Rational functions

$$X(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

$$= \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Examples

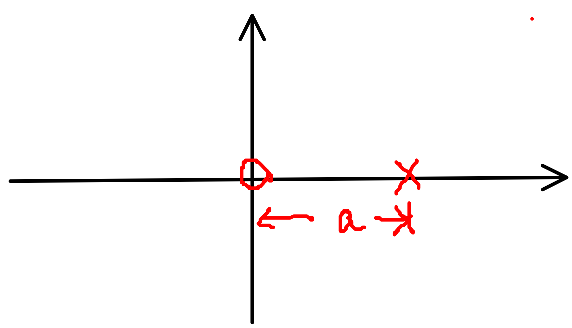
$$X(z) = \frac{1}{1 - a z^{-1}}$$

poles: $1 - a z^{-1} = 0$ $\quad a z^{-1} = 1 \quad z = a$ $\quad$ one pole at $z = a$

$z = 0$: $z^{-1} \to \infty$ $\quad X(z) = \frac{1}{\infty} \to 0$

one zero at $z = 0$

$z = \infty$: $z^{-1} = 0$ $\quad X(z) = 1$ (no pole or zero at ∞)



# of poles = # of zeros if you count poles & zeros at 0 and ∞ also

Example

$$\frac{2z^{-1}}{1-4z^{-1}+4z^{-2}}$$

$$=\frac{2z^{-1}}{(1-2z^{-1})^2}$$

$(1-2z^{-1})=0 \Rightarrow 2z^{-1}=1 \Rightarrow z=2$

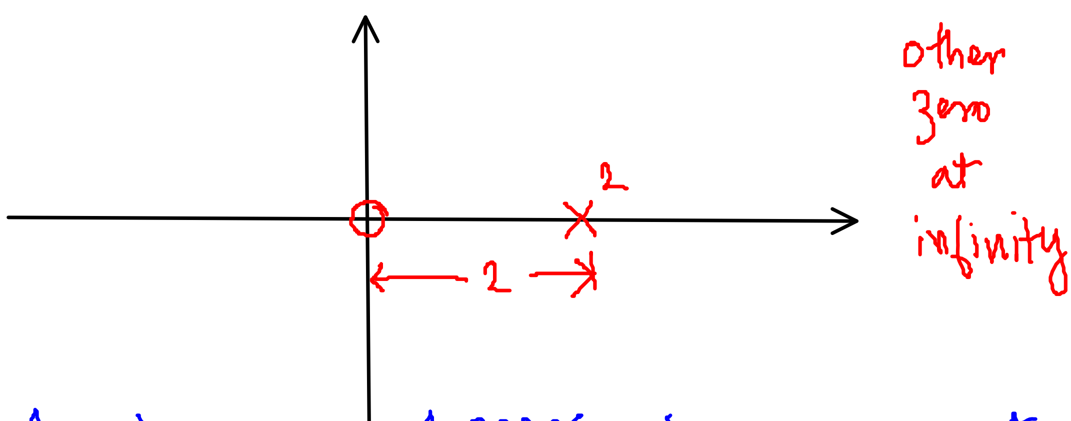
double pole at $z=2$

$z=0$? $z^{-1}\to\infty$ $X(z)\to\frac{\infty}{\infty^2}\to\frac{1}{\infty}=0$

one zero at $z=0$

$z=\infty$? $z^{-1}\to 0$ $X(z)=\frac{0}{\text{constant}}=0$

one zero at $z=\infty$



# of poles = # of zeros if you count poles & zeros at 0 and ∞ also

$$x[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n]$$

BIBO (Bounded Input Bounded output)
Stability

A system is BIBO stable if
all its poles lie inside the unit circle

[If some poles lie on the unit circle
but none lie outside the unit circle
then the system is marginally stable]

