
$\mathbb{Z}/8\mathbb{Z}$	35.7667	30.3958	32
$\mathbb{Z}/10\mathbb{Z}$	36.3729	36.7815	40
$\mathbb{Z}/7\mathbb{Z}$	42.4950	38.0609	42
$\mathbb{Z}/12\mathbb{Z}$	49.3333	42.0865	48
$\mathbb{Z}/9\mathbb{Z}$	54.3291	54.0370	54
$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$	58.7945	61.6479	64
$\mathbb{Z}/14\mathbb{Z}$	93.3043	91.1875	84

Table 3.10: Observed ratios for the second subcritical torsion group and the second supercritical torsion group for 10,000 instances of $\mathcal{Y}_2(60)$, and a comparison to the conjectured limiting ratios; there were 5,708 trials where the second subcritical torsion group was defined and 5,623 trials where the second supercritical torsion group was defined

Group	2nd Subcritical Group	2nd Supercritical Group	Cohen–Lenstra Ratio
Trivial Group	1	1	1
$\mathbb{Z}/2\mathbb{Z}$	5.2907	4.7272	4
$\mathbb{Z}/3\mathbb{Z}$	26.3274	23.5833	18
$\mathbb{Z}/4\mathbb{Z}$	39.4911	39.3056	32
$\mathbb{Z}/6\mathbb{Z}$	107.8780	108.8462	72
$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	126.3714	92.2826	96
$\mathbb{Z}/5\mathbb{Z}$	152.5172	132.6563	100

Table 3.11: Observed ratios for the third subcritical torsion group and the third supercritical torsion group for 10,000 instances of $\mathcal{Y}_2(60)$ and a comparison to the conjectured limiting ratios; there were 1,285 trials where the third subcritical torsion group was defined and 1,378 trials where the third supercritical torsion group was defined

Group	3rd Subcritical Group	3rd Supercritical Group	Cohen–Lenstra Ratio
Trivial Group	1	1	1
$\mathbb{Z}/2\mathbb{Z}$	11.0866	10.9206	8
$\mathbb{Z}/3\mathbb{Z}$	104.8182	77.1250	54

We also note that unlike the case of examining $LT(n,2)$, it does not appear to be necessary to restrict to Sylow q-subgroups for the other torsion groups. Indeed the conjectured distribution in Conjectures 3.4 and 3.5 appears in [11, 30, 51, 53], and is well-known to be a well-defined probability distribution on the set of all finite abelian groups, and that the constant of proportionality for λ_k is known to be

$$\prod_{p \text{ prime}} \prod_{i=k+1}^{\infty} (1 - 1/p^{-i}) = \prod_{i=k+1}^{\infty} \zeta(i)^{-1} < \infty,$$

where ζ denote the Riemann zeta function. Moreover, assuming Conjectures 3.4 and 3.5 tells us something about the number of phases of the torsion burst. Namely Conjecture 3.4 (resp. Conjecture 3.5) implies that there is a positive probability the kth subcritical (supercritical) torsion group is defined for any fixed k, provided there is a positive probability that there is a torsion burst. This is straightforward to compute. Let p_0 denote the asymptotic probability that $\mathcal{Y}_d(n)$ has a torsion burst (assuming such a probability exists). For $k \geq 1$, let p_k denote the asymptotic probability that the kth subcritical torsion group of $\mathcal{Y}_d(n)$ is nontrivial conditioned on the event that it exists. By Conjecture 3.4, $p_k > 0$ for all $k \geq 1$, in fact, $p_k = 1 - \prod_{i=k+1}^{\infty} \zeta(i)^{-1}$. Let q_k denote the asymptotic unconditional probability that $\mathcal{Y}_d(n)$ has a kth subcritical torsion group. Clearly, $\mathcal{Y}_d(n)$ has a kth subcritical torsion group if and only if it has a nontrivial $(k-1)$st subcritical torsion group. Thus we have the following recurrence $q_k = p_{k-1} \cdot q_{k-1}$, and $q_1 = p_0$ since a torsion burst implies that there is a nontrivial largest torsion group, so there is a defined first subcritical torsion group. Thus, if $p_0 > 0$ then $q_k > 0$ for every positive integer k.

Moreover, by linearity of expectation the asymptotic expected number of positive integers k so that $\mathcal{Y}_d(n)$ has a kth subcritical torsion group is given by

$$\sum_{i=1}^{\infty} q_i = p_0 \sum_{i=1}^{\infty} \prod_{j=1}^{i-1} p_j.$$

The same would hold for the supercritical torsion groups as well. Thus, we may compute the expected number of phases in the torsion burst. We let $\mathcal{G}^-(\mathcal{Y}_d(n))$ denote the number of k so that $\mathcal{Y}_d(n)$ has a nontrivial kth subcritical torsion subgroup and let $\mathcal{G}^+(\mathcal{Y}_d(n))$ denote the analogous value for supercritical torsion subgroups. We denote the number of phases by $\mathcal{G}(\mathcal{Y}_d(n))$ and by definition it is equal to $\mathcal{G}^-(\mathcal{Y}_d(n)) + \mathcal{G}^+(\mathcal{Y}_d(n)) + 1$. Assuming Conjectures 3.4 and 3.5 we have the following asymptotic expectation:

$$\lim_{n\to\infty} \mathbb{E}[\mathcal{G}^-(\mathcal{Y}_d(n))] = \left(\sum_{i=1}^{\infty} q_i\right) - 1.$$

The same expectation would hold for $\mathcal{G}^+(\mathcal{Y}_d(n))$. Thus the expected number of phases, assuming Conjectures 3.4 and 3.5 and that the torsion burst occurs with high probability (that is $p_0 \to 1$ as $n \to \infty$), is asymptotically given by:

$$\lim_{n\to\infty} \mathbb{E}[\mathcal{G}(\mathcal{Y}_d(n))] = 2\left(\sum_{i=1}^{\infty} q_i\right) - 2 + 1 = 2\left(\sum_{i=1}^{\infty}\prod_{j=1}^{i-1}\left(1 - \prod_{k=j+1}^{\infty}\zeta(k)^{-1}\right)\right) - 1.$$

A reasonable approximation for this value is 2.49524.

To collect data on the duration of the torsion burst experimentally we ran 1000 trials at $n = 50$, 60, 70, 80, 90, and 100. Table 3.12 summarizes the results. Additionally, we should compare the number of phases in the torsion burst with the number predicted by the Cohen–Lenstra heuristics given above.

We notice that the average duration of the torsion burst decreases with n. This, together with the conjectured asymptotic number of phases, suggests that the duration in $\mathcal{Y}_d(n)$ for which a particular nontrivial torsion group in the codimension-1 homology group persists depends on n. There are a few trivial conditions which imply that the torsion part of $H_{d-1}(Y_d(n, m))$ is isomorphic to the torsion part of $H_{d-1}(Y_d(n, m+1))$, which depend on n. For example, if the $(m+1)$st face completes the boundary of a $(d+1)$-simplex then it will not change the torsion in homology.

Similarly, if a $(d-1)$-dimensional face of the $(m+1)$st face was isolated in $Y_d(n,m)$ then the torsion in homology would not change. In the density regime we are interested in, the number of possible faces which complete a $(d+1)$-simplex boundary is expected to be $O(n)$, and the expected number of isolated $(d-1)$-dimensional faces is $O(n^d)$. Thus, at any stage in this regime the probability that we add a face which completes a tetrahedron is $O(n^{-d})$, and the probability we add a face which covers an isolated $(d-1)$-dimensional face is $O(n^{-1})$, so both of these probabilities go to zero with n. This gives a partial explanation for why a particular nontrivial torsion group is asymptotically unlikely to persist for more than one state, but the full picture remains unclear.

Table 3.12: Durations of the torsion burst (1000 trials for each n)

Vertices	Duration	Number of Phases
50	6.882 ± 6.087	2.509 ± 1.215
60	6.019 ± 6.494	2.438 ± 1.212
70	5.788 ± 6.856	2.455 ± 1.213
80	5.229 ± 4.533	2.398 ± 1.205
90	5.094 ± 4.000	2.394 ± 1.146
100	5.205 ± 6.710	2.440 ± 1.173
110	5.097 ± 4.232	2.426 ± 1.176

CHAPTER 4

SMALL SIMPLICIAL COMPLEXES WITH PRESCRIBED TORSION IN HOMOLOGY

The work in this chapter appears in [44, 45]. In the first of these papers, I prove the upper bound in Theorem 4.1 in the second I prove a refinement of this result which appears as Theorem 4.2, with the goal of counting homotopy types of simplicial complexes on n vertices.

4.1 Statement of the problem

As we discussed in Section 2.4 and saw in the previous chapter, the size of $H_{d-1}(X)_T$ where X is a simplicial complex can be quite large relative to the number of vertices of X. Indeed, as Theorem 2.5 due to [28] points out, for every $d \geq 2$ there exists a constant $k_d > 0$ so that for every n large enough there is a d-dimensional $\mathbb{Q}$-acyclic complex X on n vertices with $|H_{d-1}(X)_T| \geq \exp(k_d n^d)$. Moreover, by [28], this is essentially best possible as $|H_{d-1}(X)_T| \leq \sqrt{d+1}^{\binom{n-2}{d}}$.

In this chapter we explore the problem of *which* groups can appear as $H_{d-1}(X)_T$ for X a simplicial complex on n vertices by proving the following theorem. The lower bound is known due to [28], and here we prove the upper bound using the probabilistic method.

Theorem 4.1. *For every $d \geq 2$, there exist constants c_d and C_d so that for any finite abelian group G,*

$$c_d(\log |G|)^{1/d} \leq T_d(G) \leq C_d(\log |G|)^{1/d}.$$

Thus by this theorem we have that for every $d \geq 2$ there exists a constant $\delta_d > 0$ so that for every n, *every* abelian group of size at most $e^{\delta_d n^d}$ is realizable as $H_{d-1}(X)_T$ for X a simplicial complex on at most n vertices. From this formulation of Theorem 4.1, we can say something about the number of simplicial complexes on n vertices up to homotopy type.

Indeed if we let $h(n, d)$ denote the number of homotopy types of d-dimensional simplicial complexes on n vertices, then for $d \geq 2$ and n large enough one has that $h(n, d) \geq \exp(\delta_d n^d)$. It follows that $h(n)$, denoting the number of homotopy types of simplicial complexes on n vertices without regard to dimension, grows faster than $\exp(p(n))$ where $p(n)$ is any polynomial in n. Therefore it is natural to ask if this method of constructing simplicial complexes with prescribed torsion can be used to give an even better lower bound on $h(n)$ if we are careful about computing bounds on C_d. It turns out that we can give a better lower bound on $h(n)$ by proving the following refinement to the upper bound in Theorem 4.1.

Theorem 4.2. *For d large enough and for every finite abelian group G,*

$$T_d(G) \leq 50d \log^{1/d} |G|.$$

From this theorem we have the following corollary regarding a lower bound on $h(n)$.

Corollary 4.3. *For n large enough, $h(n) \geq \exp(\exp(0.007n))$.*

Proof of Corollary 4.3 from Theorem 4.2. From Theorem 4.2, we have that for every d large enough, every finite abelian group G of size at most $\exp(\exp(d))$ is realizable as $H_{d-1}(X)_T$ for X a d-dimensional simplicial complex on at most $50ed$ vertices. Now suppose that n is large enough, and let $d = \lfloor n/(50e) \rfloor$. Then every abelian group of size at most $\exp(\exp(\lfloor n/(50e) \rfloor))$ can be realized as $H_{d-1}(X)_T$ for X a d-dimensional simplicial complex on n vertices (add isolated vertices to get up to n, if necessary). In particular at least $\exp(\exp(0.007n))$ finite cyclic groups contribute to a different homotopy type of simplicial complexes on n vertices implying Corollary 4.3. □

The question of the number of homotopy types of simplicial complexes on n vertices appears on MathOverflow as a question of Vidit Nanda [42] where Kalai [29] conjectures that the number of homotopy types for simplicial complexes on n vertices is doubly-exponential in n. That is $h(n)$ is close to the trivial upper bound of 2^{2^n}. Corollary 4.3 proves this conjecture to show that

$$0.007n \leq \log\log h(n) \leq \log(2)n + \log\log 2.$$

While Theorem 4.1 and Theorem 4.2 appear in two separate papers, this chapter is written as a hybrid of the two. In the next three sections we give prove Theorem 4.1 however we do so in a way that keeps track of the relevant bounds, as in [44] so that in Section 4.5 we can give a short proof of Theorem 4.2 from what we have already done.

4.2 Overview of the proof

For ease of notation, we denote by $\Delta_{i,j}(X)$, for a finite simplicial complex X, the maximum degree of an i-dimensional face in j-dimensional faces, that is

$$\Delta_{i,j}(X) = \max_{\tau \in \mathrm{skel}_i(X)} |\{\sigma \in \mathrm{skel}_j(X) \mid \tau \subseteq \sigma\}|.$$

We denote by $\Delta(X)$ the maximum over i and j of $\Delta_{i,j}(X)$. Throughout the proof of Theorem 4.1, it will be important that $\Delta(X)$ is bounded, for various simplicial complexes X. Of course, if X is a simplicial complex and $\Delta_{0,1}(X)$ is bounded by some constant then $\Delta(X)$ is bounded by a constant depending on $\Delta_{0,1}(X)$. Nevertheless, in the interest of simplifying statements and proofs, it is convenient to have the notation $\Delta(X)$ and a single bound for it. Moreover if X is a pure d-dimensional simplicial complex (that is if every face of dimension less than d is contained in a d-dimensional face) then a bound on $\Delta(X)$ may be derived from a bound on $\Delta_{0,d-1}(X)$. As $\Delta_{0,d-1}$ will be relevant to the proof of Theorem 4.2, we will often keep track of this parameter in statements and proofs in this chapter.

With this notation in hand, we are ready to give an outline of the proof of the main theorem. The goal of the paper will be to provide a construction, given a dimension $d \geq 2$, which proves that the upper bound in the statement of Theorem 4.1 is correct. The first step will be to prove the following lemma. Starting with this lemma and for several others we state and prove claims about the dth homology group. These results about the dth homology group are not necessary to the proof of Theorem 4.1 nor to its refinement, but will be useful to have in Section 4.6 where the goal is to give a $\mathbb{Q}$-acyclic construction.

Lemma 4.4. *Let $d \geq 2$ be an integer. Then for every finite abelian group G there exists a d-dimensional simplicial complex $X = X(G)$ with $H_{d-1}(X)_T \cong G$ and $H_d(X) = 0$ so that the following bounds hold:*

$$|V(X)| \leq 182 d^4 \log |G|$$

and

$$\Delta_{0,(d-1)}(X) \leq 2(3^d d) + 3d^3.$$

Of course this lemma alone does not prove the upper bound in Theorem 4.1, but using this initial construction we will build a smaller complex that does. Toward explaining this "reduction step" we define for a simplicial complex X and a proper coloring c of the vertices of X, the pattern complex (X, c) according to the following definition.

Definition. If X is a simplicial complex with a coloring c of $V(X)$ we define the *pattern* of a face to be the multiset of colors on its vertices. If c is a proper coloring, in the usual graph sense that no two vertices connected by an edge receive the same color, we define the *pattern complex* (X, c) to be the simplicial complex on the set of colors of c so that a subset S of the colors of c is a face of (X, c) if and only if there is a face of X with S as its pattern. It is easy to see that this is a simplicial complex. Indeed if S is a set of colors which is a pattern for some face σ of X then for any $S' \subseteq S$, S' is a pattern for some face of σ.

