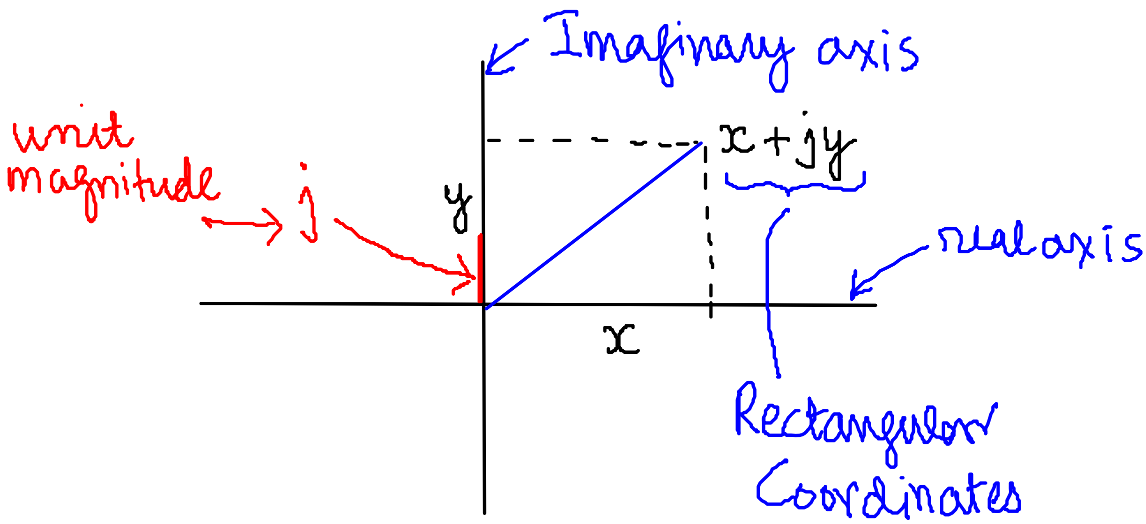
Complex Numbers

$$z = \underbrace{x}_{\text{real part}} + j\underbrace{y}_{\text{Imaginary part}}$$

$\Re e\, z = x$ real $\Im m\, z = y$ real

$j^2 = -1$ $j = \sqrt{-1}$



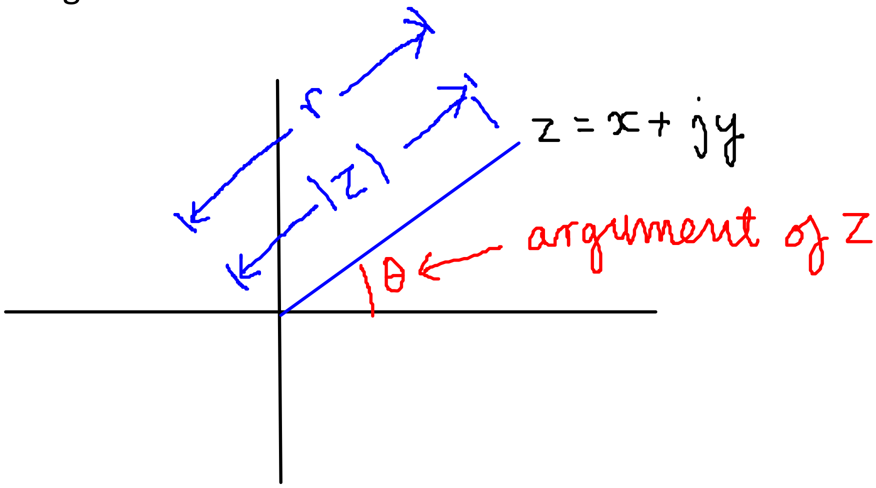
Rectangular coordinates good for addition and subtraction