General diff eq for a LTI system

$$y[n] = -\sum_{k=1}^{N} a_k\, y[n-k] + \sum_{k=0}^{M} b_k\, x[n-k]$$

$$\frac{Y(z)}{X(z)} = H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=0}^{N} a_k z^{-k}}$$

rational system function

$$\boxed{\text{If } a_k = 0 \text{ for } 1 \le k \le N \text{ then:}}$$

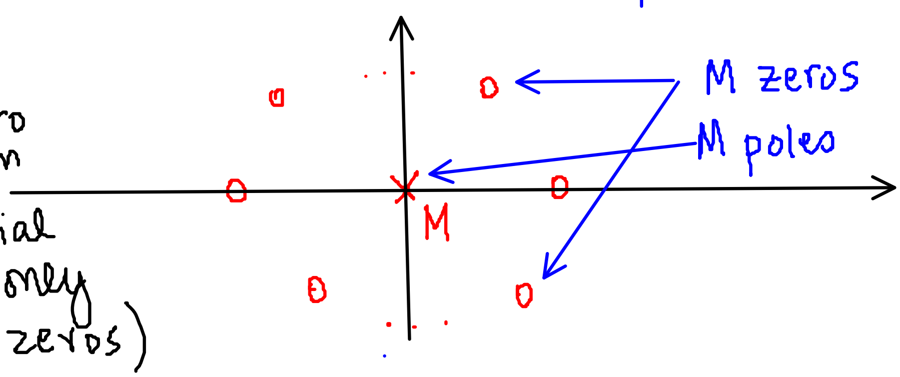
$$H(z) = \sum_{k=0}^{M} b_k z^{-k}$$

polynomial system function

FIR system

Mth order polynomial in z^{-1} → M nontrivial (zeros)

$z = 0$? $z^{-1} \to \infty$ $H(z) \to \infty^M$ M poles at $z = 0$



All zero system (all trivial poles only non-trivial zeros)

$$\boxed{b_k = 0 \text{ for } 1 \leq k \leq M}$$

IIR system

$$H(z) = \frac{b_0}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

purely recursive system

N non-trivial poles
N trivial zeros at 0 [Nth order zero at $z=0$]

all pole system

General form

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=0}^{N} a_k z^{-k}}$$

IIR system

Non trivial poles & zeros (N poles, M zeros)

pole-zero system

trivial poles & zeros are not counted

Example

$$y[n] = \frac{1}{2} y[n-1] + 2x[n]$$

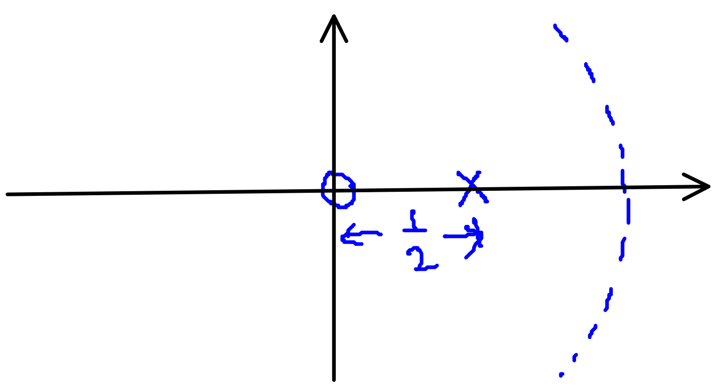
$$Y(z) = \frac{1}{2} z^{-1} Y(z) + 2X(z)$$

$$H(z) = \frac{2}{1 - \frac{1}{2} z^{-1}}$$

one non-trivial pole at $z = \frac{1}{2}$

at $z=0$: $z^{-1} \to \infty$ $\quad H(z) \to \frac{2}{\infty} \to 0$

one trivial zero at $z=0$



Stable

Example

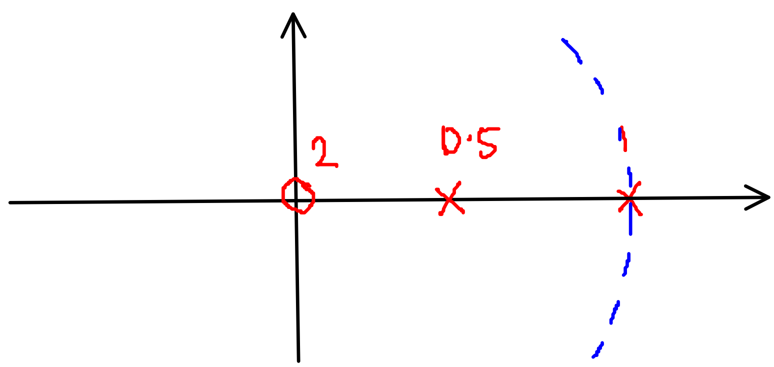
$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

