Z - Transforms Chapter 7

$$X(z) = \sum_{k=-\infty}^{\infty} x[k]\, z^{-k}$$

z transform of $x[n]$

Discrete time signal

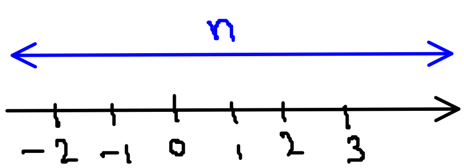
complex number

$$x[n] \longleftrightarrow X(z)$$

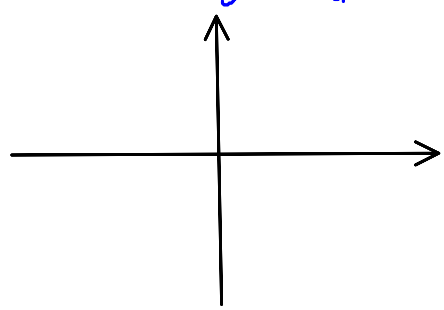
inverse z transform of $X(z)$

z transform of $x[n]$

n-Domain



Z-Domain



If $x[n]$ is causal i.e $x[n]=0$ for $n<0$

then $$X(z) = \sum_{k=0}^{\infty} x[k]\, z^{-k}$$

If $x[n]$ is causal and has finite-length $M+1$

$$X(z) = \sum_{k=0}^{M} x[k]\, z^{-k}$$

$$X(z) = \sum_{k=-\infty}^{\infty} x[k](z^{-1})^k$$

Examples

$$x[n] = \delta(n-n_0)$$

$$x[n] = \{\underset{0}{0}\ \ \underset{1}{0}\ \cdots\ 0\ \underset{n_0}{1}\ \ 0\ \ 0\ \cdots\}$$

$$X(z) = 0\times(z^{-1})^0 + 0\times(z^{-1})^1 + \cdots + 1\times(z^{-1})^{n_0} + 0\cdots$$

$$X(z) = z^{-n_0}$$

$$\boxed{\delta(n-n_0) \longleftrightarrow z^{-n_0}}$$

if $n_0 > 0$ then $z \neq 0$

$$x[n] = 2\delta(n) + 4\delta(n-1) + 6\delta(n-2) + 4\delta(n-3) + 2\delta(n-4)$$

$$x[n] = 0\ \ 0\ \ \underset{\uparrow 0}{2}\ \ \underset{1}{4}\ \ \underset{2}{6}\ \ \underset{3}{4}\ \ \underset{4}{2}\ \ \underset{5}{0}\ \ 0\ \ 0$$

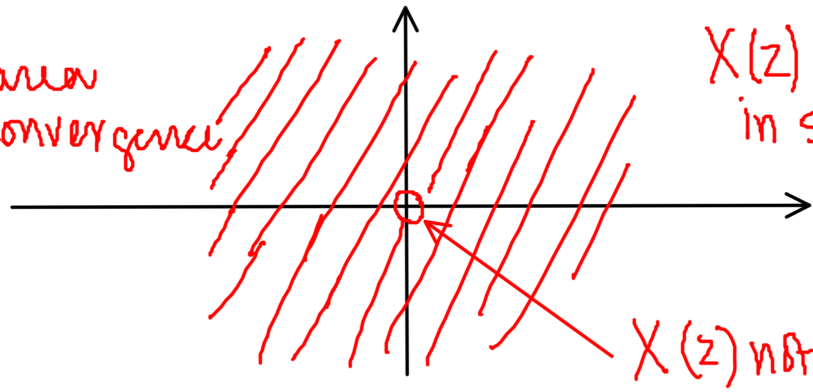
$$X(z) = 2(z^{-1})^0 + 4(z^{-1})^1 + 6(z^{-1})^2 + 4(z^{-1})^3 + 2(z^{-1})^4$$

$$= 2 + 4z^{-1} + 6z^{-2} + 4z^{-3} + 2z^{-4}$$

$$= 2 + \frac{4}{z} + \frac{6}{z^2} + \frac{4}{z^3} + \frac{2}{z^4} \qquad z \neq 0$$

$$X(z) = 2 + \frac{4}{z} + \frac{6}{z^2} + \frac{4}{z^3} + \frac{2}{z^4} \qquad z \neq 0$$

shaded area
Region of Convergence
ROC



X(z) defined in shaded area

X(z) not defined at z = 0

Inverse z transform

$$X(z) = 1 - 2z^{-1} + 3z^{-3} - z^{-5}$$

$$x[n] = \delta[n] - 2\delta[n-1] + 3\delta[n-3] - \delta[n-5]$$

z transform of infinite-length discrete signal is more tricky

example $x[n] = \left(\frac{1}{2}\right)^n u[n]$

$$x[n] = \left\{ 1, \frac{1}{2}, \left(\frac{1}{2}\right)^2, \left(\frac{1}{2}\right)^3, \cdots \right\}$$