The relevant fact about the pattern complex (X, c) that we will use is the following lemma:

Lemma 4.5. *If X is a simplicial complex of dimension at least d and c is a proper coloring of the vertices of X so that no two $(d-1)$-dimensional faces of X have the same pattern then the complex (X, c) has $H_{d-1}(X)_T \cong H_{d-1}((X, c))_T$. And moreover, $H_i(X) \cong H_i((X, c))$ for $i \geq d$.*

Proof. Suppose that $V(X)$ is colored properly by c with no two $(d-1)$-dimensional faces receiving the same pattern. We can define a simplicial map $f : X \to (X, c)$ sending each vertex v to $c(v)$. Since no two $(d-1)$-dimensional faces receive the same pattern, we also have that no two d-dimensional faces receive the same pattern. Therefore f induces a homeomorphism from $X/X^{(d-2)}$ to $(X, c)/(X, c)^{(d-2)}$. Now, taking the quotient of a CW-complex by its $(d-2)$-skeleton preserves the torsion part of the $(d-1)$st homology group and the full homology groups for all dimensions at least d. (This follows, for example, from Theorem 2.13 from [22].) Thus the conclusion follows. □

Thus the second step in the proof of Theorem 4.1 will be to show that there is a coloring of the vertices of the initial construction in a way that the pattern complex will be the final construction that we want. This is accomplished using the probabilistic method in proving the following lemma.

Lemma 4.6. *Let X be a d-dimensional simplicial complex, for $d \geq 2$, on n vertices with $\Delta(X) \leq K-1$ for some integer $K \geq 5$, then there exists a proper coloring c of $V(X)$ having at most $18K^8 d^6 \sqrt[d]{n}$ colors so that no two $(d-1)$-dimensional faces of X receive the same pattern by c.*

Now assuming Lemmas 4.4 and 4.6 we give the proof of the upper bound in Theorem 4.1.

Proof of the upper bound in Theorem 4.1. Fix $d \geq 2$, and let G be a finite abelian group. By Lemma 4.4 there exists a constant $K \geq 5$ depending only on d and a simplicial complex X with $H_{d-1}(X)_T \cong G$, $\Delta(X) \leq K-1$, and $|V(X)| \leq K \log_2 |G|$. Now by Lemma 4.6, there is a coloring c of the vertices of X with at most $18K^8 d^6 \sqrt[d]{K \log_2 |G|}$ colors so that no two $(d-1)$-dimensional faces of X receive the same pattern by c. Therefore by Lemma 4.5, $H_{d-1}((X,c))_T \cong G$, and so

$$T_d(G) \leq |V((X,c))| \leq \frac{18K^{8+d^{-1}} d^6}{\sqrt[d]{\log 2}} \sqrt[d]{\log |G|},$$

proving Theorem 4.1 with $C_d = 18K^{8+d^{-1}} d^6 / \sqrt[d]{\log 2}$. □

4.3 The Initial Construction

It is easy to see that Lemma 4.4, the first step in our proof of Theorem 4.1, is implied by the following special case:

Lemma 4.7. *Let $d \geq 2$ be an integer. Then for every $m \geq 2$ there exists a d-dimensional simplicial complex $X = X(m)$ with $H_{d-1}(X)_T \cong \mathbb{Z}/m\mathbb{Z}$ and $H_d(X) = 0$ so that the following bounds hold:*

$$|V(X)| \leq 182d^4 \log m$$

and

$$\Delta_{0,(d-1)}(X) \leq 2(3^d d) + 3d^3.$$

Proof of Lemma 4.4 from Lemma 4.7. Fix $d \geq 2$ and $G = \mathbb{Z}/m_1\mathbb{Z} \oplus \mathbb{Z}/m_2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_l\mathbb{Z}$. For each m_i, let X_i be the d-dimensional simplicial complex from Lemma 4.7. Now take X to be the disjoint union of all the X_i to get a complex which satisfies the conclusion of Lemma 4.4, since $\sum_{i=1}^{l} \log m_i = \log(m_1 m_2 \cdots m_l) = \log |G|$. $\square$

The main purpose of this section will be to prove Lemma 4.7. We will prove this theorem by giving an explicit construction which we call the sphere-and-telescope construction. The idea is to construct a space with a "repeated squares presentation" of $\mathbb{Z}/m\mathbb{Z}$ as $H_{d-1}(X)_T$. Given m, write its binary expansion as $m = 2^{n_1} + \cdots + 2^{n_k}$ with $0 \leq n_1 < n_2 < \cdots < n_k$, then $\mathbb{Z}/m\mathbb{Z}$ is given by the abelian group presentation

$$\langle \gamma_0, \gamma_1, ..., \gamma_{n_k} \mid 2\gamma_0 = \gamma_1, ..., 2\gamma_{n_k-1} = \gamma_{n_k}, \gamma_{n_1} + \gamma_{n_2} + \cdots + \gamma_{n_k} = 0 \rangle.$$

The goal is to construct a simplicial complex with this presentation as the presentation for $H_{d-1}(X)_T$ so that each γ_i is a homology class of $H_{d-1}(X)$ represented by the boundary of a d-simplex Z_i. This will be accomplished by constructing two simplicial complexes Y_1 and Y_2 and attaching them to one another to build X. The required properties of Y_1 and Y_2 will be that $H_{d-1}(Y_1) \cong \langle \gamma_0, \gamma_1, ..., \gamma_{n_k} \mid 2\gamma_0 = \gamma_1, ..., 2\gamma_{n_k-1} = \gamma_{n_k} \rangle$, $H_{d-1}(Y_2) \cong \langle \tau_1, \tau_2, ..., \tau_k \mid \tau_1 + \tau_2 + \cdots + \tau_k = 0 \rangle$, and that Y_1 and Y_2 may be attached to one another in such a way that at the level of $(d-1)$st homology γ_{n_i} is identified to τ_i for all $i \in \{1, 2, ..., k\}$.

As an example, Figure 4.1 shows the space X that we triangulate in the proof of Lemma 4.7 when $d = 2$ and $m = 25$. While we do not show the full triangulation in the figure, the space in the figure is the space that we build up to homeomorphism. On the righthand side of the figure we have the telescope portion of the construction. Each segment is a punctured projective plane, or equivalently the mapping cylinder for the degree 2 map from S^1 to S^1. With the labeling on each copy of the punctured projective plane, at the level of homology we have the relators $2\gamma_0 = \gamma_1, 2\gamma_1 = \gamma_2, 2\gamma_2 = \gamma_3$, and $2\gamma_3 = \gamma_4$ where $\gamma_0, ..., \gamma_4$ are homology classes each represented by an S^1. The lefthand side is the sphere portion of the construction, though in reality it is a multipunctured sphere. With the labeling on the cycles we have that the first homology group of this space is given by $\langle \gamma_0, \gamma_1, \gamma_3 \mid \gamma_0 + \gamma_3 + \gamma_4 = 0 \rangle$. According to the identifications of different copies of S^1 in the figure we get that the torsion part of the first homology group for this space is given by $\langle \gamma_0, ..., \gamma_4 \mid 2\gamma_0 = \gamma_1, ..., 2\gamma_3 = \gamma_4, \gamma_0 + \gamma_3 + \gamma_4 = 0 \rangle$ thus we have that the homology class $\gamma_i = 2^i \gamma_0$, and since $25 = 2^0 + 2^3 + 2^4$ we have that the torsion part of the homology group is $\mathbb{Z}/25\mathbb{Z}$. It is also worth pointing out that we do get three free homology classes by how we attach the segments of the telescope as handles to the sphere.

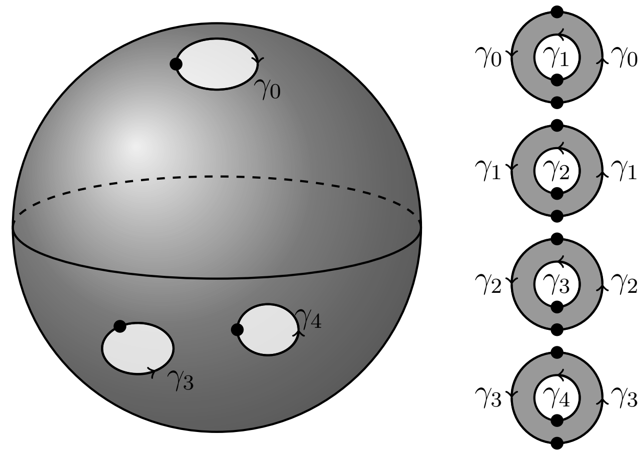

Figure 4.1: The topological space which we triangulate in our construction for $m = 25$ and $d = 2$

We will first prove Lemma 4.7 in the case that $d = 2$. The $d = 2$ case illustrates the idea of the general case while avoiding some of the more technical details, which are necessary but not particularly illuminating, in higher dimensions. The full details in arbitary dimensions are given in Section 4.3.2.

4.3.1 The 2-dimensional case

As the name of the construction suggests, we want to first build a special triangulation of a sphere.

Claim 4.8. *For any $k \in \mathbb{N}$ there exists a triangulation T of S^2 so that $\Delta(T) \leq 6$, $|V(T)| \leq 68k$, and there are k 2-dimensional faces of T, $\sigma_1, \sigma_2, ..., \sigma_k$ so that for all $i \neq j$, σ_i and σ_j do not share vertices nor is there an edge between the vertices of σ_i and the vertices of σ_j.*

Proof. To prove this claim we will define an infinite sequence of triangulations of S^2, denoted $T_0, T_1, T_2, ...$, so that $\Delta(T_i) \leq 6$ for all i, and then show that for any given k, a triangulation satisfying the conditions that we want may be found in this list of triangulations.

Let T_0 denote the tetrahedron boundary on the vertex set $\{1, 2, 3, 4\}$. To build our sequence of triangulations we will alway get T_{i+1} from T_i by triangulating a single face by deleting its interior and replacing it by the cone over its boundary. This type of retriangulation is called *stacking* or a *bistellar 0-move* and is defined in [46, 47]. Clearly stacking adds one vertex and a net total of two 2-dimensional faces at each step. However, we will be careful in choosing the face to triangulate so that $\Delta(T_i) \leq 6$ for all i. This may be accomplished by taking T_i to be on the vertex set $\{1, 2, .., 3 + i, 4 + i\}$ and triangulating the face $[2 + i, 3 + i, 4 + i]$ to get T_{i+1} on the vertex set $\{1, 2, ..., 4 + i, 5 + i\} = \{1, 2, ..., 3 + (i + 1), 4 + (i + 1)\}$ which contains the face $[2 + (i + 1), 3 + (i + 1), 4 + (i + 1)]$. Thus we may continue this process inductively. Figure 4.2 shows the first few steps in the sequence $T_0, T_1, ...$ of the sequential stacking moves retriangulating the face $[2, 3, 4]$.

We see that when we add a new vertex it has degree 3, but by always choosing the face to triangulate at T_i to be $[2+i, 3+i, 4+i]$, we see that we never triangulate a face containing a fixed vertex more than three times. Thus each vertex starts with degree 3 and that degree is increased by one at most three times. Thus $\Delta_{0,1}(T_i) \leq 6$. Also, $\Delta_{1,2}(T_i) = 2$ since T_i is a triangulated surface. Finally $\Delta_{0,2}(T_i) \leq 6$ as well since each vertex belongs to three faces when it is introduced and triangulating a face containing a given vertex increases the number of faces that vertex belongs to by one.

Now let $k \in \mathbb{N}$ be given. We claim that $T := T_{64k}$ satisfies the conditions we want our triangulation to have. Clearly, $V(T) = 4 + 64k \leq 68k$ and $\Delta(T) \leq 6$. All that remains to check is that there are k faces which are vertex-disjoint and which have no edges between them. Let G denote a graph constructed from T in the following way. The vertices of G are the 2-dimensional faces of T, with an edge $(\sigma, \tau) \in G$ if and only if there is an edge from σ to τ or σ and τ share a vertex. Thus an independent set in G points to a set of faces in T which satisfy the conditions that we need. Since $\Delta(T) \leq 6$, we have that the maximum degree of a vertex in G is at most $3 \times 6 + 3 \times 6 \times 6 = 126$. Also T has exactly $4 + 2(64k) \geq 128k$ faces. Thus G has at least $128k$ vertices and maximum degree at most 126, so there is an independent set of size at least k. This independent set gives us faces $\sigma_1, ..., \sigma_k$ which are all vertex disjoint and do not have any edges between them, and so this finishes the proof. ☐

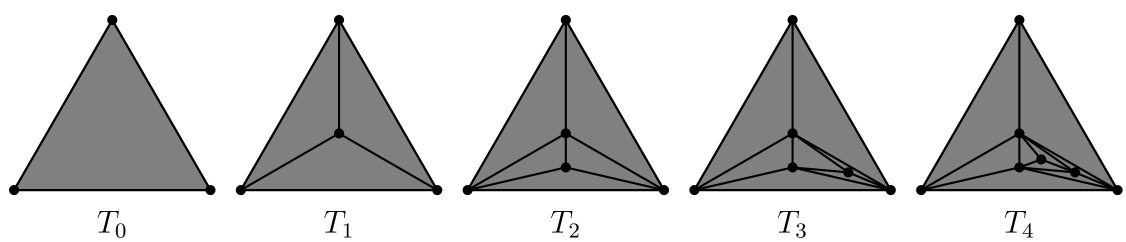

Figure 4.2: The first few triangulations of S^2 in the sequence of triangulations described in the proof of Claim 4.8

Now we are ready for the proof of Lemma 4.7 when $d = 2$. Before we prove it, we restate it to give precise constants.

Lemma (Statement of Lemma 4.7 for $d = 2$)**.** *For every integer $m \geq 2$, there exists a simplicial complex X so that $\Delta(X) \leq 16$, $|V(X)| \leq 278 \log_2 m$, $H_1(X)_T \cong \mathbb{Z}/m\mathbb{Z}$, and $H_2(X) = 0$.*

Proof. Let m be given. Write m in its binary expansion $m = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_k}$ with $0 \leq n_1 < n_2 < \cdots < n_k$. Construct the simplicial complex Y_1 with vertex set $v_0, v_1, v_2, v_3, , ..., v_{3n_k+2}$ having as its facets the follow set of faces:

$$\begin{aligned} \text{skel}_2(Y_1) \;\; = \;\; & \{[v_{3i}, v_{3i+1}, v_{3i+4}], [v_{3i+1}, v_{3i+2}, v_{3i+4}], [v_{3i+2}, v_{3i+4}, v_{3i+5}], [v_{3i}, v_{3i+2}, v_{3i+5}], \\ & [v_{3i}, v_{3i+1}, v_{3i+5}], [v_{3i+1}, v_{3i+3}, v_{3i+5}], [v_{3i+1}, v_{3i+2}, v_{3i+3}], [v_{3i}, v_{3i+2}, v_{3i+3}], \\ & [v_{3i}, v_{3i+3}, v_{3i+4}] \mid i = 0, 1, 2, ..., (n_k - 1)\} \end{aligned}$$

The faces of Y_1 are easier to see from a picture. Figure 4.3 shows a "building block" of Y_1. The full set of faces of Y_1 are the faces in Figure 4.3 as i ranges over $\{0, 1, 2, ..., (n_k - 1)\}$.