$$X(z) = \frac{1}{(1 - z^{-1})(1 - 0.5z^{-1})}$$

Two non trivial poles $z = 1$ & $z = 0.5$

at $z = 0$, $z^{-1} \to \infty$ $X(z)$ $\frac{1}{\infty^2} \to 0$

two trivial zeros at $z = 0$



Marginally stable

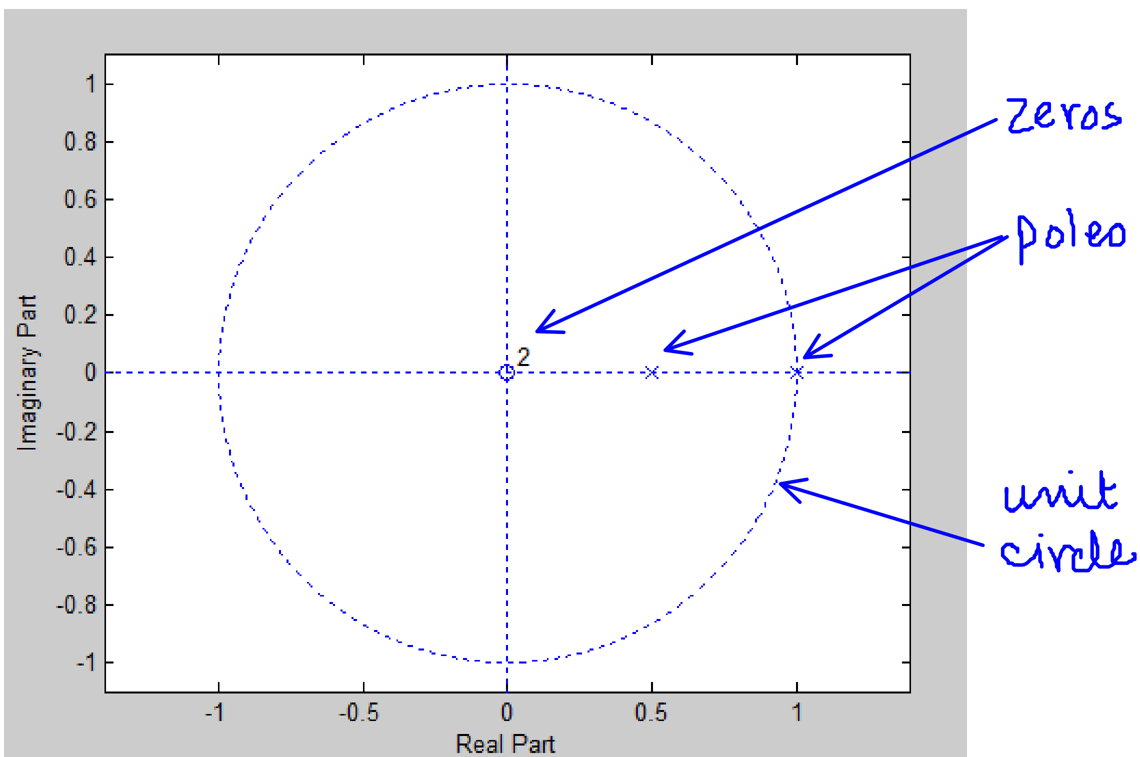
Poles & zeros using MATLAB

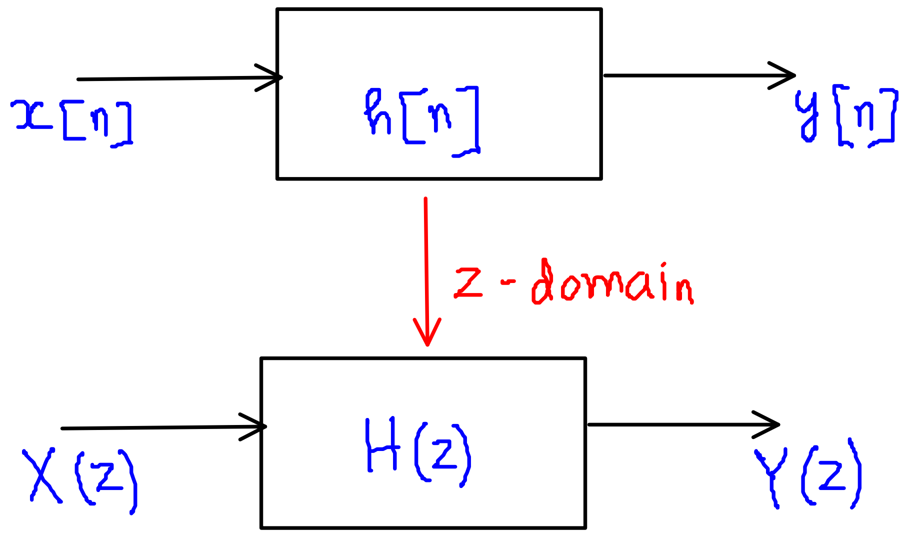
zplane(num, den)

Example

$$Y(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

```
num = [1 0 0];
den = [1 -1.5 0.5];
zplane (num, den);
```





$$Y(z) = H(z)X(z)$$

system poles & zeros

input poles & zeros

output contains poles and zeroes of H(z) and X(z)

Partial fraction expansion

$$Y(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

$a_N \neq 0$ and $M < N$ (proper fraction)

non-trivial zeros $z_1, z_2 \cdots z_M$

non-trivial poles $p_1, p_2 \cdots p_N$

$\boxed{\text{Determine } y[n]}$

a) simple poles (single order poles, non-repeated poles)

$p_1, p_2 \cdots p_N$ are all distinct

residues $A_1, A_2 \cdots A_N$

$$Y(z) = \underbrace{\frac{A_1}{1 - p_1 z^{-1}} + \frac{A_2}{1 - p_2 z^{-1}} + \cdots + \frac{A_N}{1 - p_N z^{-1}}}_{\text{partial fraction expansion}}$$

use MATLAB to find the residues

Input to MATLAB

$[r, p, k] = \text{residuez}[\text{num}, \text{den}]$

output from MATLAB