$z_1 = 3 + j4$ $z_2 = -3 + 2j$

purely imaginary

$z_1 + z_2 = (3-3) + j(4+2) = 0 + 6j = 6j$

$z_1 - z_2 = (3+3) + j(4-2) = 6 + 2j$



magnitude of $z \rightarrow |z| = \sqrt{x^2 + y^2} = r$

argument of $z \rightarrow \angle\theta = \tan^{-1} \frac{y}{x}$ radians

$z = r\angle\theta$ polar coordinates

Polar coordinates are good for
multiplication and division

$x = 10\angle\frac{\pi}{8}$ $\quad y = 5\angle\frac{\pi}{6}$

$xy = 10 \times 5 \angle\frac{\pi}{8} + \frac{\pi}{6} = 50 \angle\frac{7\pi}{24}$

$\frac{x}{y} = \frac{10}{5} \angle\frac{\pi}{8} - \frac{\pi}{6} = 2 \angle -\frac{\pi}{24}$

You can also multiply directly in rect. coord.

$(2-3j)(1+j) = 2 + 2j - 3j - \underbrace{3j^2}_{j^2 = -1} = (2+3) - j = 5 - j$

$$Z = Z_1 Z_2$$

$$|Z| = |Z_1||Z_2|$$

$$\angle Z = \angle Z_1 + \angle Z_2$$

$$Z = \frac{Z_1}{Z_2}$$

$$|Z| = \frac{|Z_1|}{|Z_2|}$$

$$\angle Z = \frac{\angle Z_1}{\angle Z_2}$$

$$z_1 = x_1 + jy_1 = r_1 \angle \theta_1$$

$$z_2 = x_2 + jy_2 = r_2 \angle \theta_2$$

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$$

$$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$$

$$z_1 z_2 = r_1 r_2 \angle \theta_1 + \theta_2$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \angle \theta_1 - \theta_2$$

Complex Conjugate

$$j^2 = -1$$

$$z = x + jy \qquad zz^* = x^2 - j^2 y^2 = x^2 + y^2$$

$$z^* = x - jy \qquad = |z|^2$$

complex conjugate of z

$$\mathrm{Re}\, z = \frac{1}{2}(z + z^*) \qquad \frac{1}{2}(x + \cancel{jy} + x - \cancel{jy}) = x$$

$$\mathrm{Im}\, z = \frac{1}{2j}(z - z^*) \qquad \frac{1}{2j}(x + jy - x + jy) = y$$

Polar to rectangular coordinates

$$r\angle\theta \rightarrow x + jy$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

Rectangular to polar coordinates

$$x + jy \rightarrow r\angle\theta$$

$$r = \sqrt{x^2 + y^2} \qquad r > 0$$

1st quad 4th quad

$$\theta = \begin{cases} \tan^{-1}\frac{y}{x} & \text{if } (x \geq 0 \;\&\; y > 0) \text{ or } (x > 0 \;\&\; y \leq 0) \\ \tan^{-1}\frac{y}{x} - \pi & \text{if } (x \leq 0 \;\&\; y < 0) \leftarrow \text{3rd quad} \\ \tan^{-1}\frac{y}{x} + \pi & \text{if } (x < 0 \;\&\; y \geq 0) \leftarrow \text{2nd quad} \end{cases}$$

$$\frac{x_1 + jy_1}{x_2 + jy_2} = \frac{r_1 \angle \theta_1}{r_2 \angle \theta_2} = \frac{r_1}{r_2} \angle \theta_1 - \theta_2$$

$$\frac{x_1 + jy_1}{x_2 + jy_2} = \frac{(x_1 + jy_1)(x_2 - jy_2)}{\underbrace{(x_2 + jy_2)}_{z_2}\underbrace{(x_2 - jy_2)}_{z_2^*}}$$

$$= \frac{x_1x_2 - jx_1y_2 + jy_1x_2 - j^2y_1y_2}{x_2^2 + y_2^2}$$

$$= \frac{(x_1x_2 + y_1y_2) + j(y_1x_2 - x_1y_2)}{x_2^2 + y_2^2}$$

$$= \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + j\,\frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2}$$