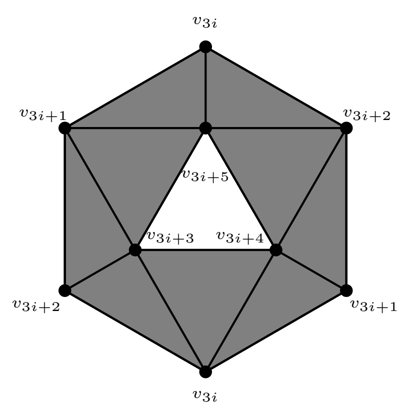

Figure 4.3: The building block for the telescope construction

By construction, each one of these building blocks is a triangulated projective plane with one face removed. Thus if we order the vertices in their natural order and let that ordering induce an orientation on all the edges and faces of Y_1 then letting γ_i

denote the 1-cycle of Y_1 represented by $[v_{3i}, v_{3i+1}] - [v_{3i}, v_{3i+2}] + [v_{3i+1}, v_{3i+2}]$ for $i \in \{0, 1, 2, ..., n_k\}$, we have that $2\gamma_i - \gamma_{i+1}$ is a 1-boundary of Y_1 for all $i \in \{0, ..., n_k - 1\}$. Now $H_1(Y_1)$ can be presented as $\langle \gamma_0, \gamma_1, ..., \gamma_{n_k} \mid 2\gamma_0 = \gamma_1, 2\gamma_1 = \gamma_2, ..., 2\gamma_{n_k-1} = \gamma_{n_k} \rangle$. (There is a bit to do to check this; one way to check is by induction on n_k, if $n_k = 1$, then Y_1 is just the triangulated projective plane with one face removed. The inductive step follows from the Mayer–Vietoris sequence; full details for this can be found in the proof of the general case.) Note that $\Delta(Y_1) \leq 10$ and $|V(Y_1)| = 3(n_k + 1)$.

Next we construct a complex Y_2 which we will attach to Y_1 in a certain way to build our complex X. Start by using Claim 4.8 to get a triangulation of S^2, T with $\Delta(T) \leq 6$, $|V(T)| \leq 136k$, and $2k$ 2-dimensional faces, σ_1, σ_2, ..., σ_{2k} which are vertex disjoint from one another and have no edges from the vertices of one to the vertices of another. Now assign an ordering to the vertices of T and give the faces of T the orientation induced by this ordering. Since T is a triangulated 2-sphere there is a 2-chain $(x_1, ..., x_l)$ (where l is the number of 2-dimensional faces of T) so that $|x_i| = 1$ for all i and so that $\partial_2(x_1, ..., x_l) = 0$ where ∂_2 denotes the top-dimensional boundary matrix of T. Now without loss of generality at least k of the faces $\sigma_1, \sigma_2, ..., \sigma_{2k}$ have a coefficient of -1 in the 2-chain x. It follows that k of these faces may be removed to create an oriented simplicial complex, which we call Y_2, which has $Y_2 = \langle \tau_1, \tau_2, ..., \tau_k | \tau_1 + \tau_2 + \cdots + \tau_k = 0 \rangle$ where each τ_i represents the positively-oriented boundary of a removed face. For each i, let $w_{3i}, w_{3i+1}, w_{3i+2}$ denote the vertices of the face boundary representing τ_i. That is, τ_i is represented by the 1-cycle $[w_{3i}, w_{3i+1}] - [w_{3i}, w_{3i+2}] + [w_{3i+1}, w_{3i+2}]$ where $w_{3i} < w_{3i+1} < w_{3i+2}$ in the vertex ordering on Y_2. Observe that $\Delta(Y_2) \leq 6$ and $|V(Y_2)| \leq 136k$.

Now Y_1 and Y_2 will be attached together in a particular way to build the complex X which has $H_1(X)_T \cong \mathbb{Z}/m\mathbb{Z}$. Let S denote the subcomplex of Y_2 induced by w_3, w_4, w_5, w_6, ..., w_{3k}, w_{3k+1}, w_{3k+2}. Since the faces we deleted from T to build Y_2 are vertex disjoint and have no edges between any two of them, S is a disjoint union of k triangle boundaries. Let $f : S \to Y_1$ be the simplicial map defined by $w_{3i} \mapsto v_{3n_i}$, $w_{3i+1} \mapsto v_{3n_i+1}$, and $w_{3i+2} \mapsto v_{3n_i+2}$. Now let $X = Y_1 \sqcup_f Y_2$, that is, Y_1 with Y_2 attached along S via f (this is defined as a topological space in Chapter 0 of [22] as the the quotient of the disjoint union of Y_1 and Y_2 by attaching each point $s \in S$ to its image $f(s)$ in Y_1). Since f is injective and S is an induced subcomplex of Y_2, X is a simplicial complex (full details for this are found in the proof of the general case as Claim 4.9).

Now we use the Mayer–Vietoris sequence to show that $H_1(X) \cong \mathbb{Z}^{k-1} \oplus \mathbb{Z}/m\mathbb{Z}$. From the Mayer–Vietoris sequence we have the following exact sequence:

$$H_1(S) \xrightarrow{\ h\ } H_1(Y_2) \oplus H_1(Y_1) \xrightarrow{\ g\ } H_1(X) \longrightarrow \tilde{H}_0(S) \longrightarrow 0$$

We claim that $(H_1(Y_1) \oplus H_1(Y_2))/\mathrm{Im}(h) \cong \mathbb{Z}/m\mathbb{Z}$. This follows since f has the effect of identifying the 1-cycle $[w_{3i}, w_{3i+1}] - [w_{3i}, w_{3i+2}] + [w_{3i+1}, w_{3i+2}]$ to the 1-cycle $[v_{3n_i}, v_{3n_i+1}] - [v_{3n_i}, v_{3n_i+2}] + [v_{3n_i+1}, v_{3n_i+2}]$, that is f identifies τ_i to γ_{n_i}. It follows that the image of h is $\langle(\tau_i, -\gamma_{n_i})_{i=1}^{k}\rangle$ therefore

$$\begin{aligned}
(H_1(Y_1) \oplus H_1(Y_2))/\mathrm{Im}(h) &\cong \langle \gamma_0, \gamma_1, ..., \gamma_{n_k}, \tau_1, ..., \tau_k \mid 2\gamma_0 = \gamma_1, 2\gamma_1 = \gamma_2, ..., \\
&\quad\ 2\gamma_{n_k-1} = \gamma_{n_k}, \tau_1 + \tau_2 + \cdots + \tau_k = 0, \\
&\quad\ \gamma_{n_1} = \tau_1, ..., \gamma_{n_k} = \tau_k \rangle \\
&\cong \langle \gamma_0, \gamma_1, ..., \gamma_{n_k} \mid 2\gamma_0 = \gamma_1, ..., 2\gamma_{n_k-1} = \gamma_{n_k}, \\
&\quad\ \gamma_{n_1} + \gamma_{n_2} + \cdots + \gamma_{n_k} = 0 \rangle \\
&\cong \mathbb{Z}/m\mathbb{Z},
\end{aligned}$$

Therefore the the image of g is isomorphic to $\mathbb{Z}/m\mathbb{Z}$ and since $\tilde{H}_0(S)$ is a free abelian group of rank $k-1$, exactness implies that $H_1(X) \cong \mathbb{Z}^{k-1} \oplus \mathbb{Z}/m\mathbb{Z}$.

We now verify that $H_2(X) = 0$. By construction we have that Y_1 has $3(n_k+1) = 3n_k+3$ vertices, $3(n_k+1)+9n_k = 12n_k+3$ edges and $9n_k$ faces. Therefore the Euler characteristic of Y_1 is 0. On the other hand Y_2 is a k-punctured 2-sphere, so the Euler characteristic of Y_2 is $2-k$. Now in X, the intersection $Y_1 \cap Y_2$ is a disjoint union of k circles, and so it has Euler characteristic 0. Therefore the Euler characteristic of X is $2-k$. We clearly have that X is connected, and we computed the first Betti number to be $k-1$ above. Thus $H_2(X) = 0$.

Finally $\Delta(X) \leq \Delta(Y_1) + \Delta(Y_2) \leq 10 + 6 = 16$. Also, $k \leq \log_2 m + 1 \leq 2\log_2 m$ and $n_k \leq \log_2 m$, and therefore $|V(X)| \leq |V(Y_1)| + |V(Y_2)| \leq 3(n_k+1) + 136k \leq 139(\log_2(m)+1) \leq 278\log_2 m$. This completes the proof. □

We credit a portion of the construction in the $d = 2$ case of Lemma 4.7 to Speyer [50]. A MathOverflow question of Palmieri [48] from 2010 asks, given a prime q, how many vertices are required to build a d-dimensional simplicial complex with q-torsion in the codimension-1 homology group. Responding to this question, Speyer gives a construction similar to the consturction here showing an upper bound of $O(\log q)$ for $d = 2$. Indeed, his construction includes the same telescope portion that we have here. However, the two constructions vary in how they add the relator $\gamma_{n_1} + \gamma_{n_2} + \cdots + \gamma_{n_k} = 0$ to the first homology group. We use a different method for our construction in order to guarantee bounded degree.

The construction to prove Lemma 4.7 in higher dimensions is similar. Given $m = 2^{n_1} + \cdots + 2^{n_k}$ with $\mathbb{Z}/m\mathbb{Z} \cong \langle \gamma_0, \gamma_1, ...\gamma_{n_k} \mid 2\gamma_0 = \gamma_1, 2\gamma_1 = \gamma_2, ..., 2\gamma_{n_k-1} = \gamma_{n_k}, \gamma_{n_1} + \cdots + \gamma_{n_k} = 0\rangle$, our goal is a simplicial complex X where each γ_i is represented by a positively-oriented d-simplex boundary all of which are disjoint from one another, so that there is a d-chain with $2\gamma_i = \gamma_{i+1}$ as its boundary for each $i \in \{0, 1, ..., n_{k-1}\}$, and a d-chain with $\gamma_{n_1} + \cdots + \gamma_{n_k}$ as its boundary.

4.3.2 The general case

The goal of this section will be to prove Lemma 4.7 for any dimension $d \geq 2$. Given m, write its binary expansion as $m = 2^{n_1} + \cdots + 2^{n_k}$, then $\mathbb{Z}/m\mathbb{Z}$ is given by the abelian group presentation $\langle \gamma_0, \gamma_1, ..., \gamma_{n_k} \mid 2\gamma_0 = \gamma_1, ..., 2\gamma_{n_k - 1} = \gamma_{n_k}, \gamma_{n_1} + \gamma_{n_2} + \cdots + \gamma_{n_k} = 0\rangle$. The goal is to construct a simplicial complex X with this presentation as the presentation for $H_{d-1}(X)_T$ and which has $|V(X)| \leq 182d^4 \log m$ and $\Delta_{0,(d-1)}(X) \leq 2(3^d d) + 3d^3$.

1. Show that there exists a simplicial complex $P = P(d)$ on $2(d+1)$ vertices so that $H_{d-1}(P) = \langle a, b \mid 2a = b\rangle$ with each of the homology classes a and b represented by an embedded, positively-oriented copy of $\partial\Delta^d$ which are vertex disjoint from one another.

2. Show that for any integer $k \geq 0$ there is a triangulation T of S^d so that $\Delta_{0,d-1}(T) \leq (d+1)\dfrac{d^2+d+1}{2}$, $|V(T)| \leq 60d^4k + (d+2)$, and with k d-dimensional faces $\sigma_1, ..., \sigma_k$ which are vertex-disjoint and nonadjacent. Non-adjacent in this context means that for $i \neq j$, there are no edges between the vertices of σ_i and the vertices of σ_j.

3. Prove Lemma 4.7 by giving an explicit construction for $m \geq 2$, with $m = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_k}$, in the following series of steps:

   (a) Attach n_k copies of P together to create a complex Y_1 with $\Delta(Y_1) \leq 2\Delta(P)$ and $H_{d-1}(Y_1) = \langle \gamma_0, \gamma_1, ..., \gamma_{n_k} \mid 2\gamma_0 = \gamma_1, 2\gamma_1 = \gamma_2, ..., 2\gamma_{n_k-1} = \gamma_{n_k}\rangle$ where each γ_i is represented by an embedded, positively-oriented copy of $\partial\Delta^d$, Z_i which are all vertex disjoint from one another. We informally refer to Y_1 as the telescope part of the construction.

   (b) Use step 2 above to construct a complex Y_2, which is a triangulation of S^d with k vertex-disjoint, nonadjacent, d-dimensional faces removed, with

$\Delta_{0,d-1}(T) \leq (d+1)\dfrac{d^2+d+1}{2}$ and $|V(T)| \leq 60d^4k+(d+2)$. This complex will have $H_{d-1}(Y_2) = \langle \tau_1, \tau_2, ..., \tau_k \mid \tau_1+\tau_2+\cdots+\tau_k = 0\rangle$ where each τ_i is represented by an embedded, positively-oriented copy of $\partial\Delta^d$, Z'_i so that $Z'_1, ..., Z'_k$ are vertex-disjoint and nonadjacent. We informally refer to Y_2 as the sphere part of the construction.

(c) Attach Y_1 to Y_2 by attaching Z_{n_i} to Z'_i for every $i \in \{1, ..., k\}$ in a way that identifies γ_{n_i} to τ_i so that we get a simplicial complex X which has

$$\begin{aligned} H_{d-1}(X)_T &\cong \langle \gamma_0, \gamma_1, ..., \gamma_{n_k}, \tau_1, ..., \tau_k \mid 2\gamma_0 = \gamma_1, 2\gamma_1 = \gamma_2, ..., \\ &\quad 2\gamma_{n_k-1} = \gamma_{n_k}, \tau_1+\tau_2+\cdots+\tau_k = 0, \gamma_{n_1} = \tau_1, ..., \gamma_{n_k} = \tau_k\rangle \\ &\cong \langle \gamma_0, \gamma_1, ..., \gamma_{n_k} \mid 2\gamma_0 = \gamma_1, ..., 2\gamma_{n_k-1} = \gamma_{n_k}, \\ &\quad \gamma_{n_1}+\gamma_{n_2}+\cdots+\gamma_{n_k} = 0\rangle \\ &\cong \mathbb{Z}/m\mathbb{Z}, \end{aligned}$$

and for which

$$\Delta_{0,d-1}(X) \leq \Delta_{0,d-1}(Y_1) + \Delta_{0,d-1}(Y_2) \leq 2\Delta_{0,d-1}(P) + \Delta_{0,d-1}(Y_2)$$

and

$$V(X) \leq V(Y_1) + V(Y_2) \leq \log_2(m)V(P) + V(Y_2).$$

The attaching maps

As the steps above indicate the construction will be built in pieces which will be attached to one another in a way that the final complex X will have $H_{d-1}(X)_T \cong \mathbb{Z}/m\mathbb{Z}$. Of course we will need to be careful in how we attach simplicial complexes to one another so that the resulting space is still a simplicial complex. We will be attaching our building blocks to each other in a fairly standard way. Given two topological spaces A and B, a subspace $S \subseteq B$, and a map $f : S \to A$, the space

$A \sqcup_f B$ may be defined by taking the quotient of the disjoint union of A and B by the equivalence relation $s \sim f(s)$ for all $s \in S$. This type of attaching is called attaching B to A along S via f and is described in, for example, Chapter 0 of [22]. In our situation, we start with A and B simplicial complexes and we want our choice of S and f to be such that $A \sqcup_f B$ is a simplicial complex. This is accomplished in the following case:

Claim 4.9. *Let A and B be simplicial complexes, with S_1 a subcomplex of A and S_2 an induced subcomplex of B, so that there is a simplicial homeomorphism $f : S_2 \to S_1$. Then $A \sqcup_f B$ is a simplicial complex.*

Proof. As f is a simplicial homeomorphism whose domain is an induced subcomplex of B we have that $A \sqcup_f B$ may be realized as a pattern complex (as defined in Section 4.2 above) with respect to a certain coloring of $A \sqcup B$. First color every vertex of B uniquely. Then color each $w \in S_1$ with the same color used for its unique preimage under f in S_2. Finally color all the vertices in $A \setminus S_1$ uniquely. Now any two faces which receive the same pattern are identified together by f since S_2 is an induced subcomplex and f is determined by its image on vertices. Thus $A \sqcup_f B$ is the pattern complex of $A \sqcup B$ with respect to this coloring we have described. So it is a simplicial complex. □

The attaching maps used in the construction will always satisfy the assumptions of Claim 4.9; thus at each step in the process the construction will be a simplicial complex. Furthermore, attaching two simplicial complexes A and B in the way described in Claim 4.9 will result in a way to express $A \sqcup_f B$ as a union of two subspaces one of which is homotopy equivalent to A and one of which is homotopy equivalent to B with their intersection being homotopy equivalent to S_2. Thus we will simplify notation slightly and write $A \cup B$ for $A \sqcup_f B$ and $A \cap B$ for $S_2 \subseteq A \sqcup_f B$ when it is clear which simplicial homeomorphism $f : S_2 \to S_1$ we are using, especially as it relates to using the Mayer–Vietoris sequence to compute the homology of $A \sqcup_f B$ from the homology of A and B.

