

Partial Fraction Expansion

Partial fraction expansion facilitates inversion of the final s -domain expression for the variable of interest back to the time domain. The goal is to cast the expression as the sum of terms, each of which has an analog in Table 10-2.

Example

$$\mathbf{F}(s) = \frac{4}{s+2} + \frac{6}{(s^2+5)^2} + \frac{8}{s^2+4s+5}.$$

The inverse transform $f(t)$ is given by

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}[\mathbf{F}(s)] \\ &= \mathcal{L}^{-1}\left[\frac{4}{s+2}\right] + \mathcal{L}^{-1}\left[\frac{6}{(s+5)^2}\right] \\ &\quad + \mathcal{L}^{-1}\left[\frac{8}{s^2+4s+5}\right]. \end{aligned}$$

- (a) The first term in Eq. (10.59), $4/(s+2)$, is functionally the same as entry #3 in Table 10-2 with $a = 2$. Hence,

$$\mathcal{L}^{-1}\left[\frac{4}{s+2}\right] = 4e^{-2t} u(t). \quad (10.60a)$$

- (b) The second term, $6/(s+5)^2$, is functionally the same as entry #6 in Table 10-2 with $a = 5$. Thus,

$$\mathcal{L}^{-1}\left[\frac{6}{(s+5)^2}\right] = 6te^{-5t} u(t). \quad (10.60b)$$

- (c) The third term, $1/(s^2+4s+5)$, is similar but not identical in form with entry #13 in Table 10-2. However, it can be rearranged to assume the proper form:

$$\frac{1}{s^2+4s+5} = \frac{1}{(s+2)^2+1}.$$

Consequently,

$$\mathcal{L}^{-1}\left[\frac{8}{(s+2)^2+1}\right] = 8e^{-2t} \sin t u(t). \quad (10.60c)$$

Combining the results represented by Eqs. (10.60a–c) gives:

$$f(t) = [4e^{-2t} + 6te^{-5t} + 8e^{-2t} \sin t] u(t). \quad (10.61)$$

1. Partial Fractions

Distinct Real Poles

Distinct Real Poles

Given a proper rational function defined by

$$\mathbf{F}(s) = \frac{\mathbf{N}(s)}{\mathbf{D}(s)} = \frac{\mathbf{N}(s)}{(s + p_1)(s + p_2) \dots (s + p_n)}, \quad (10.69)$$

with distinct real poles $-p_1$ to $-p_n$, such that $p_i \neq p_j$ for all $i \neq j$, and $m < n$ (where m and n are the highest powers of s in $\mathbf{N}(s)$ and $\mathbf{D}(s)$, respectively), then $\mathbf{F}(s)$ can be expanded into the equivalent form:

$$\begin{aligned} \mathbf{F}(s) &= \frac{A_1}{s + p_1} + \frac{A_2}{s + p_2} + \dots + \frac{A_n}{s + p_n} \\ &= \sum_{i=1}^n \frac{A_i}{(s + p_i)}, \end{aligned} \quad (10.70)$$

with expansion coefficients A_1 to A_n given by

$$\begin{aligned} A_i &= (s + p_i) \mathbf{F}(s)|_{s=-p_i}, \\ i &= 1, 2, \dots, n. \end{aligned} \quad (10.71)$$

In view of entry #3 in Table 10-2, the inverse Laplace transform of Eq. (10.70) is

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}[\mathbf{F}(s)] \\ &= [A_1 e^{-p_1 t} + A_2 e^{-p_2 t} + \dots + A_n e^{-p_n t}] u(t). \end{aligned} \quad (10.72)$$

1. Partial Fractions

Distinct Real Poles

Example

$$\mathbf{F}(s) = \frac{s^2 - 4s + 3}{s(s + 1)(s + 3)}$$

The poles of $\mathbf{F}(s)$ are $s = 0$, $s = -1$, and $s = -3$.
All three poles are real and distinct.

$$\mathbf{F}(s) = \frac{A_1}{s} + \frac{A_2}{(s + 1)} + \frac{A_3}{(s + 3)},$$

$$A_1 = s \mathbf{F}(s)|_{s=0} = \frac{s^2 - 4s + 3}{(s + 1)(s + 3)} \Big|_{s=0} = 1,$$

$$A_3 = (s + 3) \mathbf{F}(s)|_{s=-3} = \frac{s^2 - 4s + 3}{s(s + 1)} \Big|_{s=-3} = 4.$$

$$A_2 = \frac{(-1)^2 + 4 + 3}{(-1)(-1 + 3)} = -4.$$

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}[\mathbf{F}(s)] \\ &= \mathcal{L} \left[\frac{1}{s} - \frac{4}{s + 1} + \frac{4}{s + 3} \right] \\ &= [1 - 4e^{-t} + 4e^{-3t}] u(t). \end{aligned}$$

2. Partial Fractions

Repeated Real Poles

Repeated Real Poles

Expansion coefficients B_1 to B_m are determined through a procedure that involves multiplication by $(s + p)^m$, differentiation with respect to s , and evaluation at $s = -p$:

$$B_j = \left\{ \frac{1}{(m-j)!} \frac{d^{m-j}}{ds^{m-j}} [(s+p)^m \mathbf{F}(s)] \right\} \Big|_{s=-p},$$
$$j = 1, 2, \dots, m.$$