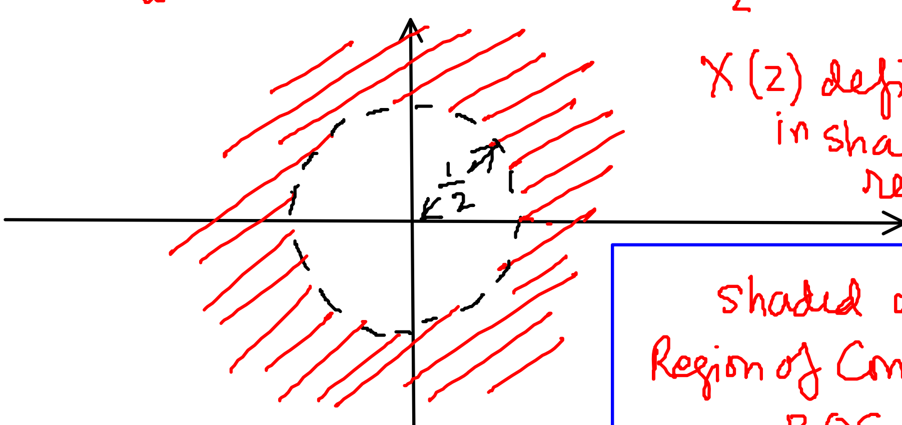
$$X(z) = 1 + \frac{1}{2} z^{-1} + \left(\frac{1}{2}\right)^2 z^{-2} + \left(\frac{1}{2}\right)^3 z^{-3} + \cdots$$

$$= 1 + \left(\frac{1}{2} z^{-1}\right)^1 + \left(\frac{1}{2} z^{-1}\right)^2 + \left(\frac{1}{2} z^{-1}\right)^3 + \left(\frac{1}{2} z^{-1}\right)^4 + \cdots$$

infinite geometric series

$$1 + A + A^2 + A^3 + \cdots = \frac{1}{1-A} \quad |A| < 1$$

$$X(z) = \frac{1}{1 - \left(\frac{1}{2} z^{-1}\right)} \quad \left|\frac{1}{2} z^{-1}\right| < 1 \quad = \frac{z}{z - \frac{1}{2}} \quad |z| > \frac{1}{2}$$



X(z) defined in shaded region

shaded area
Region of Convergence
ROC

$$\boxed{u[n] \longleftrightarrow \frac{1}{1 - z^{-1}} \quad \underset{\text{ROC}}{|z| > 1}}$$

Example

$$x[n] = \alpha^n u[n] = \begin{cases} \alpha^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