While this process of attaching A and B is well-defined and gives us the perfect setting to use the Mayer–Vietoris sequence, there is still one issue: the orientation of the faces of $A \cup B$. To compute the homology of A and the homology of B we are required to choose an orientation on each face of A and each face of B. When we attach B to A along S_2 by f we get a new simplicial complex $A \cup B$. Therefore it is necessary to choose orientations of the faces of $A \cup B$ in order to compute its homology, and in particular it is necessary to make a decision for the orientation of faces of $A \cap B$. While the choice of orientations does not affect the homology groups of a simplicial complex up to group isomorphism, it may not be easy to compute the homology groups of $A \cup B$ from the homology groups of A and the homology groups of B if the orientations chosen for $A \cap B$ cannot be made to match the initial orientations assigned to those faces in A and in B. Toward addressing this issue in the construction we give the following definition:

Definition. Given an oriented simplicial complex A and a subcomplex S, we say that S is *coherently ordered with respect to the orientation on* A (or just *coherently ordered* if A and its orientation are clear from context) provided that there is an ordering $v_1, ..., v_k$ of the vertices of S so that the orientation of each face of S induced by this ordering on the vertices matches the orientation of that face in A. We say that the ordering $v_1, ..., v_k$ is a *coherent ordering of* S.

Primarily, this definition will be used when we attach d-dimensional simplicial complexes A and B to one another so that some element of $H_{d-1}(A)$ is identified to some element of $H_{d-1}(B)$ in a way that makes $H_{d-1}(A \cup B)$ easy to compute using the Mayer–Vietoris sequence. Toward that goal we define the following:

Definition. Let A be an oriented simplicial complex and d be a positive integer. We say that $\gamma \in H_{d-1}(A)$ is *coherently represented by the d-simplex boundary Z* provided that Z is an embedded d-simplex boundary in A which is induced (i.e. the interior d-dimensional face is not in A), and which has a coherent ordering $v_0, ..., v_d$, so that γ is homologous to the $(d-1)$-cycle:

$$\sum_{i=0}^{d}(-1)^i[v_0, v_1, .., \hat{v}_i, ..., v_d].$$

Now, all of the attachments between building blocks in the construction will be along simplex boundaries which coherently represent cycles in homology. It is worth mentioning here that a coherent ordering on a d-simplex boundary is necessarily unique when $d \geq 2$ (and irrelevant if $d = 1$). Thus we will not encounter any issues with having to choose some coherent ordering. With this definition we are ready to state and prove the main attaching lemma:

Lemma 4.10. *Let $d \geq 2$ and $t \geq 1$. Let A and B be two connected, oriented simplicial complexes. Let $\gamma_1, ..., \gamma_t \in H_{d-1}(A)$ be coherently represented by the d-simplex boundaries $Z_1, Z_2, ..., Z_t$ respectively with $Z_i \cap Z_j = \emptyset$ for all $i \neq j$. Let $\tau_1, ..., \tau_t \in H_{d-1}(B)$ be coherently represented by the d-simplex boundaries $Z'_1, ..., Z'_t$ respectively with $Z'_i \cap Z'_j = \emptyset$ and no edges between the vertices of Z'_i and the vertices of Z'_j for every $i \neq j$. For each i, let $v^i_0, v^i_1, ..., v^i_d$ denote the coherent ordering for Z_i and $w^i_0, w^i_1, ..., w^i_d$ denote the coherent ordering for Z'_i. Let $f : \bigsqcup_{i=1}^{t} Z'_i \to \bigsqcup_{i=1}^{t} Z_i$ be the simplicial map defined by $w^i_j \mapsto v^i_j$ for all $i \in \{1, .., t\}$ and $j \in \{0, ..., d\}$. Then $A \sqcup_f B$ is a simplicial complex and*

$$H_{d-1}(A \sqcup_f B) \cong (H_{d-1}(A) \oplus H_{d-1}(B))/\langle(\gamma_i, -\tau_i)_{i\in[t]}\rangle \oplus \tilde{H}_{d-2}\left(\bigsqcup_{i=1}^{t} S^d\right).$$

Proof. First, $\bigsqcup_{i=1}^{t} Z'_i$ is an induced subcomplex of B since there are no edges between

Z_i' and Z_j' for any $i \neq j$. Also f is bijective as a simplicial map and so it is a simplicial homeomorphism, thus by Claim 4.9 $A \sqcup_f B$ is a simplicial complex.

Now we compute the homology. By construction, we have a portion of the Mayer–Vietoris sequence below for computing the homology of $A \cup B$ $(= A \sqcup_f B)$ from the homology of A and B.

$$H_{d-1}(A \cap B) \xrightarrow{h} H_{d-1}(A) \oplus H_{d-1}(B) \xrightarrow{g} H_{d-1}(A \cup B) \longrightarrow \tilde{H}_{d-2}(A \cap B)$$

Now h is given by sending the cycle x to $(x, -x)$ in $H_{d-1}(A) \oplus H_{d-1}(B)$ (see, for example, Chapter 4 of [22]). Furthermore since $A \cap B$ is homeomorphic to a disjoint union of t $(d-1)$-dimensional spheres, $H_{d-1}(A \cap B)$ is free. Thus h is determined by the image of the generators of $H_{d-1}(A \cap B)$. By how we have attached A and B and preserved the initial orientations on both, the image of h is $\langle (\gamma_i, -\tau_i)_{i \in [t]} \rangle$. Now if $d \geq 3$ then $\tilde{H}_{d-2}(A \cap B)$ is zero since it is the $(d-2)$nd homology group of a disjoint union of $(d-1)$-spheres. Thus for $d \geq 3$, g is surjective, so $H_{d-1}(A \cup B) \cong (H_{d-1}(A) \oplus H_{d-1}(B)) / \ker(g) = (H_{d-1}(A) \oplus H_{d-1}(B)) / \mathrm{Im}(h)$, and the lemma follows.

On the other hand if $d = 2$ then by connectedness of A and B we have the following short exact sequence:

$$0 \longrightarrow (H_1(A) \oplus H_1(B))/\mathrm{Im}(h) \xrightarrow{g'} H_1(A \cup B) \longrightarrow \tilde{H}_0(A \cap B) \longrightarrow 0$$

Here g' is the map induced by g on the quotient group $(H_1(A) \oplus H_1(B))/\mathrm{Im}(h)$, which is well-defined since $\mathrm{Im}(h) = \ker(g)$. This sequence splits since $\tilde{H}_0(A \cap B)$ is a free abelian group, so $H_1(A \cup B) \cong (H_1(A) \oplus H_1(B))/\mathrm{Im}(h) \oplus \tilde{H}_0(A \cap B)$. Thus the claim follows in this case too.

□

Building the space from Y_1 and Y_2.

As discussed in the outline of the proof of Lemma 4.7, the ultimate goal of the construction is two simplicial complexes Y_1 and Y_2 which satisfy certain properties and which may be attached in a way to build a complex which has a prescribed finite cyclic group as the torsion group of a prescribed homology group. Before fully constructing Y_1 and Y_2 we give sufficient conditions that will allow them to be attached to form X with $H_{d-1}(X)_T \cong \mathbb{Z}/m\mathbb{Z}$:

Lemma 4.11. *Fix $d \geq 2$, and let $(n_1, ..., n_k)$ be a list of k nonnegative integers with $n_1 < n_2 < \cdots < n_k$. Suppose that Y_1 is a connected, oriented, simplicial complex with $H_{d-1}(Y_1) = \langle \gamma_0, \gamma_1, ..., \gamma_{n_k} \mid 2\gamma_0 = \gamma_1, 2\gamma_1 = \gamma_2, ..., 2\gamma_{n_k-1} = \gamma_{n_k} \rangle$ where each γ_i is coherently represented by a d-simplex boundary denoted Z_i, with $Z_i \cap Z_j = \emptyset$ for all i, j. Suppose that Y_2 is a connected, oriented, simplicial complex with $H_{d-1}(Y_2) = \langle \tau_1, ..., \tau_k \mid \tau_1 + \tau_2 + \cdots + \tau_k = 0 \rangle$ where each τ_i is coherently represented by a d-simplex boundary denoted Z_i', so that for all $i \neq j$, $Z_i' \cap Z_j' = \emptyset$ and there are no edges between the vertices of Z_i' and the vertices of Z_j'. For each i, let $v_0^i, v_1^i, ..., v_d^i$ denote the coherent ordering for Z_i and $w_0^i, w_1^i, ..., w_d^i$ denote the coherent ordering for Z_i'. Let $f : \bigsqcup_{i=1}^{k} Z_i' \to \bigsqcup_{i=1}^{k} Z_{n_i}$ be the simplicial map defined by $w_j^i \mapsto v_j^{n_i}$ for all $i \in \{1, .., k\}$ and $j \in \{0, ..., d\}$. Then $X = Y_1 \sqcup_f Y_2$ is a simplicial complex and $H_{d-1}(X)_T$ is isomorphic to the cyclic group of order $2^{n_1} + 2^{n_2} + \cdots + 2^{n_k}$ and $H_d(X) \cong H_d(Y_1) \oplus H_d(Y_2)$.*

Proof. Let $X = Y_1 \sqcup_f Y_2$, then by Lemma 4.10, X is a simplicial complex, and:

$$\begin{aligned}
H_{d-1}(X)_T &\cong [(H_{d-1}(Y_1) \oplus H_{d-1}(Y_2))/\langle(\gamma_{n_i}, \tau_i)_{i\in[k]}\rangle]_T \\
&\cong \langle \gamma_0, \gamma_1, ..., \gamma_{n_k}, \tau_1, ..., \tau_k \mid 2\gamma_0 = \gamma_1, 2\gamma_1 = \gamma_2, ..., 2\gamma_{n_k-1} = \gamma_{n_k}, \\
&\qquad \tau_1 + \tau_2 + \cdots + \tau_k = 0, \gamma_{n_1} = \tau_1, ..., \gamma_{n_k} = \tau_k \rangle_T \\
&\cong \langle \gamma_0, \gamma_1, ..., \gamma_{n_k} \mid 2\gamma_0 = \gamma_1, ..., 2\gamma_{n_k-1} = \gamma_{n_k}, \gamma_{n_1} + \gamma_{n_2} + \cdots + \gamma_{n_k} = 0 \rangle_T \\
&\cong [\mathbb{Z}/(2^{n_1} + 2^{n_2} + \cdots + 2^{n_k})\mathbb{Z}]_T
\end{aligned}$$

$$= \mathbb{Z}/(2^{n_1} + 2^{n_2} + \cdots + 2^{n_k})\mathbb{Z}$$

In fact, from Lemma 4.10, we actually have if $d > 2$ then $H_{d-1}(X) \cong \mathbb{Z}/(2^{n_1} + 2^{n_2} + \cdots + 2^{n_k})\mathbb{Z}$, and if $d = 2$ then $H_1(X) \cong \mathbb{Z}^{k-1} \oplus \mathbb{Z}/(2^{n_1} + 2^{n_2} + \cdots + 2^{n_k})\mathbb{Z}$.

Regarding the dth homology group we have the following exact sequence, if $d \geq 3$, from the Mayer–Vietoris sequence and the fact that $Y_1 \cap Y_2$ is a disjoint union of k d-spheres:

$$0 \longrightarrow H_d(Y_1) \oplus H_d(Y_2) \xrightarrow{f} H_d(X) \xrightarrow{g} \mathbb{Z}^k \xrightarrow{h} \mathbb{Z} \oplus \mathbb{Z}^{k-1} \longrightarrow \mathbb{Z}/(2^{n_1} + 2^{n_2} + \cdots + 2^{n_k})\mathbb{Z}$$

By exactness we have that the image of h is isomorphic to $\mathbb{Z}^k$, and so the kernel of h is trivial. Thus g is the zero map, so f is an isomorphism.

If $d = 2$, then a basic Euler characteristic argument proves that $H_2(X) \cong H_2(Y_1) \oplus H_2(Y_2)$. Indeed the intersection of Y_1 and Y_2 in X has Euler characteristic zero so $\chi(X) = \chi(Y_1) + \chi(Y_2)$ and moreover the portion of the Mayer–Vietors sequence below induces an isomorphism for the middle homomorphism when $i \neq 0, 1, 2$.

$$H_i(Y_1 \cap Y_2) \longrightarrow H_i(Y_1) \oplus H_i(Y_2) \longrightarrow H_i(X) \longrightarrow H_{i-1}(Y_1 \cap Y_2).$$

It follows that $\beta_2(Y_1) - \beta_1(Y_1) + \beta_0(Y_1) + \beta_2(Y_2) - \beta_1(Y_2) + \beta_0(Y_2) = \beta_2(X) - \beta_1(X) + \beta_0(X)$. Since $\beta_1(Y_1) = 1$, $\beta_1(Y_2) = k - 1$, $\beta_1(X) = k - 1$, and Y_1, Y_2, and X are connected we have $\beta_2(X) = \beta_2(Y_1) + \beta_2(Y_2)$.

Regarding the torsion part of $H_2(X)$, we have the following exact sequence which implies that $H_2(X)_T \cong H_2(Y_1)_T \oplus H_2(Y_2)_T$ and finishes the proof.

$$0 \longrightarrow H_2(Y_1) \oplus H_2(Y_2) \longrightarrow H_2(X) \longrightarrow \mathbb{Z}^k.$$

□

Now if we have Y_1 and Y_2 attached to one another satisfying the statement of Lemma 4.11, then the clearly the resulting complex X has $\Delta_{0,d-1}(X) \leq \Delta_{0,d-1}(Y_1) + \Delta_{0,d-1}(Y_2)$ and $|V(X)| \leq |V(Y_1)| + |V(Y_2)|$. But as the proof of Lemma 4.11 shows, this is irrelevant to checking that X has the prescribed torsion in homology. Nevertheless, these properties are both critical to the second step in the proof of the main theorem. Moving forward it will be our goal to build Y_1 and Y_2 satisfying the assumptions of Lemma 4.11, but with bounded degree and logarithmically many vertices.