(10.79)

For the m , $m - 1$, and $m - 2$ terms, Eq. (10.79) reduces to

$$B_m = (s+p)^m \mathbf{F}(s) \Big|_{s=-p}, \quad (10.80a)$$

$$B_{m-1} = \left\{ \frac{d}{ds} [(s+p)^m \mathbf{F}(s)] \right\} \Big|_{s=-p}, \quad (10.80b)$$

$$B_{m-2} = \left\{ \frac{1}{2!} \frac{d^2}{ds^2} [(s+p)^m \mathbf{F}(s)] \right\} \Big|_{s=-p}. \quad (10.80c)$$

2. Partial Fractions

Repeated Real Poles

Example

$$F(s) = \frac{N(s)}{D(s)} = \frac{s^2 + 3s + 3}{s^4 + 11s^3 + 45s^2 + 81s + 54}$$

Solution: In theory, any polynomial with real coefficients can be expressed as a product of linear and quadratic factors (of the form $(s + p)$ and $(s^2 + as + b)$, respectively). The process involves long division, but it requires knowledge of the roots of the polynomial, which can be determined through the application of numerical techniques. In the present case, a random check reveals that $s = -2$ and $s = -3$ are roots of $D(s)$. Given that $D(s)$ is fourth order, it should have four roots, including possible duplicates.

Since $s = -2$ is a root of $D(s)$, we should be able to factor out $(s + 2)$ from it. Long division gives

$$\begin{aligned} D(s) &= s^4 + 11s^3 + 45s^2 + 81s + 54 \\ &= (s + 2)(s^3 + 9s^2 + 27s + 27). \end{aligned}$$

Next, we factor out $(s + 3)$ by

$$\begin{aligned} D(s) &= (s + 2)(s + 3)(s^2 + 6s + 9) \\ &= (s + 2)(s + 3)^3. \end{aligned}$$

Cont.

2. Partial Fractions

Repeated Real Poles

Example cont.

$$\mathbf{F}(s) = \frac{\mathbf{N}(s)}{\mathbf{D}(s)} = \frac{s^2 + 3s + 3}{s^4 + 11s^3 + 45s^2 + 81s + 54}$$

Hence, $\mathbf{F}(s)$ has a distinct real pole at $s = -2$ and a thrice repeated pole at $s = -3$. The given expression can be rewritten as

$$\mathbf{F}(s) = \frac{s^2 + 3s + 3}{(s + 2)(s + 3)^3}.$$

Through partial fraction expansion, $\mathbf{F}(s)$ can be decomposed into

$$\mathbf{F}(s) = \frac{A}{s + 2} + \frac{B_1}{s + 3} + \frac{B_2}{(s + 3)^2} + \frac{B_3}{(s + 3)^3},$$

with

$$A = (s + 2) \mathbf{F}(s)|_{s=-2} = \frac{s^2 + 3s + 3}{(s + 3)^3} \Big|_{s=-2} = 1,$$

$$B_3 = (s + 3)^3 \mathbf{F}(s)|_{s=-3} = \frac{s^2 + 3s + 3}{s + 2} \Big|_{s=-3} = -3,$$

$$B_2 = \frac{d}{ds} [(s + 3)^3 \mathbf{F}(s)] \Big|_{s=-3} = 0,$$

and

$$B_1 = \frac{1}{2} \frac{d^2}{ds^2} [(s + 3)^3 \mathbf{F}(s)] \Big|_{s=-3} = -1.$$

Hence,

$$\mathbf{F}(s) = \frac{1}{s + 2} - \frac{1}{s + 3} - \frac{3}{(s + 3)^3},$$

and application of Eq. (10.81) leads to

$$\mathcal{L}^{-1}[\mathbf{F}(s)] = \left[e^{-2t} - e^{-3t} - \frac{3}{2} t^2 e^{-3t} \right] u(t).$$

3. Distinct Complex Poles

- Procedure similar to “Distinct Real Poles,” but with complex values for s
- Complex poles always appear in conjugate pairs
- Expansion coefficients of conjugate poles are conjugate pairs themselves

Example
$$\mathbf{F}(s) = \frac{4s + 1}{(s + 1)(s^2 + 4s + 13)}$$

$$s^2 + 4s + 13 = (s + 2 - j3)(s + 2 + j3)$$

$$\mathbf{F}(s) = \frac{A}{s + 1} + \frac{\mathbf{B}_1}{s + 2 - j3} + \frac{\mathbf{B}_2}{s + 2 + j3}$$

$$A = (s + 1) \mathbf{F}(s)|_{s=-1} = \frac{4s + 1}{s^2 + 4s + 13} \Big|_{s=-1} = -0.3, \quad (10.87a)$$

$$\begin{aligned} \mathbf{B}_1 &= (s + 2 - j3) \mathbf{F}(s)|_{s=-2+j3} \\ &= \frac{4s + 1}{(s + 1)(s + 2 + j3)} \Big|_{s=-2+j3} \\ &= \frac{4(-2 + j3) + 1}{(-2 + j3 + 1)(-2 + j3 + 2 + j3)} \\ &= \frac{-7 + j12}{-18 - j6} = 0.73e^{-j78.2^\circ}, \end{aligned} \quad (10.87b)$$

and

$$\begin{aligned} \mathbf{B}_2 &= (s + 2 + j3) \mathbf{F}(s)|_{s=-2-j3} \\ &= \frac{4s + 1}{(s + 1)(s + 2 - j3)} \Big|_{s=-2-j3} = 0.73e^{j78.2^\circ}. \end{aligned} \quad (10.87c)$$

Note that \mathbf{B}_2 is the complex conjugate of \mathbf{B}_1 .