$$\frac{1}{2-3j}\,\frac{1}{1+j} = \frac{1}{2+2j-3j-3j^2}$$

$$= \frac{1}{5-j}$$

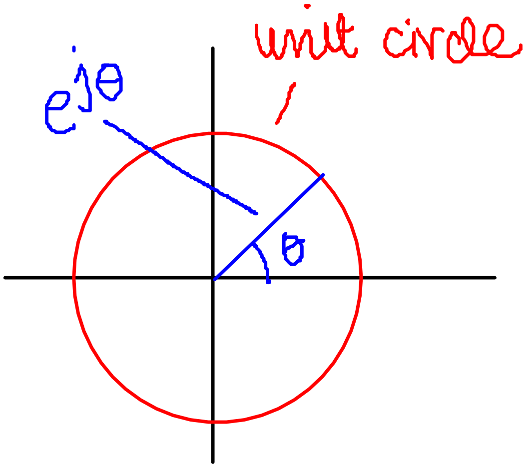
$$= \frac{1}{5-j}\,\frac{5+j}{5+j}$$

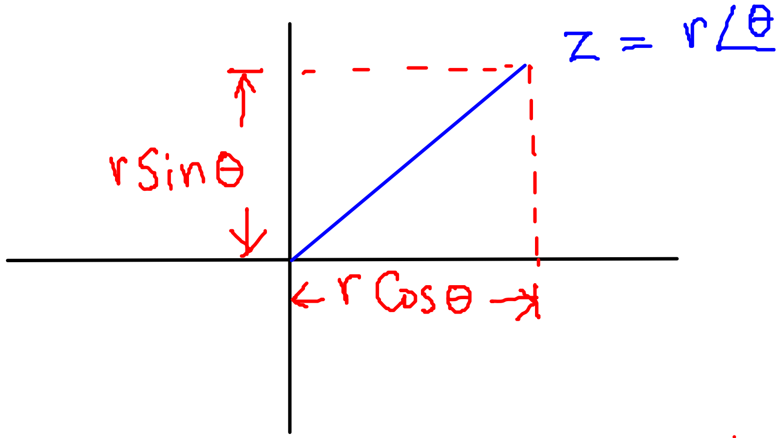
$$= \frac{5+j}{5^2+1^2} = \frac{5+j}{26}$$