The triangulated sphere Y_2

In this section we describe how to build the space Y_2. The complex Y_2 will be a triangulated d-dimensional sphere with k top-dimensional faces removed where k is the Hamming weight of m. However we want a bound on the degree of Y_2 and have the number of vertices be linear in k. Toward that goal we prove the following fact about triangulations of d-dimensional spheres.

Claim 4.12. *Let $d, k \in \mathbb{N}$, with $d \geq 2$, there exists a connected, oriented simplicial complex Y_2, so that $H_{d-1}(Y_2)$ is presented as the abelian group $\langle \tau_1, ..., \tau_k \mid \tau_1 + \tau_2 + \cdots + \tau_k = 0 \rangle$ where each τ_i is coherently represented by a d-dimplex boundary Z_i, and so that for $i \neq j$, $Z_i \cap Z_j = \emptyset$ and there are no edges from the vertices of Z_i to the vertices of Z_j. Moreover,*

$$|V(Y_2)| \leq (d+2) + 60d^4k,$$

$$\Delta_{0,(d-1)}(Y_2) \leq (d+1)\frac{d^2+d+1}{2},$$

and

$$H_d(Y_2) = 0.$$

Proof. We will define a sequence $T_0, T_1, T_2, ..., T_i, ...$ of triangulations of S^d with $\Delta_{0,(d-1)}(T_i) \leq (d+1)\frac{d^2+d+1}{2}$ for all i, but with $|V(T_i)| \to \infty$ as $i \to \infty$. The complex Y_2 will be obtained by finding an appropriately large triangulation of S^d from this sequence and deleting k top-dimensional faces which are vertex-disjoint and have no edges from the vertices of one to the vertices of another. We define the sequence T_i inductively exactly as in the $d = 2$ case. Let T_0 be the $(d+1)$-simplex boundary with vertex set $\{v_0, ..., v_{d+1}\}$. At step $i \geq 1$, we perform a bistellar 0-move at the face $[v_{i+1}, ..., v_{d+i+1}]$ with a new vertex v_{d+i+2}. Recall that a bisteller 0-move (also known as *stacking*) at a d-dimensional face is defined, first in [46, 47], as the retriangulation obtained by deleting the d-dimensional face and replacing it by the cone over its boundary. The bistellar 0-move adds the face $[v_{i+2}, ..., v_{d+i+2}]$ to the complex so we may continue inductively. Now, when a vertex is introduced in this inductive process it has $(d-1)$-dimensional-face-degree $\binom{d+1}{d-1}$, and each bistellar 0-move increases this degree by at most $\binom{d}{d-2}$. However, we only retriangulate a top-dimensional face containing any particular vertex at most $d+1$ times. Thus, for all i, $\Delta_{0,(d-1)}(T_i) \leq \binom{d+1}{d-1} + (d+1)\binom{d}{d-2} \leq \frac{(d+1)^2}{2} + (d+1)\frac{d^2}{2}$.

Now for each i define a graph Γ_i with the vertices of Γ_i equal to the d-dimensional faces of T_i and (σ, τ) an edge of Γ_i if there is an edge in T_i from some vertex of σ to some vertex of τ or if $\sigma \cap \tau \neq \emptyset$. For every i, we have that the maximum degree of Γ_i is at most

$$\begin{aligned}
&(d+1)2\Delta_{0,(d-1)}(T_i) + (d+1)\Delta_{0,1}(T_i)2\Delta_{0,(d-1)}(T_i) \\
&\leq 2(d+1)^2\frac{d^2+d+1}{2} + 4(d+1)^3\frac{d^2+d+1}{2} \\
&= (d+1)^2(d^2+d+1)(2d+3)
\end{aligned}$$

The first summand follows since at there are $(d+1)$ vertices which define any d-dimensional faces, each belongs to at most $\Delta_{0,(d-1)}(T_i)$ codimension-1 faces, and $\Delta_{(d-1),d}(T_i) = 2$. The second summand follows since we have at most $\Delta_{0,1}(T_i)$ vertices which are adjacent to any fixed vertex. Moreover $\Delta_{0,1}(T_i) \leq 2(d+1)$ for every i as every vertex starts with edge-degree at most $(d+1)$, each bistellar 0-move increases the edge-degree of a vertex by at most 1, and we only retriangulate at any fixed vertex at most $(d+1)$ times.

By construction, we have $f_d(T_i) = f_d(T_{i-1}) + d$, with T_0 having $\binom{d+2}{d+1} = d+2$ top-dimensional faces. Thus, Γ_i has $d+2+di$ vertices, and so for every i, Γ_i has an independent set of size

$$\alpha(\Gamma_i) \geq \frac{d+2+di}{(d+1)^2(d^2+d+1)(2d+3)}.$$

Thus if we let

$$i_0 = \frac{2(d+1)^2(d^2+d+1)(2d+3)k}{d},$$

Then Γ_{i_0} has an independent set of size $2k$. This corresponds to a collection S of $2k$ top-dimensional faces in T_{i_0} which are all vertex-disjoint from one another and which have no edges between the vertices of one and the vertices of another. Since T_{i_0} is a triangulated manifold with orientation induced by some ordering on the vertices there is a d-chain $x = (x_1, ..., x_l)$ (where $l = f_d(T_{i_0})$) with $x_j = \pm 1$ for all j and $\partial_d x = 0$, where ∂_d denotes the top boundary matrix of T_{i_0}. Without loss of generality at least half of the faces in S have coefficient -1 in x, let $\tau_1, \tau_2, ..., \tau_k$ be a collection of k such faces. Delete these faces from T_{i_0} to get Y_2. Now we have

$$|V(Y_2)| = |V(T_{i_0})| = (d+2) + i_0 \leq (d+2) + 60d^4k$$

and

$$\Delta_{0,(d-1)}(Y_2) \leq (d+1)\frac{d^2+d+1}{2}.$$

Finally since Y_2 is a triangulated multipunctured d-dimensional sphere, $H_d(Y_2) = 0$. □

The triangulated telescope Y_1

In this section we will describe how to build, for any dimension $d \geq 2$ and any integer n, a d-dimensional, connected, oriented simplicial complex Y_1 so that $H_{d-1}(Y_1)$ is presented by $\langle \gamma_0, \gamma_1, \gamma_2, ...\gamma_n \mid 2\gamma_0 = \gamma_1, 2\gamma_1 = \gamma_2, \cdots, 2\gamma_{n-1} = \gamma_n \rangle$ where each γ_i is coherently represented by a d-simplex boundary, which are vertex disjoint from one another, and so that $\Delta(Y_1) \leq M$ and $|V(Y_1)| \leq Mn$ for some constant M depending only on d.

This will be accomplished by first constructing a simplicial complex $P = P(d)$ with an orientation so that $H_{d-1}(P) = \langle a, b \mid 2a = b \rangle$ where a and b are coherently represented by a pair of vertex-disjoint d-simplex boundaries. We may then attach n copies of these complexes "end-to-end" to build Y_1, with $\Delta(Y_1) \leq 2\Delta(P)$ and $|V(Y_1)| \leq |V(P)|n$. We now build $P(d)$.

Lemma 4.13. *Fix $d \geq 2$. Then there exists an oriented simplicial complex P, depending only on d, on $2(d+1)$ vertices with its orientation induced by an ordering on the vertices, so that $H_{d-1}(P)$ is presented as $\langle a, b \mid 2a = b \rangle$ where a and b are coherently represented by a pair of vertex-disjoint d-simplex boundaries, and so that $H_d(P) = 0$.*

Proof. The proof will be by induction on d. For $d = 2$ our complex will be the pure simplicial complex on vertex set $\{1, 2, 3, 4, 5, 6\}$ with orientation induced by the natural ordering on the vertices and top dimensional faces $[1, 2, 6]$, $[1, 3, 6]$, $[3, 5, 6]$, $[2, 4, 6]$, $[2, 3, 4]$, $[1, 3, 4]$, $[1, 4, 5]$, $[1, 2, 5]$, and $[2, 3, 5]$. This complex is given as Figure 4.4. Of course it matches Figure 4.3, but the vertices have been relabeled as the focus here is only to describe the building block, but not how they are attached to one another.

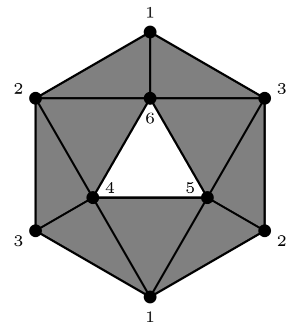

Figure 4.4: The simplicial complex $P(2)$

Observe that $\partial_2([1,2,6]+[2,4,6]+[2,3,4]-[1,3,4]-[1,4,5]+[1,2,5]+[2,3,5]-[3,5,6]-[1,3,6]) = 2([1,2]-[1,3]+[2,3])-([4,5]-[4,6]+[5,6])$. Thus $H_1(P)$ can be presented as $\langle a, b \mid 2a = b\rangle$ where a is represented by $[1,2]-[1,3]+[2,3]$ and b is represented by $[4,5]-[4,6]+[5,6]$. That is, a is coherently represented by the triangle boundary on vertex set $\{1,2,3\}$ and b is coherently represented by the triangle boundary on vertex set $\{4,5,6\}$. This completes the base case.

We are now ready to prove the inductive step; to build $P(d+1)$ from $P(d)$. Begin with $P(d)$ a d-dimensional oriented simplicial complex, with the orientation induced by an ordering on the vertices, with $H_{d-1}(P(d))$ presented as $\langle a, b \mid 2a = b\rangle$ with a and b coherently represented by vertex-disjoint d-simplex boundaries. Write $a_0, a_1..., a_d$ for the coherently-ordered vertices of the simplex boundary representing a and $b_0, b_1, ..., b_d$ for the coherently-ordered vertices of the simplex boundary representing b. Now take the suspension of $P(d)$ with two suspension vertices w_1 and w_2. When taking the suspension $SP(d)$ keep the ordering on the vertices of $P(d)$ and add w_1 followed by w_2 to the beginning of the ordering.

With respect to this ordering we have that the map $\phi : C_i(P(d)) \to C_{i+1}(SP(d))$ given by sending the generator $[v_0, ..., v_i]$ to $[w_1, v_0, ..., v_i] - [w_2, v_0, ..., v_i]$ induces an isomorphism from $H_i(P(d))$ to $H_{i+1}(SP(d))$ for $i > 0$, in particular for $i = d - 1$ (This is routine to check, but we do prove it as Claim 4.14 below.). Thus, by the inductive hypothesis $H_d(SP(d))$ is generated by a', b' with the relator $2a' = b'$ and where a' is represented by

$$\sum_{i=0}^{d}(-1)^i[w_1, a_0, ..., \hat{a}_i, ..., a_d] - \sum_{i=0}^{d}(-1)^i[w_2, a_0, ..., \hat{a}_i, ..., a_d],$$

and b' is represented by

$$\sum_{i=0}^{d}(-1)^i[w_1, b_0, ..., \hat{b}_i, ..., b_d] - \sum_{i=0}^{d}(-1)^i[w_2, b_0, ..., \hat{b}_i, ..., b_d].$$

While we have that $H_d(SP(d)) = \langle a', b' \mid 2a' = b' \rangle$, we are not done since a' and b' are not represented by $(d + 1)$-simplex boundary. To fix this we add the $(d + 1)$-dimensional faces $[w_1, a_0, ..., a_d]$ and $[w_2, b_0, ..., b_d]$ along with the necessary d-dimensional faces $[a_0, ..., a_d]$ and $[b_0, ..., b_d]$ to our complex. Now we have that

$$\partial_{d+1}[w_1, a_0, ..., a_d] = [a_0, ..., a_d] + \sum_{i=0}^{d}(-1)^{i+1}[w_1, a_0, ..., \hat{a}_i, ..., a_d],$$

and

$$\partial_{d+1}[w_2, b_0, ..., b_d] = [b_0, ..., b_d] + \sum_{i=0}^{d}(-1)^{i+1}[w_2, b_0, ..., \hat{b}_i, ..., b_d].$$

Thus after adding in these two new $(d+1)$-dimensional faces and two d-dimensional faces we have the new relators in the codimension-1 homology group of our complex given by

$$[a_0, ..., a_d] = \sum_{i=0}^{d}(-1)^i[w_1, a_0, ..., \hat{a}_i, ..., a_d]$$

and

$$[b_0, ..., b_d] = \sum_{i=0}^{d}(-1)^i[w_2, b_0, ..., \hat{b}_i, ..., b_d]$$

Using these relators and the representatives for a' and b' listed above, we have that the codimension-1 homology group is generated by

$$a' = [a_0, ..., a_d] + \sum_{i=0}^{d}(-1)^{i+1}[w_2, a_0, ..., \hat{a}_i, ..., a_d]$$

and

$$b' = -[b_0, ..., b_d] - \sum_{i=0}^{d}(-1)^{i+1}[w_1, b_0, ..., \hat{b}_i, ..., b_d]$$

with the relator $2a' = b'$. Moreover we have that a' and $-b'$ are coherently represented by the boundary of $(d+1)$-simplicies, namely the boundaries of $[w_2, a_0, ..., a_d]$ and $[w_1, b_0, ..., b_d]$ respectively. We are not quite finished yet since the orientation has been reversed, but if we simply reverse the order of b_0 and b_1 then we have the oriented simplicial complex $P(d+1)$ that we want. Alternatively, we could observe that there is a sign change every time we increase the dimension and modify $P(2)$ by switching the labels 4 and 5 if we are building to an odd dimension d.

Regarding $H_d(P(d))$ we have that $P(2)$ is homotopy equivalent to the circle, and for $d \geq 3$, we have that $P(d)$ is homotopy equivalent to the suspension of $P(d-1)$, and so induction implies that for all d, $H_d(P(d)) = 0$. □

In the previous proof we make use of the following claim, which we prove here for the sake of completeness.

Claim 4.14. *Suppose X is an oriented simplicial complex with orientation induced by an ordering on the vertices of X. If we take the suspension of X, denoted SX, with the two suspension vertices w_1 and w_2 added to the beginning of the vertex ordering, then the map $\phi : C_i(X) \to C_{i+1}(SX)$ given by sending each generator $[v_0, ..., v_i]$ to $[w_1, v_0, ..., v_i] - [w_2, v_0, ..., v_i]$ induces an isomorphism from $H_i(X)$ to $H_{i+1}(X)$ for all $i > 0$.*

Proof. Let ∂_i denote the ith boundary map of X and ∂'_i denote the ith boundary map of X'. By the choice of ordering we have for each $i > 0$ that the matrix ∂'_{i+1} is given by

$$\partial'_{i+1} = \begin{pmatrix} -\partial_i & 0 & 0 \\ 0 & -\partial_i & 0 \\ I & I & \partial_{i+1} \end{pmatrix},$$

where the columns are indexed by $(i+1)$-dimensional faces which contain w_1, followed by $(i+1)$-dimensional faces that contain w_2, followed by $(i+1)$-dimensional faces present in X, and the rows are indexed by i-dimensional faces that contain w_1, followed by i-dimensional faces that contain w_2, followed by i-dimensional faces present in X. With respect to this basis, the map ϕ sends an arbitrary vector $v = (a_0, ..., a_k) \in C_i(X)$ to the vector $(a_0, ..., a_k, -a_0, ..., -a_k, 0, ...0)$ in $C_{i+1}(SX)$, where k is the number of i-dimensional faces in X. To simplify notation we denote $\phi(v)$ with $(v, -v, 0)$.