3. Distinct Complex Poles (Cont.)

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}[\mathbf{F}(s)] \\ &= \mathcal{L}^{-1}\left(\frac{-0.3}{s+1}\right) + \mathcal{L}^{-1}\left(\frac{0.73e^{-j78.2^\circ}}{s+2-j3}\right) \\ &\quad + \mathcal{L}^{-1}\left(\frac{0.73e^{j78.2^\circ}}{s+2+j3}\right) \\ &= [-0.3e^{-t} + 0.73e^{-j78.2^\circ}e^{-(2-j3)t} \\ &\quad + 0.73e^{j78.2^\circ}e^{-(2+j3)t}] u(t). \end{aligned}$$

Next, we combine the last two terms:

$$\begin{aligned} &0.73e^{-j78.2^\circ}e^{-(2-j3)t} + 0.73e^{j78.2^\circ}e^{-(2+j3)t} \\ &= 0.73e^{-2t}[e^{j(3t-78.2^\circ)} + e^{-j(3t-78.2^\circ)}] \\ &= 2 \times 0.73e^{-2t} \cos(3t - 78.2^\circ) \\ &= 1.46e^{-2t} \cos(3t - 78.2^\circ). \end{aligned} \tag{10.89}$$

Hence, the final time-domain solution is

$$f(t) = [-0.3e^{-t} + 1.46e^{-2t} \cos(3t - 78.2^\circ)] u(t). \tag{10.90}$$

4. Repeated Complex Poles:

Same procedure as for repeated real poles

Table 10-3: Transform pairs for four types of poles.

Pole	$\mathbf{F}(s)$	$f(t)$
1. Distinct real	$\frac{A}{s + a}$	$Ae^{-at} u(t)$
2. Repeated real	$\frac{A}{(s + a)^n}$	$A \frac{t^{n-1}}{(n-1)!} e^{-at} u(t)$
3. Distinct complex	$\left[\frac{Ae^{j\theta}}{s + a + jb} + \frac{Ae^{-j\theta}}{s + a - jb} \right]$	$2Ae^{-at} \cos(bt - \theta) u(t)$
4. Repeated complex	$\left[\frac{Ae^{j\theta}}{(s + a + jb)^n} + \frac{Ae^{-j\theta}}{(s + a - jb)^n} \right]$	$\frac{2At^{n-1}}{(n-1)!} e^{-at} \cos(bt - \theta) u(t)$

Example 10-10: Interesting Transform!

Determine the time-domain equivalent of the Laplace transform

$$\mathbf{F}(s) = \frac{se^{-3s}}{s^2 + 4}.$$

Solution: We start by separating out the exponential e^{-3s} from the remaining polynomial fraction. We do so by defining

$$\mathbf{F}(s) = e^{-3s} \mathbf{F}_1(s),$$

where

$$\begin{aligned} \mathbf{F}_1(s) &= \frac{s}{s^2 + 4} \\ &= \frac{s}{(s + j2)(s - j2)} \\ &= \frac{\mathbf{B}_1}{s + j2} + \frac{\mathbf{B}_2}{s - j2} \end{aligned}$$

with

$$\begin{aligned} \mathbf{B}_1 &= (s + j2) \mathbf{F}_1(s)|_{s=-j2} \\ &= \left. \frac{s}{s - j2} \right|_{s=-j2} \\ &= \frac{-j2}{-j4} = \frac{1}{2} \end{aligned}$$

and

$$\mathbf{B}_2 = \mathbf{B}_1^* = \frac{1}{2}.$$

$$\mathbf{F}(s) = e^{-3s} \mathbf{F}_1(s)$$

$$= \frac{e^{-3s}}{2(s + j2)} + \frac{e^{-3s}}{2(s - j2)}.$$

Property #3a in Table 10-2:

$$e^{-a(t-T)} u(t - T) \longleftrightarrow \frac{e^{-Ts}}{s + a}$$

Hence:

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}[\mathbf{F}(s)] \\ &= \mathcal{L}^{-1} \left[\frac{1}{2} \frac{e^{-3s}}{s + j2} + \frac{1}{2} \frac{e^{-3s}}{s - j2} \right] \\ &= \left[\frac{1}{2} (e^{-j2(t-3)} + e^{j2(t-3)}) \right] u(t - 3) \\ &= [\cos(2t - 6)] u(t - 3). \end{aligned}$$

s-Domain Circuit Models

The s-domain transformation of circuit elements incorporates initial conditions associated with any energy storage that may have existed in capacitors and inductors at $t = 0^-$.