$$= \frac{5}{26} + j\frac{1}{26}$$

$$e^{j\theta} = \cos\theta + j\sin\theta \qquad |e^{j\theta}| = 1$$

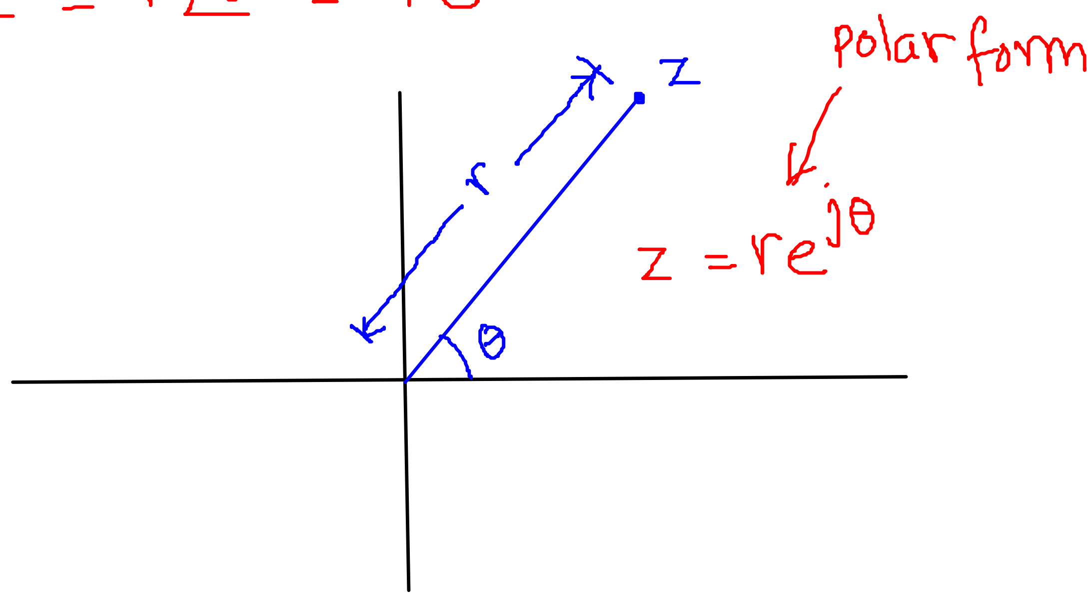
← complex exponential





$$z = r\cos\theta + jr\sin\theta = r(\cos\theta + j\sin\theta) = re^{j\theta}$$

$$z = r\angle\theta = re^{j\theta}$$



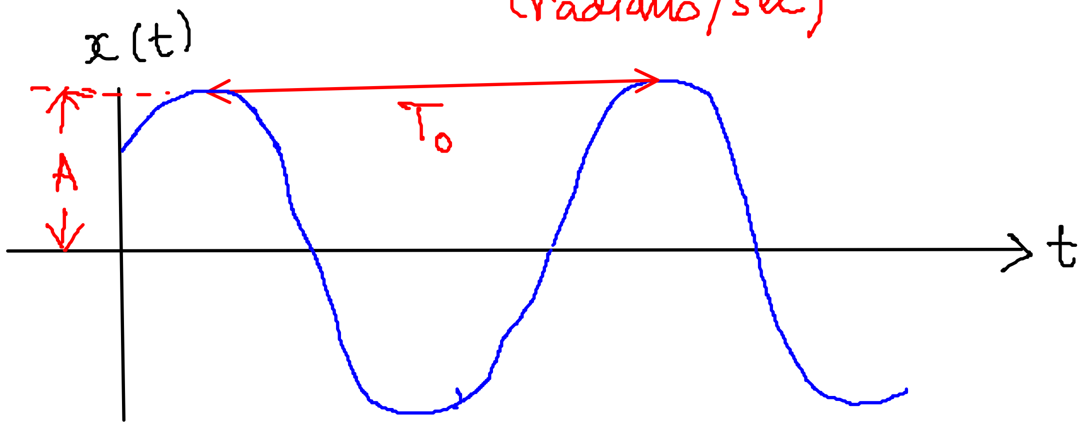
Sinusoids

$$x(t) = A\cos(\omega_0 t + \phi)$$

Amplitude $A > 0$

radial frequency (radians/sec)

phase shift (radians)



$$T_0 = \frac{2\pi}{\omega_0} = \frac{1}{f_0}$$

period

radial frequency units radians/sec

cyclic frequency

units Hz cycles/sec

Converting sin to cos

$$\sin\theta = \cos\left(\theta - \frac{\pi}{2}\right)$$

Converting cos to sin

$$\cos\theta = \sin\left(\theta + \frac{\pi}{2}\right)$$

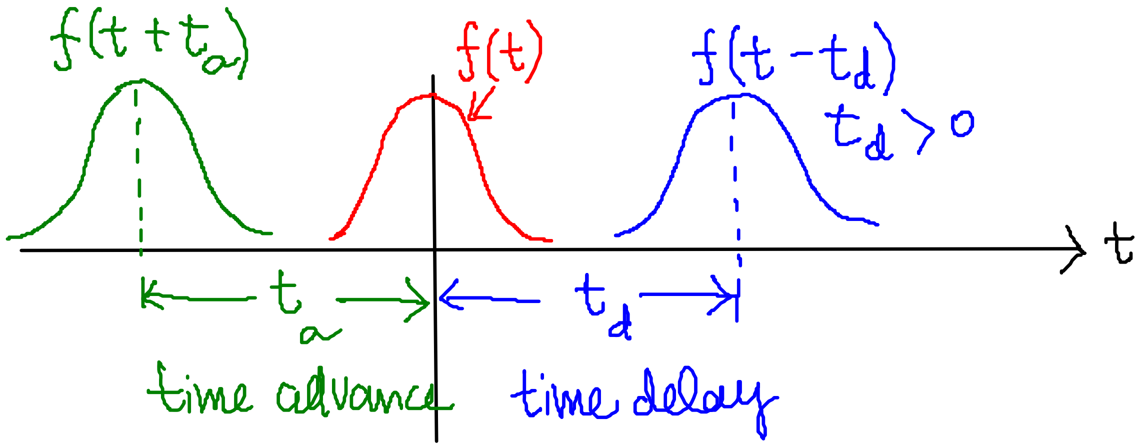
We will measure phase shift with respect to a cosine