Now we prove that ϕ is well-defined on homology groups by showing that ϕ sends cycles to cycles and boundaries to boundaries. Suppose that $\partial_i(v) = 0$, then by the construction of ∂'_{i+1} given above we have that $\partial'_{i+1}(\phi(v)) = \partial'_{i+1}((v, -v, 0)) = 0$, so ϕ sends cycles to cycles. Next, suppose that v is in the image of ∂_i, then there exists u so that $\partial_i u = v$. Thus, $\partial'_{i+1}((-u, u, 0)) = (v, -v, 0) = \phi(v)$. It follows that ϕ is a well-defined homomorphism on homology groups.

Now we check that ϕ is injective. Suppose $\phi(v) = 0$. Then $(v, -v, 0)$ belongs to $\mathrm{Im}(\partial'_{i+2})$. Thus there exists u which we write as (u_1, u_2, u_3) so that $\partial'_{i+2}((u_1, u_2, u_3)) = (v, -v, 0)$. Thus $\partial_{i+1}(-u_1) = v$, so $v \in \mathrm{Im}(\partial_{i+1})$. Thus at the level of homology groups $v = 0$, so ϕ is injective.

Now we show that ϕ is surjective. Let $z \in H_{i+1}(SX)$. Since every $(i+1)$-dimensional face of X is contained in at least one (in fact at least 2) $(i+2)$-dimensional faces in SX, we may write z as $x + y$ where x is a sum of generating $(i+1)$-chains that all contain the vertex w_1 and y is a sum of generating $(i+1)$-chains which all contain the vertex w_2. That is we may write $z = (x, y, 0)$. Since $\partial'_{i+1}(z) = 0$ we have $\partial'_{i+1}((x, y, 0)) = (-\partial_i x, -\partial_i y, x + y) = (0, 0, 0)$. Thus $x = -y$ in the sense that after deleting w_1 from every chain in the support of x and deleting w_2 from every chain in the support of y we have that x and $-y$ are the exact same i-chain. Thus $\phi(x) = (x, -x, 0) = (x, y, 0) = z$, proving that ϕ in onto, and completing the proof that ϕ is an isomorphism. $\square$

Now that we have the construction for $P(d)$ we may apply Lemma 4.10 to construct the complex Y_1 that we need.

Lemma 4.15. *Fix $d \geq 2$, and let $P = P(d)$ denote the complex constructed in Lemma 4.13, and let n be a positive integer. Then there exists a connected, oriented simplicial complex Y_1 with $\Delta_{0,d-1}(Y_1) \leq 2\Delta_{0,d-1}(P)$, $|V(Y_1)| \leq n|V(P)|$, $H_d(Y_1) = 0$, and $H_{d-1}(Y_1)$ presented by $\langle \gamma_0, ..., \gamma_n \mid 2\gamma_0 = \gamma_1, 2\gamma_1 = \gamma_2, ..., 2\gamma_{n-1} = \gamma_n \rangle$ where each γ_i is coherently represented by a d-simplex boundary Z_i so that the for all $i \neq j$, $Z_i \cap Z_j = \emptyset$. Moreover, $H_d(Y_1) = 0$.*

Proof. Fix n and d. Take n copies of P denoted $P_1, P_2, ..., P_n$ with $H_{d-1}(P_i) = \langle a_i, b_i \mid 2a_i = b_i \rangle$, a_i coherently represented by d-simplex boundary α_i and b_i coherently represented by d-simplex boundary β_i. Now use Lemma 4.10 to attach P_i to P_{i+1} by the order-preserving simplicial homeomorphism $f_i : \beta_i \to \alpha_{i+1}$ for every $i \in \{1, ..., n-1\}$. This results in a connected, oriented simplicial complex Y_1 which has $H_{d-1}(Y_1)$ presented by $\langle \gamma_0, ..., \gamma_n \mid 2\gamma_0 = \gamma_1, 2\gamma_1 = \gamma_2, ..., 2\gamma_{n-1} = \gamma_n \rangle$ so that each γ_i is coherently represented by a d-simplex boundary which are all vertex disjoint from one another, and so that $H_d(Y_1) = 0$. It is clear from the fact that we use n copies of P to build Y_1 that $|V(Y_1)| \leq n|V(P)|$ (in fact $|V(Y_1)| = (d+1)(n+1)$). Furthermore no vertex belongs to more than two copies of P, and so $\Delta_{0,(d-1)}(Y_1) \leq 2\Delta_{0,(d-1)}(P)$. $\square$

Finishing the proof

Proof of Lemma 4.7. Let $d, m \geq 2$ be given. Write $m = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_k}$ with $0 \leq n_1 < n_2 < \cdots < n_k$. Using Lemma 4.15 we construct a complex Y_1 with $H_{d-1}(Y_1) = \langle \gamma_0, \gamma_1, \cdots, \gamma_{n_k} \mid 2\gamma_0 = \gamma_1, 2\gamma_1 = \gamma_2, \cdots, 2\gamma_{n-1} = \gamma_n \rangle$ and $H_d(Y_1) = 0$ where each γ_i is coherently represented by a d-simplex boundary all of which are vertex disjoint from one another and so that

$$|V(Y_1)| \leq 2(d+1)\log_2(m),$$

and

$$\Delta_{0,(d-1)}(Y_1) \leq 2\Delta_{0,d-1}(P) \leq 2f_{d-1}(P) \leq 2(3^d d).$$

Note that the bound on $f_{d-1}(P)$ of $3^d d$ follows from the fact that $P(2)$ has 9 triangles and $P(d+1)$ is obtained by taking the suspension of $P(d)$ and adding in two new $(d+1)$-dimensional faces, so $f_d(P(d))$ satisfies the recurrence

$$f_d(P(d)) = 2f_{d-1}(P(d-1)) + 2.$$

Thus $f_d(P) \leq 3^d$ and each top-dimensional face contains d codimension-1 faces.

Next, we use Lemma 4.12 to construct Y_2 with the so that

$$|V(Y_2)| \leq (d+2) + 60d^4(\log_2(m) + 1),$$
$$\Delta_{0,(d-1)}(Y_2) \leq (d+1)\frac{d^2+d+1}{2},$$

and with $H_{d-1}(Y_2) = \langle \tau_1, ..., \tau_k \mid \tau_1 + \tau_2 + \cdots + \tau_k = 0 \rangle$ and $H_d(Y_2) = 0$ where each τ_i is coherently represented by a d-simplex boundary Z_i, and so that for $i \neq j$, $Z_i \cap Z_j = \emptyset$ and there are no edges from the vertices of Z_i to the vertices of Z_j. Finally, we use Lemma 4.11 to attach Y_1 to Y_2 in a way that identifies the oriented simplex boundary representing γ_{n_i} to the oriented simplex boundary representing τ_i for every i to obtain X with

$$H_{d-1}(X)_T \cong \mathbb{Z}/m\mathbb{Z},$$

$$H_d(X) = 0,$$

$$|V(X)| \leq (d+2) + 60d^4(\log_2(m)+1) + 2(d+1)\log_2(m) \leq 182d^4\log(m),$$

and

$$\Delta_{0,(d-1)}(X) \leq 2(3^d d) + (d+1)\frac{d^2+d+1}{2} \leq 2(3^d d) + 3d^3.$$

□

4.4 The Final Construction

In this section we use the probabilistic method, in particular the Lovász Local Lemma, to show that there exists a coloring c of the initial construction X, so that the pattern complex (X, c) is the final construction which proves the upper bound of Theorem 4.1. We begin by stating (the symmetric version of) the Lovász Local Lemma as it is stated in Chapter 5 of [1].

Lovász Local Lemma (Erdős and Lovász [17])**.** *Let $A_1, A_2, ..., A_n$ be events in an arbitrary probability space. Suppose that each event A_i is mutually independent of a set of all the other events A_j but at most t, and that $\Pr[A_i] \leq p$ for all $1 \leq i \leq n$. If $ep(t+1) \leq 1$ then $\Pr[\bigwedge_{i=1}^{n} \overline{A_i}] > 0$*

We are now ready for the final step, the proof of lemma 4.6. In proof, we implicitly treat $\sqrt[d]{n}$ as an integer, when really we mean $\lceil \sqrt[d]{n} \rceil$.

Proof of Lemma 4.6. We will find three colorings c_1, c_2, c_3 of $V(X)$ so that c_1 is a proper coloring of $X^{(1)}$, c_2 has no pair of intersecting $(d-1)$-dimensional faces receiving the same pattern, and c_3 has no pair of disjoint $(d-1)$-dimensional faces receiving the same pattern. We then let $c = (c_1, c_2, c_3)$, and c will have the required properties with $|c| = |c_1||c_2||c_3|$ and we will show that this is bounded by the value in the statement. Finding c_1 is easy; there is a proper coloring of $X^{(1)}$ by at most

K colors since the vertex degree is bounded above by $K-1$, choose such a proper coloring for c_1. To show that c_2 and c_3 exist, we will use the Lovász Local Lemma.

We first show that there exists a coloring c_2 on at most $3d^5K^5$ colors so that no pair of intersecting $(d-1)$-dimensional faces receive the same pattern by c_2. We may define a graph H from X by letting the vertices of H be the $(d-1)$-dimensional faces of X with $(v,w) \in E(H)$ if and only if $v \cap w \neq \emptyset$, a coloring of $V(X)$ induces a coloring of H by the patterns on the $(d-1)$-dimensional faces. We wish to show that there is a coloring of $V(X)$ by at most $3d^5K^5$ colors which induces a proper coloring on H. Consider the probability space of coloring of $V(X)$ by coloring each vertex uniformly at random from among a set of $3d^5K^5$ colors. For a fixed $(d-1)$-dimensional face σ, let A_σ denote the event that some neighbor of σ in H receives the same pattern as σ.

By the bounded degree condition on X, for a $(d-1)$-dimensional face σ the number of faces τ so that $(\sigma, \tau) \in E(H)$ is at most dK. Now σ and τ are most likely to receive the same pattern if they share $(d-1)$ vertices. Thus the probability of A_σ is bounded above by $dK/(3d^5K^5)$ by a union bound. We also observe that if the distance between σ and τ in H is at least 4, then A_σ and A_τ are mutually independent as they are on disjoint vertex sets. It follows that each A_σ is mutually independent from all A_τ but at most $(dK)^4$. Thus we may apply the Lovász Local Lemma since

$$e\frac{dK}{3d^5K^5}(d^4K^4+1) \leq 3d^5K^5/(3d^5K^5) = 1.$$

It follows that there is a coloring so that no A_σ holds, i.e. a coloring so that no two intersecting $(d-1)$-dimensional faces receive the same pattern. Let c_2 be one of these colorings.

We now handle the disjoint $(d-1)$-dimensional faces, again using the Lovász Local Lemma. Consider the probability space of colorings of $V(X)$ by coloring each vertex uniformly at random from among a set of $6K^2d\sqrt[d]{n}$ colors. For σ and τ disjoint

$(d-1)$-dimensional faces, let $A_{(\sigma,\tau)}$ denote the event that σ and τ receive the same pattern. We have that

$$\Pr(A_{\sigma,\tau}) \leq \frac{d^d}{6^d K^{2d} d^d n} = \frac{1}{6^d K^{2d} n}.$$

Now for (σ, τ) and (σ', τ'), $A_{(\sigma,\tau)}$ and $A_{(\sigma',\tau')}$ are mutually independent if the two pairs are on disjoint vertex sets. It follows that for any (σ, τ) the number of (σ', τ') so that $A_{(\sigma,\tau)}$ is not disjoint from $A_{(\sigma',\tau')}$ is at most $2(dK(K/d)n)$ (pick one of the at most dK faces adjacent to σ for σ' and then any other of the at most $(K/d)n$ faces for τ, then reverse the roles of σ and τ). Thus the Lovász Local Lemma may be used since

$$e\frac{1}{6^d K^{2d} n}(2K^2 n + 1) \leq \frac{6K^2 n}{6K^2 n} = 1.$$

It follows that there is a coloring by at most $6K^2 d\sqrt[d]{n}$ colors so that no disjoint $(d-1)$-dimensional faces receive the same pattern, let c_3 be one of these colors. We let $c = (c_1, c_2, c_3)$ then c is on at most $K(3d^5K^5)(6K^2d\sqrt[d]{n}) = 18K^8d^6\sqrt[d]{n}$ colors and has the required properties. □

4.5 Proof of Theorem 4.2

We are now ready to prove the refined version of the prescribed torsion theorem in order to obtain the doubly-exponential lower bound on $h(n)$. As in the case of the proof of Theorem 4.1, we will build an initial construction and then take a random quotient of it. By how we have organized this chapter we already have then initial construction that will work from Lemma 4.4, we just have to find more efficient way to take a quotient of it.

Given a finite abelian group G and a dimension d we construct $X = X(G, d)$ as in the proof of Lemma 4.4. Now we wish to prove the existence of an efficient proper coloring of its vertices so that no two $(d-1)$ dimensional faces receive the same pattern. In the proof of Lemma 4.6, this was accomplished using a three-step approach. Namely, we found a proper coloring, then a coloring in which no pair of intersecting $(d-1)$-dimensional faces received the same pattern, and finally a coloring in which no pair of non-intersecting $(d-1)$-dimensional faces received the same pattern. The product of these three colorings therefore had all three properties.

In the case of the proof of Theorem 4.2 we will take a two-step approach to the coloring instead. The first step will handle the proper-coloring condition and the pairs of intersecting $(d-1)$-dimensional faces simultaneously. Furthermore this first coloring will be explicitly described. For the second step we will remember the colors used in the first step and use the Lovász Local Lemma as in the proof of Lemma 4.6. This step is given by the following lemma.

Lemma 4.16. *Let X be a d-dimensional simplicial complex, for $d \geq 2$, on n vertices with $\Delta_{0,d-1}(X) \leq L$ for some L. If there exists a proper coloring c of the vertices of X having at most K colors so that no pair of intersecting $(d-1)$-dimensional faces receive the same pattern by c then there exists a second proper coloring c' of $V(X)$ having at most $K(3eL^2n)^{1/d}$ colors so that no pair of $(d-1)$-dimensional faces of X receive the same pattern by c'.*

Proof. Unlike the proof of Lemma 4.6, we begin by assuming there is a coloring c on K colors which is proper and which has no two intersecting $(d-1)$-dimensional faces receiving the same pattern. For c', which handles the disjoint $(d-1)$-dimensional faces, we will take the product of c with a second coloring c_2 found using the Lovász Local Lemma by considering the random process of coloring every vertex independently from a set of $(3eL^2n)^{1/d}$ colors. Moreover, we will also use the fact that we have colored the vertices properly by c. For a vertex-disjoint pair of $(d-1)$-dimensional faces (σ, τ), let $A_{\sigma,\tau}$ be the event that σ and τ receive the same pattern by the coloring $c' = (c, c_2)$ (unlike the proof of Lemma 4.6, we will not have that no two disjoint faces receive the same pattern by c_2 alone, only that they will not receive the same pattern by the *product* of the two colorings). If σ and τ receive different patterns by c, then c_2 is irrelevant to guaranteeing that they do not receive the same pattern by c'. Thus we only have to consider vertex-disjoint pairs (σ, τ) where σ and τ receive the same pattern by c. Since c is a proper coloring of $V(X^{(1)})$ we have that the identical colorings of σ and τ induce a bijection $\varphi : V(\sigma) \to V(\tau)$ by sending each vertex in σ to the unique vertex in τ that received the same coloring under c. Thus σ, τ receive the same pattern by c' if and only if for every $v \in \sigma$, $c_2(v) = c_2(\varphi(v))$. For each v, this occurs with probability $(3eL^2n)^{-1/d}$. Thus for every vertex-disjoint pair (σ, τ) we have

$$\Pr(A_{\sigma,\tau}) \leq \frac{1}{((3eL^2n)^{1/d})^d} = \frac{1}{3eL^2n}$$

Now, we need to bound the number of (σ', τ') which have $A_{\sigma,\tau}$ not independent from $A_{\sigma',\tau'}$. This is bounded above by $2dL(Ln/d) = 2L^2n$ (pick one of the at most dL faces σ' which share a vertex with σ, then pick any of the at-most Ln/d faces for τ and multiply by two since we may reverse the role of σ and τ). Now, we have

$$e\frac{1}{3eL^2n}(2L^2n + 1) \leq 1,$$

so the Lovász Local Lemma applies. Thus there is a coloring c_2, so that the resulting product c' of c and c_2 is a proper coloring and has that no two $(d-1)$-dimensional faces receive the same pattern. □

We now have everything we need to prove Theorem 4.2

Proof of Theorem 4.2. Let $d \geq 2$ and G a finite abelian group be given. Let $X = X(G, d)$ be as in the proof of Lemma 4.4. Then $|V(X)| \leq 182d^4 \log |G|$, $\Delta_{0,(d-1)}(X) \leq 2(3^d d) + 3d^3$, and $H_{d-1}(X)_T \cong G$. We first show that we may properly color X with at most $5(d+1)$ colors so that no intersecting pair of $(d-1)$-dimensional faces of X receive the same pattern.