Resistor in the s-Domain

Application of the Laplace transform to Ohm's law,

$$\mathcal{L}[v] = \mathcal{L}[Ri], \quad (10.91)$$

leads to

$$\mathbf{V} = R\mathbf{I}, \quad (10.92)$$

where by definition,

$$\mathbf{V} = \mathcal{L}[v] \quad \text{and} \quad \mathbf{I} = \mathcal{L}[i]. \quad (10.93)$$

Hence the correspondence between the time and s-domains is

$$v = Ri \iff \mathbf{V} = R\mathbf{I}. \quad (10.94)$$

Inductor in the s-Domain

$$f' = \frac{df}{dt} \iff \mathbf{sF(s)} - f(0^-)$$

$$v = L \frac{di}{dt} \iff \mathbf{V} = sL\mathbf{I} - L i(0^-).$$

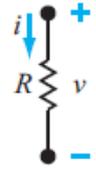
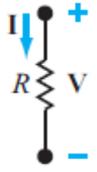
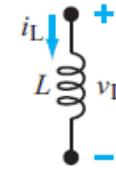
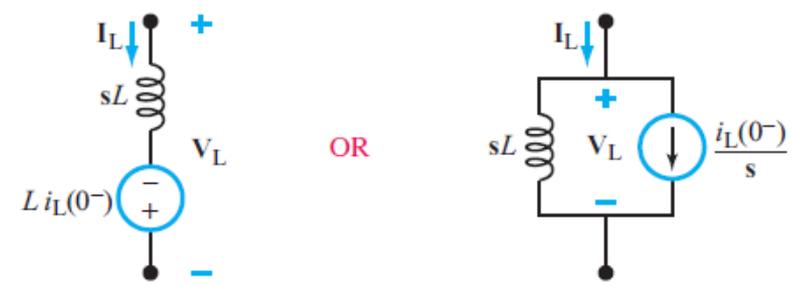
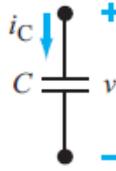
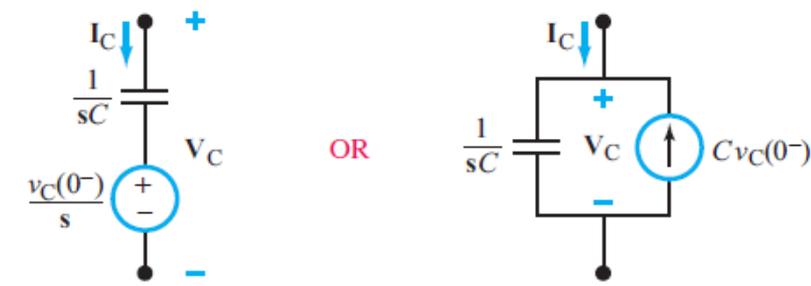
Capacitor in the s-Domain

$$i = C \frac{dv}{dt} \iff \mathbf{I} = sC\mathbf{V} - C v(0^-),$$

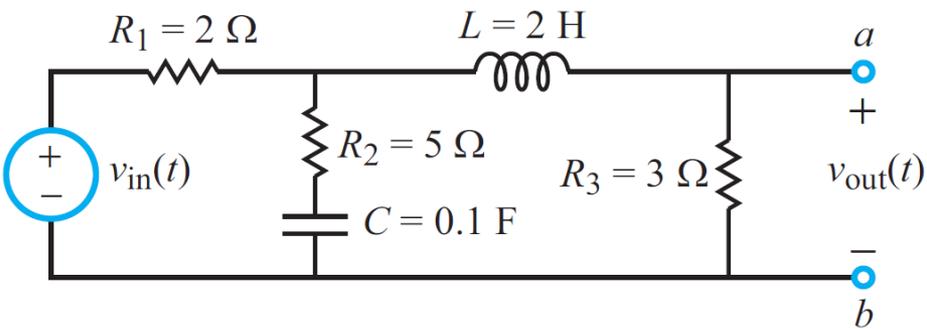
Under zero initial conditions:

$$\mathbf{Z}_R = R, \quad \mathbf{Z}_L = sL, \quad \text{and} \quad \mathbf{Z}_C = \frac{1}{sC}.$$

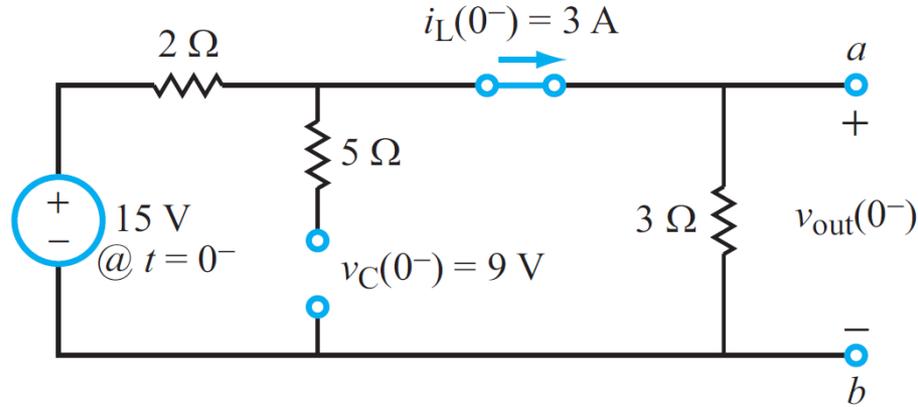
Table 10-4: Circuit models for R , L , and C in the s -domain.