$$\cos(\omega_0 t + \underset{\substack{\uparrow \\ \text{phase shift}}}{\phi})$$

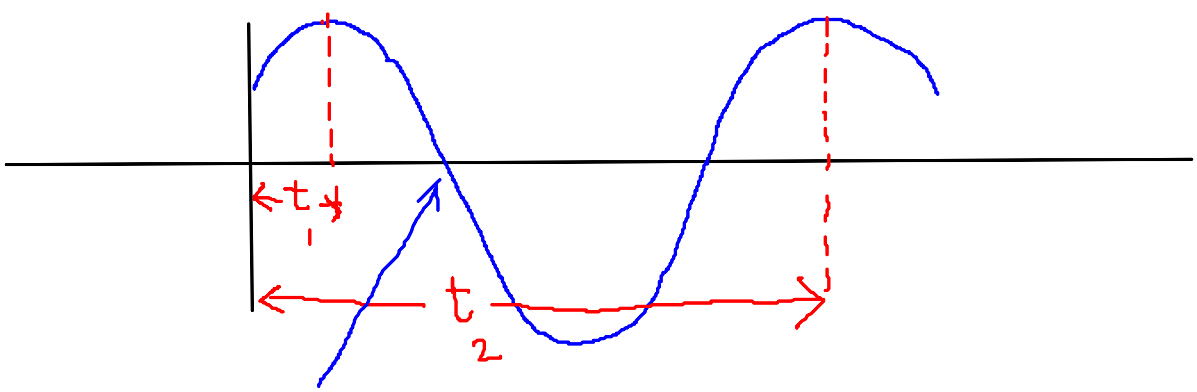
To find phase shift of a sine convert to cosine

$$\sin(\omega_0 t + \phi) = \cos(\omega_0 t + \underbrace{\phi - \frac{\pi}{2}})$$

$$\text{phase shift} = \phi - \frac{\pi}{2}$$



Determining phase shift from time delay or advance



$$\cos(\omega_0(t-t_1)) = \cos(\omega_0 t \underbrace{-\omega_0 t_1}_{\phi_1}) \qquad \phi_1 = -\omega_0 t_1$$

$$\cos(\omega_0(t-t_2)) = \cos(\omega_0 t \underbrace{-\omega_0 t_2}_{\phi_2}) \qquad \phi_2 = -\omega_0 t_2$$

Choosing different maxima give us phase shifts that differ by multiples of 2π

$$\phi_i = \phi_j \pm 2\pi n$$

Find any ϕ_j then add or subtract multiples of 2π so that ϕ_i satisfies

$$-\pi < \phi_i \leq \pi$$

principal value of the phase shift

2-5 Complex exponentials & Phasors

Complex exponential signal

$$z(t) = A e^{j(\omega_0 t + \phi)}$$

amplitude $A = |z(t)| = |A| \overbrace{|e^{j(\omega_0 t + \phi)}|}^{=1}, A > 0$

$$z(t) = \underbrace{A \cos(\omega_0 t + \phi)}_{\text{sinusoid}} + j A \sin(\omega_0 t + \phi)$$

$$A \cos(\omega_0 t + \phi) = \mathcal{Re}\, z(t)$$

$$z(t) = A e^{j(\omega_0 t + \phi)}$$

$$z(t) = \underbrace{A e^{j\phi}}_{X} e^{j\omega_0 t}$$

$X \rightarrow$ phasor, also known as complex amplitude

$$z(t) = X e^{j\omega_0 t}$$