To show that such a coloring of X exists we will properly color the square, in the graph-power sense, of the 1-skeleton of X. Recall that the square of a graph H is the graph obtained by starting with H and adding in an edge between any pair of vertices at distance two from one another. If we properly color the square of the 1-skeleton of X then we have a proper coloring of X in which no pair of intersecting faces of any dimension receive the same pattern. We claim that we can color the square of the 1-skeleton of X properly using at most $5(d+1)$ colors.

It suffices to consider each connected component of X, i.e. to assume that G is a cyclic group of order $m = 2^{n_1} + \cdots + 2^{n_k}$ with $0 \leq n_1 < \cdots < n_k$. In this case, as in the proof of Lemma 4.7, we have that X is build from two complexes Y_1 and Y_2. To color X we will first color Y_1 and then color $Y_2 \setminus Y_1$. By construction, the 1-skeleton of Y_1 is a subgraph of the graph obtained by starting with $n_k + 1$ copies of the $(d+1)$-clique, $K_0, K_1, K_2, ..., K_{n_k}$ and for each $i \in \{0, ..., n_k - 1\}$ adding in all possible edges from K_i to K_{i+1}. This graph can be colored using at most $3(d+1)$ colors so that no vertices at distance at most 2 from one another receive the same color. Thus this gives a coloring of Y_1 with the same property. Now we turn our attention to $Y_2 \setminus Y_1$. Recall that Y_2 is a punctured, triangulated d-dimensional sphere obtained from a sequence $T_0, T_1, ...$ of triangulated d-spheres as in Claim 4.12. The repeated stacking moves with define this sequence of triangulations give an ordering to the vertices of Y_2. Indeed we start with T_0 on vertex set $\{v_1, v_2, ..., v_{d+2}\}$ and at step i we add vertex a new vertex to T_i which is adjacent to exactly the last $d+1$ vertices in the ordered vertex set of T_i. Thus, if we consider this order on the vertices of Y_2 restricted to $Y_2 \setminus Y_1$ we have that for any i, the number of $j < i$ so that $v_j \in Y_2$ is within two steps of v_i is at most $2(d+1)$. Therefore, after starting by properly coloring of the square of Y_1 with at most $3(d+1)$ colors, we may use a greedy approach with no more than $2(d+1)+1$ colors to color the rest of X so that the square of X is properly colored by at most $5d+6$ colors.

Finally we invoke Lemma 4.16 with $K = 5d+6$ and $L = 2(3^d d) + 3d^3$ to get a proper coloring c of X, having at most $(5d+6)(546(3^d 2d + 3d^3)^2 ed^4)^{1/d}(\log|G|)^{1/d}$ colors so that no pair of $(d-1)$-dimensional faces receive the same pattern. As d tends to infinity, this approaches $9(5d+6)(\log|G|)^{1/d}$, so for d large enough, $T_d(G) \leq 50d(\log|G|)^{1/d}$. $\square$

We make two observations here. The first is that the constant of 50 in the statement of Theorem 4.2 is clearly not best possible. Indeed from the proof it can be improved to $45+\epsilon$ for any $\epsilon > 0$. However, it can be improved further when we note that the 45 comes from $9 * 5$ where the 9 is from the bound of $3^d d$ on $f_{d-1}(P(d))$. However, as we noted in the proof of Lemma 4.7, this bound comes from the fact that $f_d(P(d)) = 2f_{d-1}P(d-1) + 2$ where $f_2(P(2)) = 9$. Trivially, for all d, we have that $f_d(P(d)) \leq 3^d$, however for any $\epsilon > 0$ we can find d large enough so that $f_d(P(d)) \leq (2+\epsilon)^d$. Thus we can improve the $45+\epsilon$ to $5*(2+\epsilon)^2$, and thus for any $\epsilon > 0$ we have that for d large enough $T_d(G) \leq (20+\epsilon)d(\log|G|)^{1/d}$. This implies a lower bound on $h(n)$ of $\exp(\exp(0.018n))$ for all n sufficiently large.

The second observation is that we have a lower bound on $T_d(G)$ of $\Omega(d(\log|G|)^{1/d})$. Indeed, from [28] we have that if X is a d-dimensional simplicial complex on n vertices the we have that $|H_{d-1}(X)_T| \leq \sqrt{d+1}^{\binom{n-2}{d}}$, thus $T_d(G) \geq (1-o_d(1))\frac{d}{e}(\log|G|)^{1/d}$. It follows that $T_d(G) = \Theta(d(\log|G|)^{1/d})$ where the implied constants are absolute and, asymptotically in d, not that far apart, between $\frac{1}{e}$ and 20.

4.6 Toward a $\mathbb{Q}$-acyclic version

Observe that we have shown that how to construct for any group G a simplicial complex X with G as the torsion part of $H_{d-1}(X)$. However the construction we give here generally is not a $\mathbb{Q}$-acyclic complex. Indeed X will typically not have complete $(d-1)$-skeleton and there is no reason to expect that the random quotient will not produce any free $(d-1)$-cycles. On the other hand, our initial construction will have $\beta_d = 0$, and taking a quotient as described in Lemma 4.5 will preserve this top Betti number. Therefore X will have no top homology, one of the three conditions required for X to be a $\mathbb{Q}$-acyclic complex. Given this, it is perhaps reasonable to hope that the following conjecture, which would be the $\mathbb{Q}$-acyclic version of Theorem 4.1, is true.

Conjecture 4.17. *For every $d \geq 2$, there exists a constant C_d so that for any finite abelian group G there exists a d-dimensional $\mathbb{Q}$-acyclic complex X which has $H_{d-1}(X) \cong G$ and at most $C_d(\log|G|)^{1/d}$ vertices.*

Almost immediately from the proof of Theorem 4.1 we have the following partial result

Proposition 4.18. *For every $d \geq 2$, there exists a constant C_d so that for any finite abelian group G there exists a d-dimensional $\mathbb{Q}$-acyclic complex X which has $G \leq H_{d-1}(X)$ and at most $C_d(\log|G|)^{1/d}$ vertices.*

Proof. Fix $d \geq 2$ and let G be an arbitrary finite abelian group. By Lemma 4.4 there exists a constant K, depending only on d, and a simplicial complex X on at most $K\log_2(|G|)$ vertices with $\Delta(X) \leq K$, $H_{d-1}(X)_T \cong G$ and $H_d(X) = 0$. By Lemma 4.6 there exists a coloring c of X with at most $K'\sqrt[d]{\log|G|}$ colors so that no two $(d-1)$-dimensional faces receive the same pattern, where K' depends only on d. By Lemma 4.5, $H_{d-1}(X)_T \cong H_{d-1}((X,c))_T$ and $H_d((X,c)) = H_d(X)$. Thus (X,c) has at most $K' \sqrt[d]{\log |G|}$ vertices, $H_{d-1}((X,c))_T \cong G$, and $H_d((X,c)) = 0$. Now add to (X,c) every face of dimension at most $(d-1)$ on $V((X,c))$. Call this resulting complex X_1. Adding faces of dimension at most $(d-1)$ will not change the top boundary map of (X,c) and so $H_{d-1}(X_1)_T \cong G$ and $H_d(X_1) = 0$. Moreover, since X_1 has complete $(d-1)$-skeleton $\tilde{H}_i(X_1) = 0$ for $i < d-1$. Now if $\beta_{d-1}(X_1) = 0$, then we are done. Otherwise there exists a d-dimensional face on $V(X_1)$ which may be added to X_1 to decrease β_{d-1}, thus we may inductively add faces which drop β_{d-1} until we get a complex which has $\beta_{d-1} = 0$. This process will never increase β_d or affect the lower homology groups, however we may increase the torsion. Nevertheless, G will always embed in the torsion part of $H_{d-1}(X_1)$ at each step. This finishes the proof with $C_d = K'$. □

4.7 Toward a deterministic version

Observe that even though the proof of Theorem 4.1 uses the probabilistic method, it does give an algorithm for construction a small simplicial complex with prescribed torsion in homology. The algorithm is as follows:

Algorithm: Prescribed Torsion Construction

Input: A finite abelian group G and a dimension $d \geq 2$.

Output: A d-dimensional simplicial complex Y with $H_{d-1}(Y)_T \cong G$ and $n(Y) \leq C_d \sqrt[d]{\log}|G|$.

Steps

1. Build the simplicial complex X in the proof of Theorem 4.4.

2. Color the vertices of X so that no two adjacent vertices receive the same color and no two $(d-1)$-dimensional faces receive the same pattern in a way that minimizes the number of colors used. Call this coloring c.

3. Return (X, c).

The probabilistic method in the proof of Theorem 4.1 was only required to show that Step 2 can be carried out with at most $C_d \sqrt[d]{\log |G|}$. But observe that we could take *any* simplicial complex X with $H_{d-1}(X)_T \cong G$ and then color the vertices properly by a coloring c so that no two $(d-1)$-dimensional faces receive the same pattern and the resulting pattern complex (X, c) will have the right torsion group.

Therefore, if we could find a rule for coloring the vertices of the construction described in the proof of Lemma 4.4 in a way that satisfies the hypothesis of Lemma 4.5 using the minimum number of colors possible we would have a deterministic version of Theorem 4.1. However, such a coloring seems difficult to find.

On the other hand we could apply, say, a modified greedy algorithm to build a small simplicial complexes with prescribed torsion in homology. While we don't yet have a proof that such an approach will lead to a complex of minimal size, Table 4.1 gives experimental results in the case that G is a cyclic group to this approach. Note that in this table we include both the number of vertices in the initial construction, as the column $|V(X)|$, and in the final construction, under the heading $|V((X, c))|$.

A motivation to obtain such an algorithm would be to provide explicit construction for any finite abelian group. With Theorems 4.1 and 4.2 we understand the asymptotic behavior of $T_d(G)$, but cannot get meaningful bounds on $T_d(G)$ for G of any reasonable size. For example taking the proof of Theorem 4.2 as a black box, we have that $T_5(\mathbb{Z}/10^{2018}\mathbb{Z}) \leq 62{,}744$; a greedy approach instead gives a bound of 112.

m	$d = 2$		$d = 3$		$d = 4$		$d = 5$	
	$\|V(X)\|$	$\|V((X,c))\|$	$\|V(X)\|$	$\|V((X,c))\|$	$\|V(X)\|$	$\|V((X,c))\|$	$\|V(X)\|$	$\|V((X,c))\|$
10^{10}	115	46	147	39	180	43	214	61
10^{25}	273	69	352	51	434	52	518	76
10^{50}	561	101	710	66	869	62	1031	82
10^{100}	1106	142	1406	82	1722	70	2046	86
10^{250}	2789	223	3524	109	4307	86	5110	91
10^{500}	5576	307	7042	134	8605	99	10208	95
10^{1000}	11131	432	14067	168	17196	118	20403	103
10^{2018}	22461	609	28384	209	34698	138	41169	112

Table 4.1: Greedy Approach to Bound $T_d(\mathbb{Z}/m\mathbb{Z})$

CHAPTER 5

COMMENTS AND FUTURE RESEARCH DIRECTIONS

In the concluding remarks of the previous chapters we discussed several open problems about the Linial–Meshulam model. Here we collect those open problems, and, where possible, discuss some related partial or conditional results. We also introduce a random abelian group model that appears similar to the random abelian group model given by $H_{d-1}(Y)$ with $Y \sim Y_d(n,p)$. We present some early results about this model and discuss work in progress and how the random abelian group model may inspire new techniques to studying homology of random simplicial complexes.

5.1 Toward existence and uniqueness of the torsion burst

At time of writing, the torsion burst has only been observed experimentally; no proof of its existence is known. Indeed it is not even clear what the right conjecture to make is. Naively, one might conjecture that if $p = c_d/n$ where c_d is the constant discussed in [3, 35] so that c_d/n is the sharp threshold for top-dimensional homology to appear in $Y_d(n,p)$, then with high probability $Y \sim Y_d(n,p)$ has a nontrivial torsion subgroup in $H_{d-1}(Y)$. However, this is likely not correct. Recall that to see the torsion burst, one apparently has to consider the discrete-time stochastic process version of the Linial–Mesulam model, setting p to be exactly c_d/n and generating $Y \sim Y_d(n,p)$ rarely results in $H_{d-1}(Y)$ having a nontrivial torsion group. Instead, the right conjecture related to the torsion burst is likely the hitting-time conjecture, Conjecture 3.2.

Of course, establishing the existence of the torsion burst is not the only open problem. Conjectures discussed in Chapter 3 about Cohen–Lenstra heuristics and the size of the groups within the torsion burst remain open as well, and seem much more difficult than establishing existence of the torsion burst.

On the other hand, uniqueness of the torsion burst appears to more tractable. Uniqueness of the torsion burst refers to the conjecture of Łuczak and Peled [37], given as Conjecture 2.3, that if $|np - c_d|$ is bounded away from zero then $Y \sim Y_d(n,p)$ has $H_{d-1}(Y)$ torsion-free. This conjecture naturally splits into (at least) two problems: showing that there is no torsion before the phase transition and showing that there is no torsion after the phase transition.