Time-Domain	s-Domain
<p>Resistor</p>  <p>$v = Ri$</p>	 <p>$V = RI$</p>
<p>Inductor</p>  <p>$v_L = L \frac{di_L}{dt}$</p> <p>$i_L = \frac{1}{L} \int_{0^-}^t v_L dt + i_L(0^-)$</p>	 <p>OR</p> <p>$V_L = sLI_L - L i_L(0^-)$</p> <p>$I_L = \frac{V_L}{sL} + \frac{i_L(0^-)}{s}$</p>
<p>Capacitor</p>  <p>$i_C = C \frac{dv_C}{dt}$</p> <p>$v_C = \frac{1}{C} \int_{0^-}^t i_C dt + v_C(0^-)$</p>	 <p>OR</p> <p>$V_C = \frac{I_C}{sC} + \frac{v_C(0^-)}{s}$</p> <p>$I_C = sC V_C - C v_C(0^-)$</p>

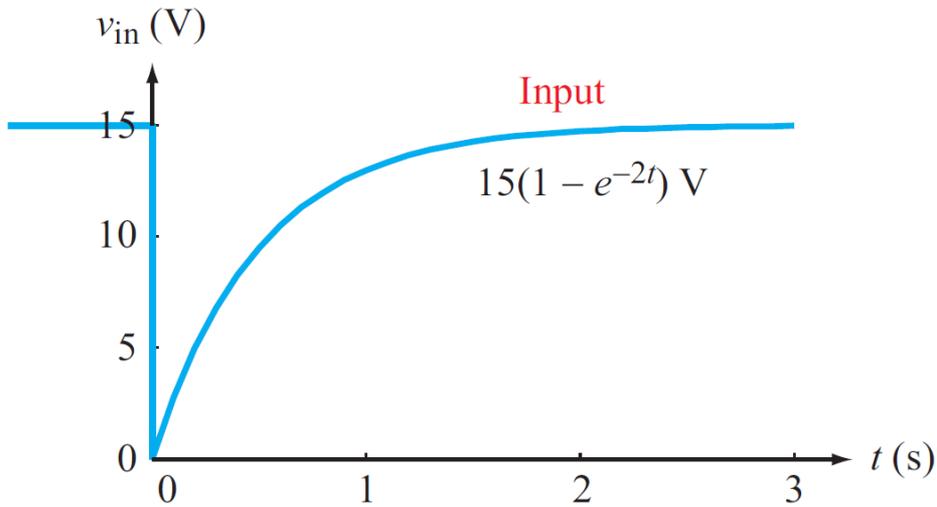
Example 10-11: Interrupted Voltage Source



(a) Time domain



(d) At $t = 0^-$



(b) Waveform of $v_{in}(t)$

Initial conditions:

$$v_C(0^-) = 9 \text{ V}, \quad i_L(0^-) = 3 \text{ A}, \quad v_{out}(0^-) = 9 \text{ V}$$

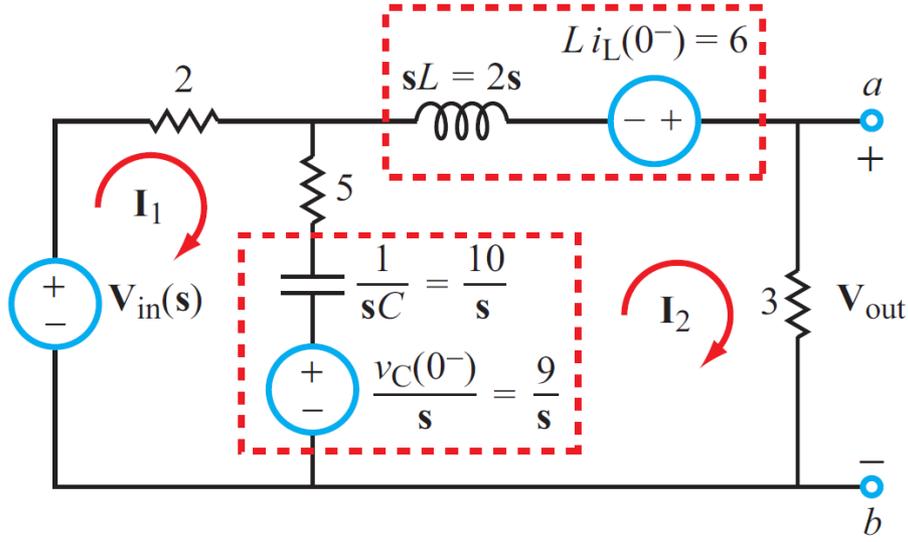
Voltage Source

$$v_{in}(t) = \begin{cases} 15 \text{ V} & \text{for } t \leq 0^- \\ 15(1 - e^{-2t}) u(t) \text{ V} & \text{for } t \geq 0. \end{cases}$$

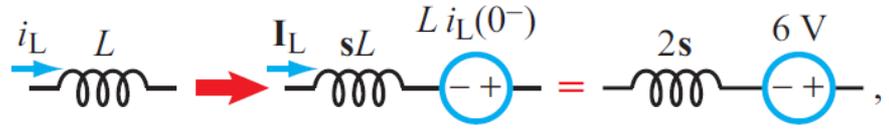
$$V_{in}(s) = \frac{15}{s} - \frac{15}{s + 2} \quad (\text{s-domain})$$

Cont.