$$X(z) = \sum_{n=0}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^n$$

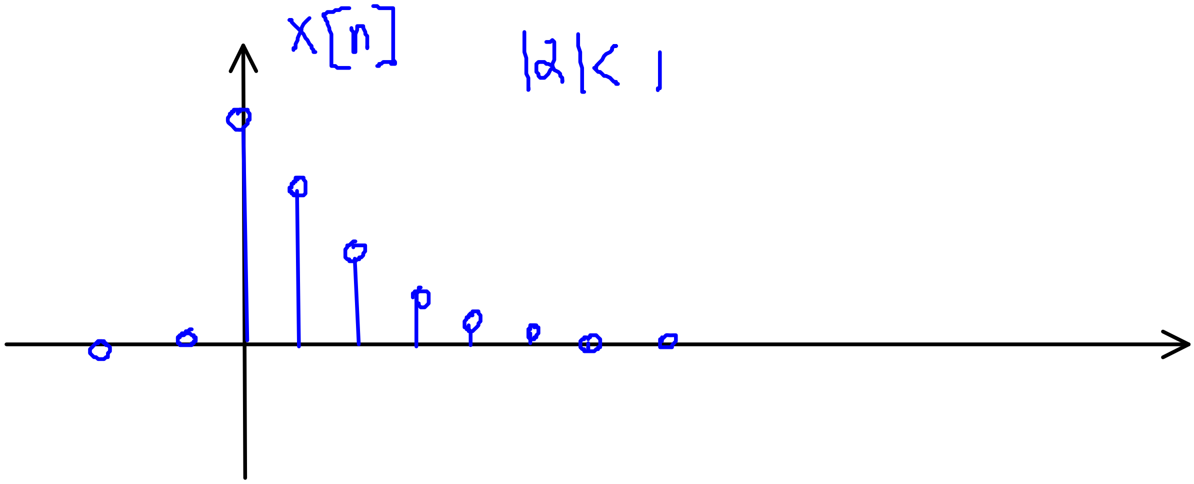
infinite geometric series

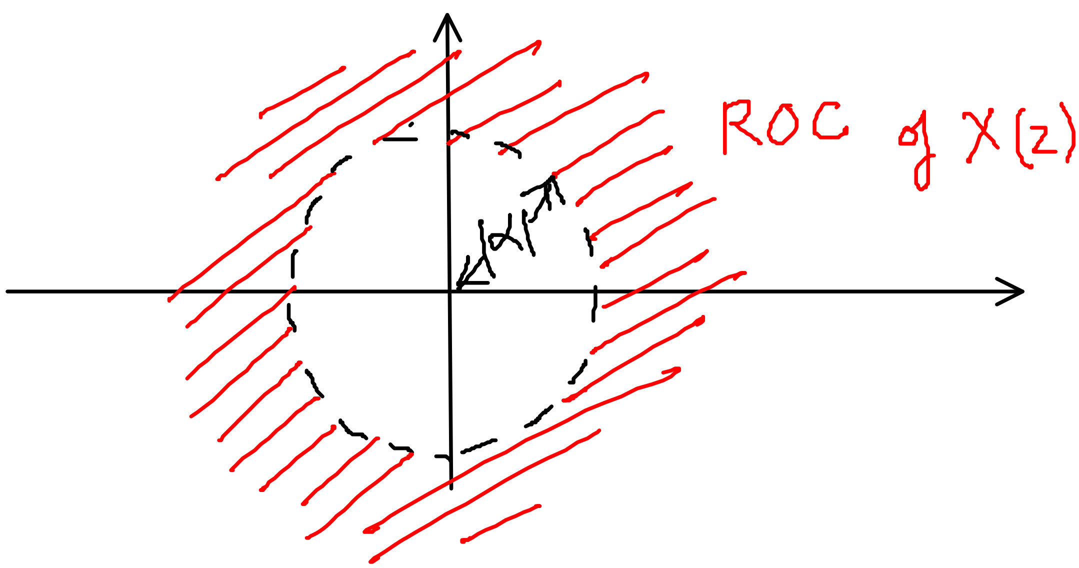
$$1 + A + A^2 + A^3 + \cdots = \frac{1}{1-A} \quad |A| < 1$$

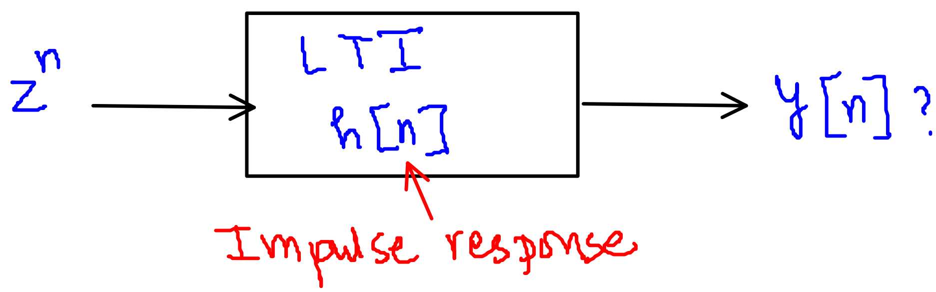
$$X(z) = \frac{1}{1-(\alpha z^{-1})} \quad |\alpha z^{-1}| < 1$$

$$= \frac{1}{1-\alpha z^{-1}} \quad |z| > |\alpha|$$

$$\boxed{\alpha^n u[n] \longleftrightarrow \frac{1}{1-\alpha z^{-1}} \qquad \underset{\text{ROC}}{|z| > |\alpha|}}$$





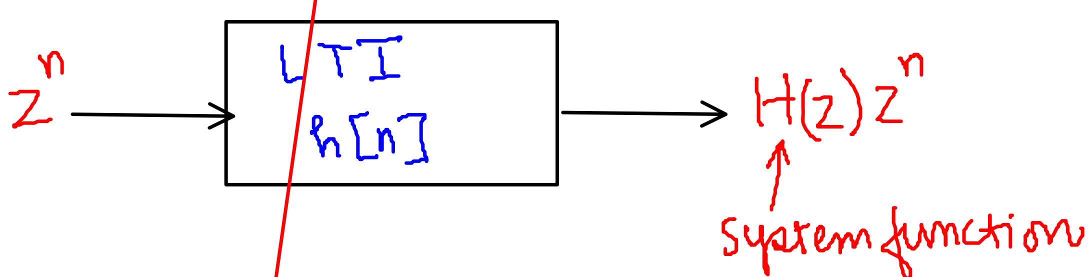


$$y[n] = h[n] * z^n$$

$$= \sum_{k=0}^{\infty} h[k]\, z^{(n-k)}$$

$h[n] = 0$ for $n < 0$
Causal system

$$y[n] = \underbrace{\left(\sum_{k=0}^{\infty} h[k]\, z^{-k} \right)}_{H(z)} z^n$$



$$\boxed{h[n] \longleftrightarrow H(z)}$$

recap

$$\delta[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow h[n]$$

impulse response

$$z^n \longrightarrow \boxed{\text{LTI}} \longrightarrow H(z)\, z^n$$

system function

$$\boxed{h[n] \longleftrightarrow H(z)}$$

$$x[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n]$$

An LTI system can be fully described by a:

Difference equation:

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$

or

Impulse response

$$h[n]$$

n-domain

or

System Function

$$H(z)$$

z-domain