For showing that there is no torsion before the phase transition, we can look to the work of Linial and Peled in [35] on establishing that $c < c_d$ and $p = c/n$ implies that $H_d(Y;\mathbb{R})$, for $Y \sim Y_d(n,p)$, is generated by $(d+1)$-simplex boundaries. Their proof relies on spectral methods that, at least presently, don't appear to work in the case where $\mathbb{R}$ is replaced by $\mathbb{Z}/q\mathbb{Z}$ for some prime q. It remains an open problem to show that $H_d(Y;\mathbb{Z}/q\mathbb{Z})$ is generated by $(d+1)$-simplex boundaries for $Y \sim Y_d(n,c/n)$ and $c < c_d$. Of course by the universal coefficient theorem and Linial and Peled's result on homology with coefficients in $\mathbb{R}$ this is equivalent to proving that there is no q-torsion in $H_{d-1}(Y)$ for $Y \sim Y_d(n,c/n)$ and $c < c_d$. Moreover, the following conjecture together with the work of [35] would imply that there is no torsion before the phase transition:

Conjecture 5.1. *For $c < c_d$ and $Y \sim Y_d(n,c/n)$ the expected rank of $H_{d-1}(Y)_T = o(n^d)$.*

The proof of the main result of [35] is obtained by first using spectral techniques to show that with high probability $\beta_d(Y;\mathbb{R}) = o(n^d)$ where $Y \sim Y_d(n, c/n)$ and $c < c_d$. From there, [35] establish that the minimal cores of Y are all boundaries of $(d+1)$-simplicies using a structural result from [4] about cores in $Y_d(n, c/n)$. By the universal coefficient theorem and the main result of [35], Conjecture 5.1 would establish that $\beta_d(Y, \mathbb{Z}/q\mathbb{Z}) = o(n^d)$ simultaneously for every prime q. From here, one could use the exact same argument from the structural result on cores from [4] as applied in [35] to show that $H_d(Y; \mathbb{Z}/q\mathbb{Z})$ is generated by $(d+1)$-simplex boundaries simultaneously over all primes q. This would imply that there is no torsion in the $(d-1)$st homology group before the phase transition.

Moreover, the conjectured Cohen–Lenstra heuristics discussed in Chapter 3 are much stronger than Conjecture 5.1. Indeed if we fix the prime q and consider the distribution on q-groups where the probability of every q-group G is inversely proportional to $|\operatorname{Aut}(G)|$, then with high probability a q-group sampled according to that distribution rank less than $f(n)$ for any function $f(n)$ that tends to infinity arbitrarily slowly with n.

5.2 Toward the integer homology threshold

The very first paper on the Linial–Meshulam model [32] established that $2\log n/n$ is the sharp threshold for the first homology group of $Y \sim Y_2(n, p)$ with $\mathbb{Z}/2\mathbb{Z}$ coefficients to vanish. This paper was shortly followed by [40] which extends the result to higher dimensions and to any fixed finite ring R in place of $\mathbb{Z}/2\mathbb{Z}$. However, establishing a sharp threshold for codimension-1 homology with integer coefficients to vanish in $Y_d(n, p)$ remains open for all $d \geq 3$; the $d = 2$ case is proved in [37].

Now, by the universal coefficient theorem, and the result of [40] we know that for $c > d$ and $Y \sim Y_d(n, c\log n/n)$, $H_{d-1}(Y)$ is a finite group. Indeed, $H_{d-1}(Y, \mathbb{Z}/2\mathbb{Z}) = 0$ in this regime so by the universal coefficient theorem $H_{d-1}(Y)$ has no free part (and no 2-torsion). However, since [40] relies on the ring of coefficients being fixed, we cannot rule out the possibility that $H_{d-1}(Y)$ is finite group whose size grows with n. On the other hand Conjecture 2.3 would imply that $H_{d-1}(Y)$ is torsion free for $Y \sim Y_d(n, f(n))$ for any $f(n)$ that is $\omega(1/n)$, in particular for $f(n) = c \log n/n$ and $c > d$. Therefore Conjecture 2.3 plus the result of [40] would establish $d \log n/n$ as the sharp threshold for integer homology to vanish in $Y_d(n, p)$.

5.3 The fundamental group

The fundamental group of $Y_2(n, p)$ has been studied by for example [5, 10, 12, 13, 23]. Within the context of the torsion burst and generalizing the Erdős–Rényi phase transition, in [43] I prove the following result about the fundamental group of $Y \sim Y_2(n, c/n)$ where γ_2 is the constant from the collapsibility threshold of [2, 4] and c_2 is the constant from the homology threshold of [3, 35].

Theorem 5.2. *If $c < \gamma_2$ and $Y \sim Y_2(n, c/n)$, then with high probability $\pi_1(Y)$ is a free group and if $c > c_2$ and $Y \sim Y_2(n, c/n)$ then with high probability $\pi_1(Y)$ is not a free group.*

The lower bound of γ_2 follows by strengthening the 2-dimensional case of the main result of [4]. In [4], Linial and his collaborators show that if $c < \gamma_2$ and $Y \sim Y_2(n, c/n)$, then the probability that Y is 2-collapsible given that it does not contain a tetrahedron boundary tends to 1. This is strengthened in [43] to say that even if $Y \sim Y_2(n, c/n)$ contains tetrahedron boundaries, with high probability it is 2-collapsible after deleting a face from each of its tetrahedron boundaries.

The upper bound comes from combining the main result of [3, 35] that $c > c_2$ implies that $Y \sim Y_2(n, c/n)$ has $\beta_2(Y) = \Theta(n^2)$ with the a group cohomology result of Costa and Farber [13]. In [13], Costa and Farber show that for $p = o(n^{-46/47})$,

$Y \sim Y_2(n,p)$ is *asphericable*, that is, the complex obtained from Y by deleting on face from every tetrahedron boundary contained in Y is an aspherical simplicial complex.

An aspherical topological space is a space X with $\pi_i(X) = 0$ for $i \geq 2$. Covering all the background on homotopy groups is outside the scope of this thesis, but the relevant fact in the present context is that a 2-dimensional aspherical simplicial complex with nontrivial second homology group has a non-free fundamental group. In fact the fundamental group of such a complex has cohomological dimension equal to two, which is much stronger than just saying that it is not a free group. We refer the reader to [8] for more background on group cohomology.

Once they have shown that $Y \sim Y_2(n, c/n)$ is asphericable when $p = o(n^{-46/47})$, Costa and Farber show that for $c > 3$ and $Y \sim Y_2(n, c/n)$ has fundamental group with cohomological dimension two since removing a face from each tetrahedron does not affect the fundamental group and the resulting complex is aspherical with positive second Betti number. By using the result of [3, 35], this 3 can be reduced to c_2.

The behavior of the fundamental group for $Y \sim Y_2(n, c/n)$ and $c \in [\gamma_2, c_2]$, remains open. If $\pi_1(Y)$ remains free all the way up to c_2/n, then we have that after the removal of one face from every tetrahedron the resulting complex is homotopy equivalent to a graph. This follows directly from the result of [13][1]. However, it is known due to [2] that for $c > \gamma_2$ with high probability $Y \sim Y_2(n, c/n)$ is not 2-collapsible to a graph even after the removal of one face from every tetrahedron. Thus we have the question of what is the threshold (if it even exists) for the property that $Y \sim Y_2(n,p)$ is not homotopy equivalent to a graph. This particular question can be generalized to higher dimensions:

[1] In this regime Y with one face deleted from every tetrahedron boundary is a $K(\pi, 1)$, that is a space with a contractible universal cover, which are known to be unique up to homotopy type (see for example Theorem 1B.8 of [22]) if $\pi_(Y)$ is free in this case then Y is homotopy equivalent to a graph, as a bouquet of m circles is a $K(\pi, 1)$ for the free group on m generators.

Question 5.3. *Is there a sharp threshold for the property that $Y \sim Y_d(n,p)$ is not homotopy equivalent to a $(d-1)$-dimensional simplicial complex after removal of one face from every embedded $(d+1)$-simplex boundary. What is the threshold if it exists?*

We can also ask the following stronger question which is equivalent when $d = 2$, but not in general.

Question 5.4. *Is there a sharp threshold for the property that $Y \sim Y_d(n,p)$ is not homotopy equivalent to a bouquet of $(d-1)$-dimensional spheres after removal of one face from every embedded $(d+1)$-simplex boundary? What is the threshold if it exists?*

5.4 A random abelian group model

When conducting experiments for [27] which are discussed in Chapter 3, we wanted a way to see what happens in higher dimensions. While we were able to generate complexes on up to 270 vertices in 2 dimensions, we could only practically go to 50 vertices in 3 dimensions and even smaller in 4 dimensions and higher. The problem is that as d increases, but n stays the same the number of top-dimensional faces grows exponentially. To work around this issue I considered something of a abstracted version of Linial–Meshulam complexes. Ultimately, this model appears to be quite similar in many ways to $Y_d(n,p)$ and in this section we discuss some experiments in this model and some results which are analogous to results in the Linial–Meshulam model.

5.4.1 Setup and connection to Linial–Meshulam model

Suppose that X is a simplicial complex of dimension d and we want to write down its dth boundary matrix $\partial_d(X)$. First, assign an ordering to the vertices of X and then let that ordering induce an orientation and the lexicographic ordering on higher dimensional faces. Then if $\sigma = [v_0, v_1, ..., v_d]$ for $v_0 < v_1 < \cdots < v_d$ in the vertex ordering, then the boundary of $[v_0, v_1, ..., v_d]$ is given by

$$\partial_d([v_0, v_1, ..., v_d]) = \sum_{i=0}^{d} (-1)^i [v_0, v_1, ..., \hat{v}_i, ..., v_d].$$

Moreover with the lexicographic ordering induced on $(d-1)$-dimensional faces we have

$$[v_1, ..., v_d] > [v_0, v_2, v_3, ..., v_d] > [v_0, v_1, v_3, ..., v_d] > \cdots > [v_0, v_1, ..., v_{d-1}]$$

Thus the boundary matrix $\partial_d(X)$ with respect to the given orientation and the lexicographic ordering is a matrix in which each column has exactly $d+1$ nonzero entries and furthermore from top to bottom the nonzero entries in any column alternate between 1 and -1.

With this as the motivation, we define the *pseudosimplicial random abelian group model*. For $d \geq 1$, $n \in \mathbb{N}$, and $p = p(n) \in [0, 1]$, let $A_d(n, p)$ denote the probability space on abelian groups where we sample by starting with an ordered generating set $a_1 < a_2 < \cdots < a_n$ and include each relator of the form, $a_{i_0} - a_{i_1} + a_{i_2} - \cdots + (-1)^d a_{i_d}$ where $i_0 < i_1 < \cdots i_d$, independently with probability p.

This model of random abelian group is apparently new in this context, however a $\mathbb{Z}/2\mathbb{Z}$-version of it has been studied extensively as the random k-Xorsat model. In random k-Xorsat, one generates a random matrix with n columns and m rows where each row is chosen uniformly at random from all row vectors in $(\mathbb{Z}/2\mathbb{Z})^n$ with exactly k 1's. Moreover, as Linial and Peled point out in [35], several thresholds from random d-dimensional complexes coincide with analogous thresholds in random $(d+1)$-Xorsat. For example, [3,35] show that $p = c_d/n$ is the threshold for top-dimensional homology to emerge in $Y_d(n, p)$, that is if the number of faces is larger than $\frac{c_d}{n}\binom{n}{d+1} = \frac{c_d}{d+1}\binom{n-1}{d}$ then with high probability there is nontrivial top homology. Thus if the density of d-dimensional faces to $(d-1)$-dimensional faces exceeds $\frac{c_d}{d+1}$ then with high probability there is nontrivial top homology. Analogously, [14, 49] studies the satisfiability threshold for $(d+1)$-Xorsat with m rows and n columns is at $m/n = \frac{c_d}{d+1}$, that is if A is the $m \times n$ matrix generated in the random $(d+1)$-Xorsat model and b is a uniform random vector in $(\mathbb{Z}/2\mathbb{Z})^m$, then with high probability $Ax = b$ has a solution if $m/n > \frac{c_d}{d+1}$ and does not have a solution if $m/n < \frac{c_d}{d+1}$. Moreover, the model $A_d(n, p)$ has an natural underlying $(d+1)$-uniform hypergraph on n vertices and every edge included independently with probability p. Within a $(d+1)$-uniform hypergraph we can say that a vertex belonging to exactly one hyperedge is free and can define an elementary collapse to be the process of deleting a free vertex and the unique hyperedge that contains it. It is known due to [41] that $\frac{\gamma_d}{d+1}$ is the density threshold for a random hypergraph to be collapsible to isolated vertices in this sense. This matches exactly the density threshold for d-collapsibility in $Y_d(n, p)$ due to [2, 4].

5.4.2 Experimental data

Given the similarities between random k-Xorsat, random hypergraphs, and random simplicial complexes outlined above, it is perhaps not surprising that experimental evidence points to many connections between $Y_d(n, p)$ and $A_d(n, p)$. Indeed both models exhibit a torsion burst. In the discrete-time stochastic process version of $A_d(n, p)$ there is a window near the place where the ratio of relators to generators passes $c_d/(d+1)$ where there are interesting torsion groups in the group. Just as in $Y_d(n, p)$, this torsion appear to be short-lived, huge, and Cohen–Lenstra distributed.

5.4.3 Early results

One reason for introducing this model was to try to prove results analogous to results we would like to prove in the random simplicial complex model. Ideally, proving something in $A_d(n,p)$ will be easier than proving its analogue and will also inspire new techniques to study $Y_d(n,p)$. This seems to be the case for working on proving Conjecture 2.3. The analogue for Conjecture 2.3 in $A_d(n,p)$ is the following.

Conjecture 5.5. *For every $d \geq 2$ and $p = p(n)$ such that $|n^d p - d!c_d|$ is bounded away from 0, $A_d(n,p)$ is torsion-free with high probability.*

This conjecture is formulated with the goal in mind of having the ratio of relators to generators being different from $c_d/(d+1)$, which is the critical ratio of d-dimensional faces to $(d-1)$-dimensional faces where we see the torsion burst in $Y_d(n,p)$. In $A_d(n,p)$ if $p = \dfrac{d!c_d}{n^d}$, then the expected number of relators is

$$\begin{aligned} \frac{d!c_d}{n^d}\binom{n}{d+1} &\approx \frac{d!c_d}{n^d}\frac{n^{d+1}}{(d+1)!} \\ &= \frac{c_d}{d+1}n. \end{aligned}$$

As n is always the number of generators, this value of p corresponds to the critical density where we see torsion in our experiments. Regarding this conjecture the following result can be proved in $A_d(n,p)$. The proof, which comes from a collaboration with Elliot Paquette, is omitted here. While the analogous result is not yet known in $Y_d(n,p)$, the hope is to use similar methods developed in the setting of $A_d(n,p)$ to prove such a result.

Theorem 5.6. *For every $d \geq 1$, if $p = \omega(1/n^d)$ then with high probability $A_d(n,p)$ is torsion-free.*

Moreover with this result, one can also show that $p = d!d\log n/n^d$ is the threshold for $A_d(n,p)$ to be trivial[2] proving the analogue for $p = d\log n/n$ to be the threshold

[2] Note that by the setup of $A_d(n,p)$, when d is odd $A_d(n,1) = \mathbb{Z}$, thus in the case that d is odd we actually have that this is the threshold for the group to become isomorphic to $\mathbb{Z}$ rather than literally the trivial group.

for integer homology in $Y_d(n, p)$ to vanish. While it seems like more techniques will need to be developed to prove the analogue of Theorem 5.6 in $Y_d(n, p)$, it does appear that the following conjecture can be proved by adapting the methods used to prove Theorem 5.6.

Conjecture 5.7. *For $d \geq 2$ and $c > d - 1/2$, $H_{d-1}(Y)$ is torsion-free with high probability for $Y \sim Y_d(n, c \log n/n)$.*

The $1/2$ in the statement comes from the techniques we are attempting to use to prove it, but as this is work in progress I omit the details here. This conjecture together with the result of Meshulam and Wallach [40] implies that $d \log n/n$ is indeed the threshold for $H_{d-1}(Y)$ with integer coefficients to vanish for $Y \sim Y(n, p)$.

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