Partial Fraction Expansion by hand

$$Y(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

$a_N \neq 0$, $M < N$ (proper fraction)

a) Simple poles (non-repeated poles):

$p_1, p_2, \cdots p_N$ all distinct

$$Y(z) = \underbrace{\frac{A_1}{1 - p_1 z^{-1}} + \frac{A_2}{1 - p_2 z^{-1}} + \cdots + \frac{A_N}{1 - p_N z^{-1}}}_{\text{Partial Fraction Expression}}$$

Example:

$$Y(z) = \frac{4}{(1-2z^{-1})(1-4z^{-1})}$$

$$p_1 = 2, \quad p_2 = 4$$

$$\frac{4}{(1-2z^{-1})(1-4z^{-1})} = \frac{A_1}{1-2z^{-1}} + \frac{A_2}{1-4z^{-1}}$$

Determine A_1:

multiply throughout by $1-2z^{-1}$

$$\frac{4}{(1-2z^{-1})(1-4z^{-1})}(1-2z^{-1}) = \frac{A_1}{1-2z^{-1}}(1-2z^{-1}) + \frac{A_2}{1-4z^{-1}}(1-2z^{-1})$$

$$\frac{4}{1-4z^{-1}} = A_1 + A_2\frac{(1-2z^{-1})}{(1-4z^{-1})}$$

$$\frac{4}{1-4z^{-1}} = A_1 + A_2 \frac{(1-2z^{-1})}{(1-4z^{-1})}$$

replace z by 2

$$\frac{4}{1-\frac{4}{2}} = A_1 + A_2 \frac{\left(1-\frac{2}{2}\right)}{\left(1-\frac{4}{2}\right)}$$

$$\frac{4}{1-2} = -4$$

$$A_1 = -4$$

Determine A_2:

$$A_2 = Y(z)(1-P_2 z^{-1})\Big|_{z=P_2}$$

$$A_2 = \frac{4}{(1-2z^{-1})(1-4z^{-1})} \times (1-4z^{-1})\Big|_{z=4}$$

$$A_2 = \frac{4}{1-2z^{-1}}\Big|_{z=4} = \frac{4}{1-\frac{2}{4}} = 8$$

2

$$\frac{4}{(1-2z^{-1})(1-4z^{-1})} = \frac{\overbrace{-4}^{A_1}}{1-2z^{-1}} + \frac{\overbrace{8}^{A_2}}{1-4z^{-1}}$$

$$A_n = Y(z)(1-P_n z^{-1})\Big|_{z=P_n}$$

Multiple Poles:

$$Y(z) = \frac{4}{(1-2z^{-1})\underbrace{(1-4z^{-1})^2}_{\text{double pole}}}$$

$$\frac{4}{(1-2z^{-1})(1-4z^{-1})^2} = \frac{A_1}{(1-2z^{-1})} + \frac{A_{21}}{(1-4z^{-1})} + \frac{A_{22}}{(1-4z^{-1})^2}$$

Determine A_1 as before.

$$A_1 = Y(z)(1-2z^{-1})\Big|_{z=2}$$

$$= \frac{4}{(1-4z^{-1})^2}\Big|_{z=2} = \frac{4}{(1-\frac{4}{2})^2} = \frac{4}{(1-2)^2} = 4$$

Determine A_{22} by multiplying both sides by $(1-2z^{-1})^2$

$$\frac{4}{1-2z^{-1}} = \frac{A_1(1-4z^{-1})^2}{(1-2z^{-1})} + A_{21}(1-4z^{-1}) + A_{22}$$

$$\frac{4}{1-2z^{-1}} = \frac{A_1(1-4z^{-1})^2}{(1-2z^{-1})} + A_{21}(1-4z^{-1}) + A_{22}$$

replace z by 4

$$\frac{4}{1-\frac{2}{4}} = \frac{A_1(1-4z^{-1})^2}{(1-2z^{-1})} + A_{21}(1-4z^{-1}) + A_{22}$$