residues $A_1 \cdots A_N$ correspond to poles $p_1 \cdots p_N$

$r = [A_1 \; A_2 \; \cdots\cdots \; A_N]$

$p = [p_1 \; p_2 \; \cdots \; p_N]$

$k = [\,]$ if $N > M$

$$Y(z) = \underbrace{\frac{A_1}{1-P_1 z^{-1}} + \frac{A_2}{1-P_2 z^{-1}} + \cdots + \frac{A_N}{1-P_N z^{-1}}}_{\text{partial fraction expansion}}$$

once the residues are known you can use table look up to find the inverse z transform of each term

$$A_k P_k^n u(n) \longleftrightarrow \frac{A_k}{1-P_k z^{-1}} \leftarrow \text{from table}$$

$$y[n] = A_1 P_1^n u[n] + A_2 P_2^n u[n] + \cdots$$

$$\boxed{y[n] = \left[A_1 P_1^n + A_2 P_2^n \cdots A_N P_N^n\right] u[n]}$$

Note: if $|P_1|, |P_2| \cdots |P_N|$ lie within the unit circle the every term decays with n and $y[n]$ remains bounded

otherwise

$y[n]$ increases exponentially with n

Example

$$Y(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

$y[n]$?

MATLAB

```
num = [1 0 0];
den = [1 -1.5 0.5];
[r p k] = residuez(num,den);

>> r

r =

     2
    -1

>> p

p =

    1.0000
    0.5000

>> k

k =

     0
```

residues

poles

$$Y(z) = \frac{2}{1 - z^{-1}} - \frac{1}{1 - 0.5z^{-1}}$$

$$Y(z) = \frac{2}{1 - z^{-1}} - \frac{1}{1 - 0.5z^{-1}}$$

$$y[n] = 2u[n] - (0.5)^n u[n]$$
$$= (2 - 0.5^n)u[n]$$

MATLAB

zplane (num, den)