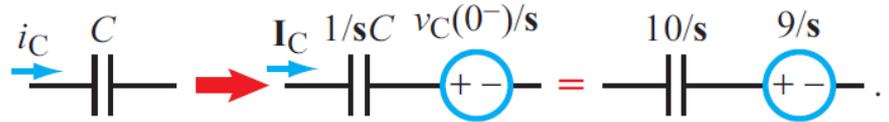
Example 10-11: Interrupted Voltage Source (cont.)



(e) s-domain



and



$$\left(2 + 5 + \frac{10}{s}\right) \mathbf{I}_1 - \left(5 + \frac{10}{s}\right) \mathbf{I}_2 = \mathbf{V}_{in} - \frac{9}{s},$$

and

$$-\left(5 + \frac{10}{s}\right) \mathbf{I}_1 + \left(8 + 2s + \frac{10}{s}\right) \mathbf{I}_2 = \frac{9}{s} + 6.$$

$$\mathbf{I}_2 = \frac{42s^3 + 162s^2 + 306s + 300}{s(s+2)(14s^2 + 51s + 50)}$$

$$= \frac{42s^3 + 162s^2 + 306s + 300}{14s(s+2)(s^2 + 51s/14 + 50/14)}$$

The roots of the quadratic term in the denominator are

$$s_1 = \left[-\frac{51}{14} - \sqrt{\left(\frac{51}{14}\right)^2 - 4 \times \frac{50}{14}} \right] / 2$$

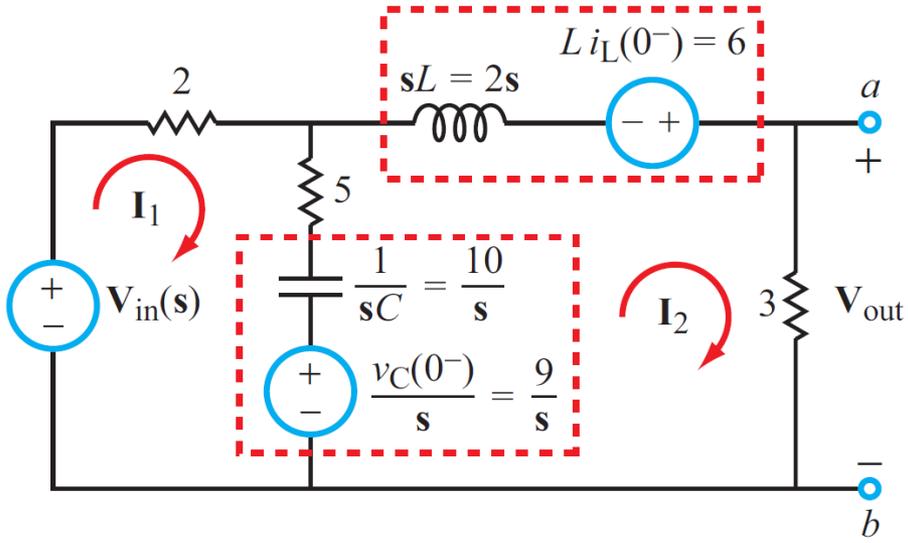
$$= -1.82 - j0.5$$

and

$$s_2 = -1.82 + j0.5.$$

Cont.

Example 10-11: Interrupted Voltage Source (cont.)



(e) s-domain

$$\mathbf{I}_2 = \frac{42s^3 + 162s^2 + 306s + 300}{14s(s + 2)(s + 1.82 + j0.5)(s + 1.82 - j0.5)} \quad (10.107)$$

The expression for \mathbf{I}_2 now is ready for expansion in the form of partial fractions as

$$\mathbf{I}_2 = \frac{A_1}{s} + \frac{A_2}{s + 2} + \frac{\mathbf{B}}{s + 1.82 + j0.5} + \frac{\mathbf{B}^*}{s + 1.82 - j0.5} \quad (10.108)$$

$$A_1 = \mathbf{sI}_2|_{s=0} = \frac{42s^3 + 162s^2 + 306s + 300}{14(s + 2)(s^2 + 51s/14 + 50/14)} \Big|_{s=0} = 3,$$

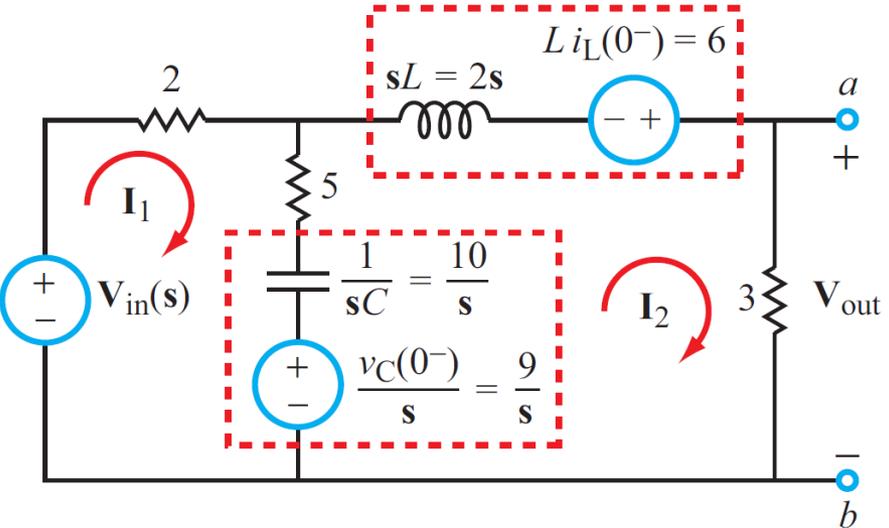
$$A_2 = (s + 2)\mathbf{I}_2|_{s=-2} = \frac{42s^3 + 162s^2 + 306s + 300}{14s(s^2 + 51s/14 + 50/14)} \Big|_{s=-2} = 0,$$