(the $(1-4z^{-1})^2$ and $(1-4z^{-1})$ terms each go to 0)

$$A_{22} = \frac{4}{1-\frac{1}{2}} = 8$$

Determine A_{21}

$$\frac{4}{(1-2z^{-1})(1-4z^{-1})^2} = \frac{A_1}{(1-2z^{-1})} + \frac{A_{21}}{(1-4z^{-1})} + \frac{A_{22}}{(1-4z^{-1})^2}$$

multiply by $(1-4z^{-1})^2$

$$\frac{4}{1-2z^{-1}} = \frac{A_1(1-4z^{-1})^2}{(1-2z^{-1})} + A_{21}(1-4z^{-1}) + A_{22}$$

$$\frac{4}{1-2\bar{z}^{1}} = \frac{A_1\left(1-4\bar{z}^{1}\right)^2}{\left(1-2\bar{z}^{1}\right)} + A_{21}\left(1-4\bar{z}^{1}\right) + A_{22}$$

take derivative $\frac{d}{d\bar{z}^{1}}$

$$\frac{d}{d\bar{z}^{1}}\left(\frac{4}{1-2\bar{z}^{1}}\right) = \frac{d}{d\bar{z}^{1}}\frac{A_1\left(1-4\bar{z}^{1}\right)^2}{\left(1-2\bar{z}^{1}\right)} + A_{21}(-4) + 0$$

replace z by 4

$$\left.\frac{d}{d\bar{z}^{1}}\left(\frac{4}{1-2\bar{z}^{1}}\right)\right|_{z=4} = \left.\frac{d}{d\bar{z}^{1}}\frac{A_1\left(1-4\bar{z}^{1}\right)^2}{\left(1-2\bar{z}^{1}\right)}\right|_{z=4} - 4A_{21}$$

(the first term on the right is 0)

$$A_{21} = -\frac{1}{4}\frac{d}{d\bar{z}^{1}}\left(\frac{4}{1-2\bar{z}^{1}}\right)\Bigg|_{z=4}$$

$$= -\frac{1}{4}\left[-\frac{4}{\left(1-2\bar{z}^{1}\right)^2}(-2)\right]\Bigg|_{z=4} = \frac{-2}{\left(1-\frac{2}{4}\right)^2} = -8$$

$$\frac{4}{(1-2z^{-1})(1-4z^{-1})^2}$$

$$= \frac{4}{(1-2z^{-1})} - \frac{8}{(1-4z^{-1})} + \frac{8}{(1-4z^{-1})^2}$$

$$Y(z) = \frac{A_{11}}{(1-p_1 z^{-1})} + \underbrace{\frac{A_{12}}{(1-p_1 z^{-1})^2}} + \cdots$$

double pole

$$A_{12} = Y(z)(1-p_1 z^{-1})^2 \Big|_{z=p_1}$$

$$A_{11} = \frac{1}{(-p_1)} \frac{d}{dz^{-1}} \left\{ Y(z)(1-p_1 z^{-1})^2 \right\} \Big|_{z=p_1}$$

Summary:

Single pole P_n:

$$Y(z) = \cdots + \frac{A_n}{1 - P_n z^{-1}} + \cdots$$

$$A_n = Y(z)(1 - P_n z^{-1})\Big|_{z = P_n}$$

Double pole P_n:

$$Y(z) = \cdots + \frac{A_{n1}}{(1 - P_n z^{-1})} + \frac{A_{n2}}{(1 - P_n z^{-1})^2} + \cdots$$

$$A_{n1} = \frac{1}{(-P_n)} \frac{d}{d z^{-1}} \left\{ Y(z)(1 - P_n z^{-1})^2 \right\}\Big|_{z = P_n}$$

$$A_{n2} = Y(z)(1 - P_n z^{-1})^2\Big|_{z = P_n}$$