$$\begin{aligned} \mathbf{B} &= (s + 1.82 + j0.5)\mathbf{I}_2|_{s=-1.82-j0.5} \\ &= \frac{42s^3 + 162s^2 + 306s + 300}{14s(s + 2)(s + 1.82 - j0.5)} \Big|_{s=-1.82-j0.5} \\ &= 5.32e^{-j90^\circ}. \end{aligned}$$

$$\mathbf{I}_2 = \frac{3}{s} + \frac{5.32e^{-j90^\circ}}{s + 1.82 + j0.5} + \frac{5.32e^{j90^\circ}}{s + 1.82 - j0.5}.$$

Cont.

Example 10-11: Interrupted Voltage Source (cont.)



(e) s-domain

$$I_2 = \frac{3}{s} + \frac{5.32e^{-j90^\circ}}{s + 1.82 + j0.5} + \frac{5.32e^{j90^\circ}}{s + 1.82 - j0.5}$$

$$\frac{3}{s} \iff 3u(t),$$

and from Table 10-3,

$$\frac{Ae^{j\theta}}{s + a + jb} + \frac{Ae^{-j\theta}}{s + a - jb} \iff 2Ae^{-at} \cos(bt - \theta) u(t)$$

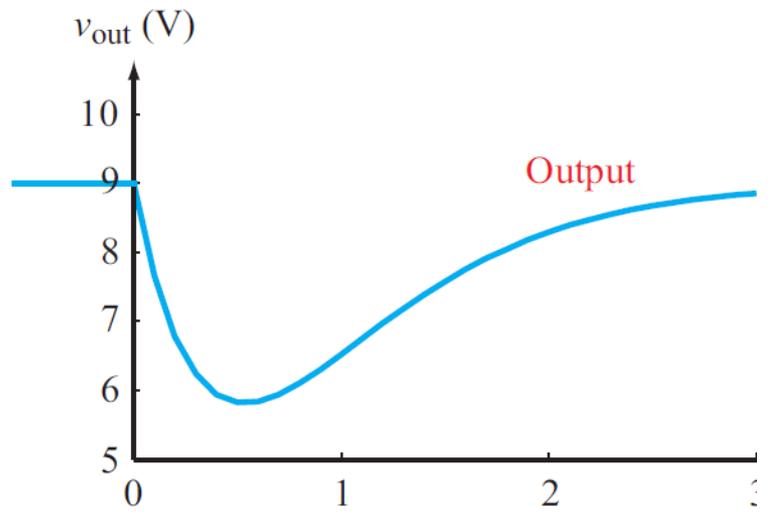
$$i_2(t) = [3 + 10.64e^{-1.82t} \cos(0.5t + 90^\circ)] u(t)$$

$$= [3 - 10.64e^{-1.82t} \sin 0.5t] u(t) \text{ A,}$$

and the corresponding output voltage is

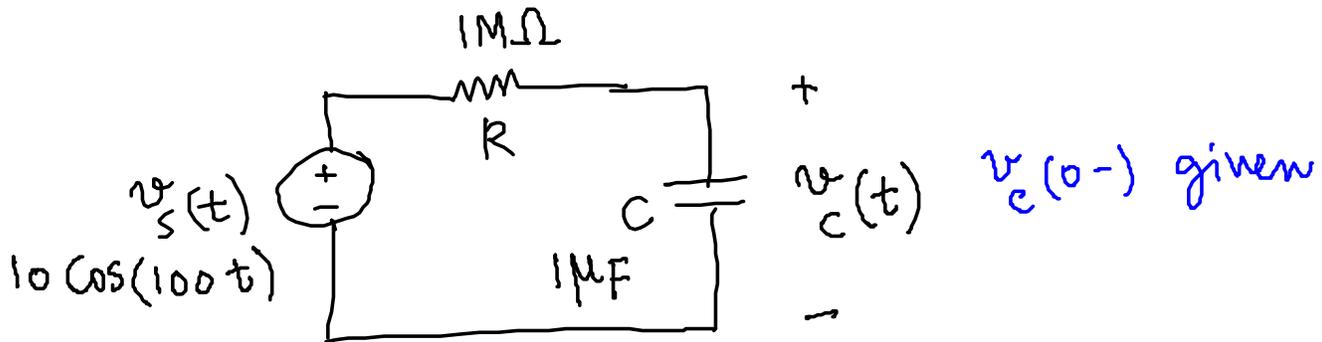
$$v_{out}(t) = 3i_2(t)$$

$$= [9 - 31.92e^{-1.82t} \sin 0.5t] u(t) \text{ V.}$$

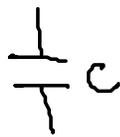
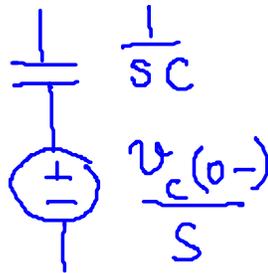


(c) Waveform of $v_{out}(t)$

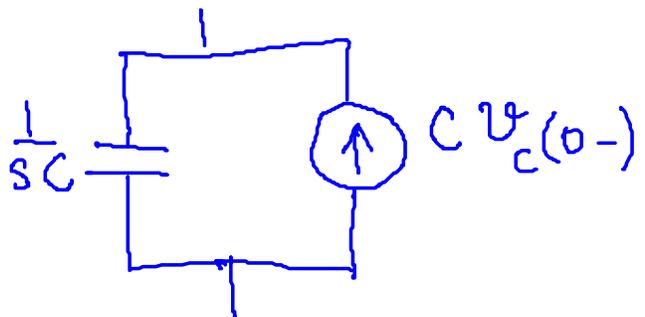
Example RC circuit



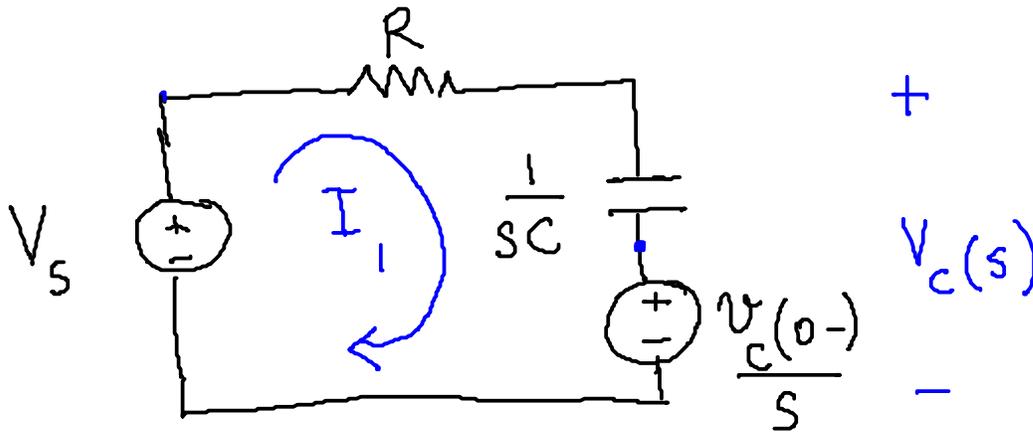
S Domain



or



Try mesh analysis



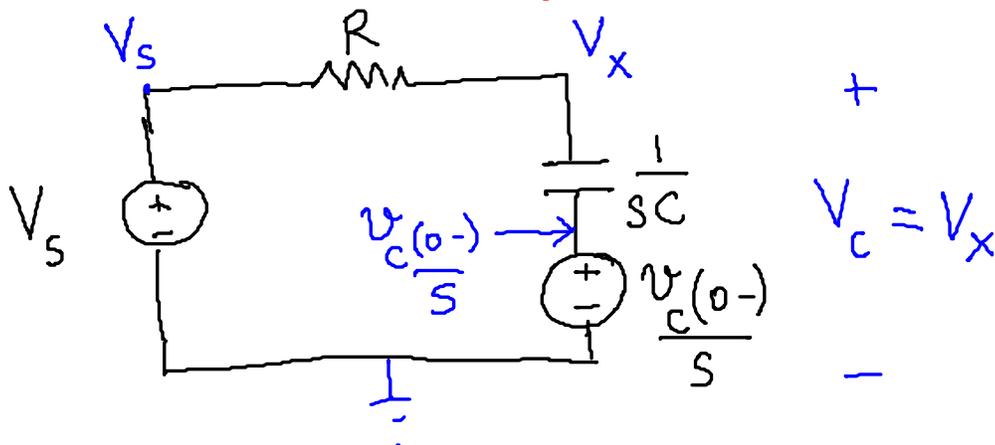
$$-V_s + RI_1 + \frac{1}{sC}I_1 + \frac{v_c(0-)}{s} = 0$$

$$\left(R + \frac{1}{sC}\right)I_1 = V_s - \frac{v_c(0-)}{s}$$

$$I_1 = \frac{V_s - \frac{v_c(0-)}{s}}{R + \frac{1}{sC}}$$

$$V_c(s) = \frac{1}{sC}I_1 + \frac{v_c(0-)}{s}$$

Node Analysis slightly easier



$$\frac{V_s - V_x}{R} + \frac{\frac{v_c(0-)}{s} - V_x}{\frac{1}{sC}} = 0 \quad V_x = V_c$$

$$\frac{1}{sCR} (V_s - V_c) + \frac{v_c(0-)}{s} - V_c = 0$$

$$\left(1 + \frac{1}{sCR}\right) V_c = \frac{1}{sCR} V_s + \frac{v_c(0-)}{s}$$

$$\left(s + \frac{1}{CR}\right) V_c = \frac{1}{CR} V_s + v_c(0-)$$

$$V_c = \underbrace{\frac{1}{CR} \frac{V_s}{s + \frac{1}{CR}}}_{\text{zero state response}} + \underbrace{\frac{v_c(0-)}{s + \frac{1}{CR}}}_{\text{zero input response}}$$

Same as lecture 62's result