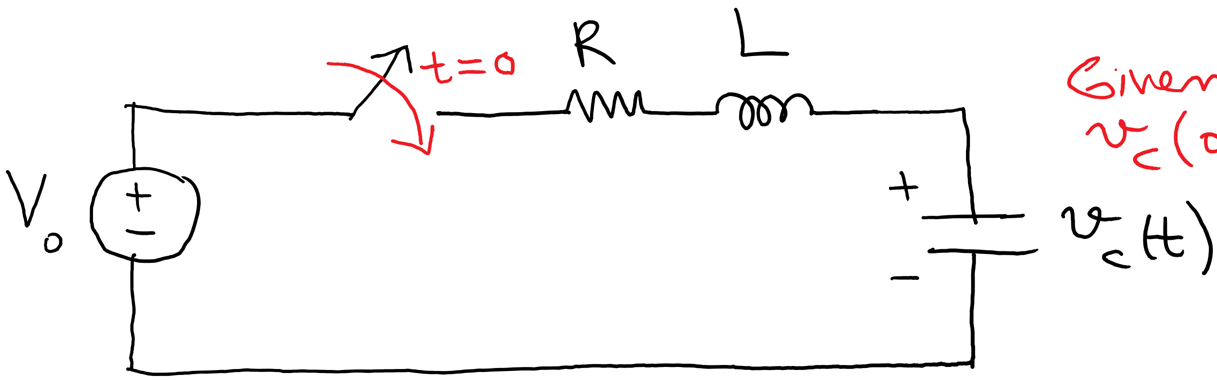
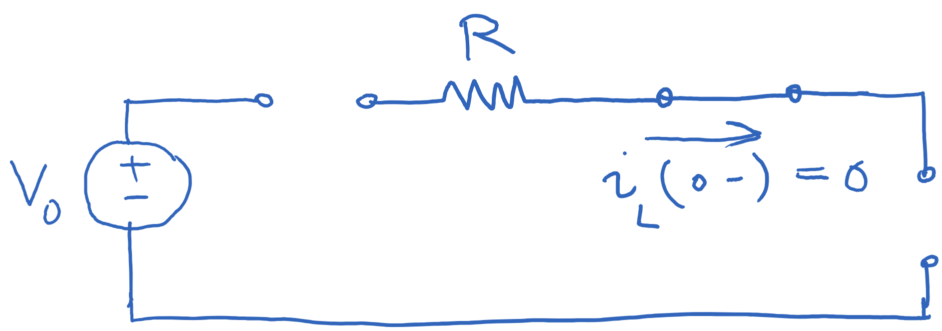
RLC Second Order Circuits

Series RLC circuit

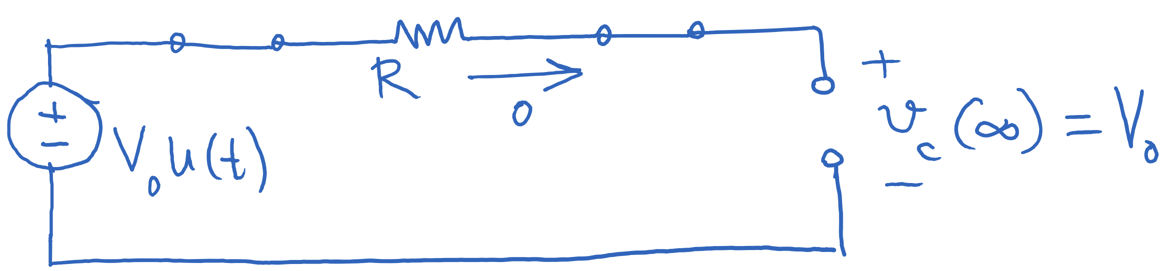


Given
$v_c(0-) = 0$

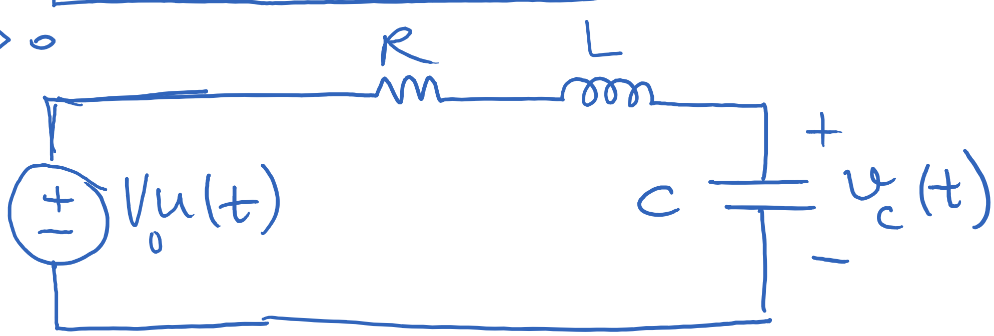
$t = 0-$ dc-conditions

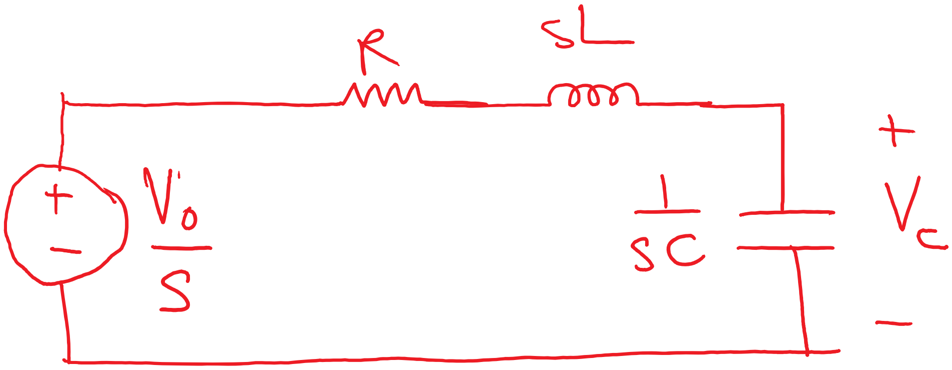


$t \to \infty$ dc-conditions.



$t > 0$





$$V_c = \frac{\frac{1}{sC}}{R + sL + \frac{1}{sC}} \frac{V_0}{s}$$

$$V_c = \underbrace{\frac{1}{LC} \frac{1}{s^2 + s\left(\frac{R}{L}\right) + \left(\frac{1}{LC}\right)}}_{\substack{\text{Transfer function} \\ H(s)}} \underbrace{\frac{V_0}{s}}_{\text{input}}$$

$\frac{R}{L} \rightarrow 2\alpha$

$\frac{1}{LC} \rightarrow \omega_0^2$

define $\alpha = \frac{R}{2L}$ $\qquad \omega_0 = \frac{1}{\sqrt{LC}}$

$$H(s) = \frac{\omega_0^2}{s^2 + 2\alpha s + \omega_0^2}$$

α : Damping Coefficient

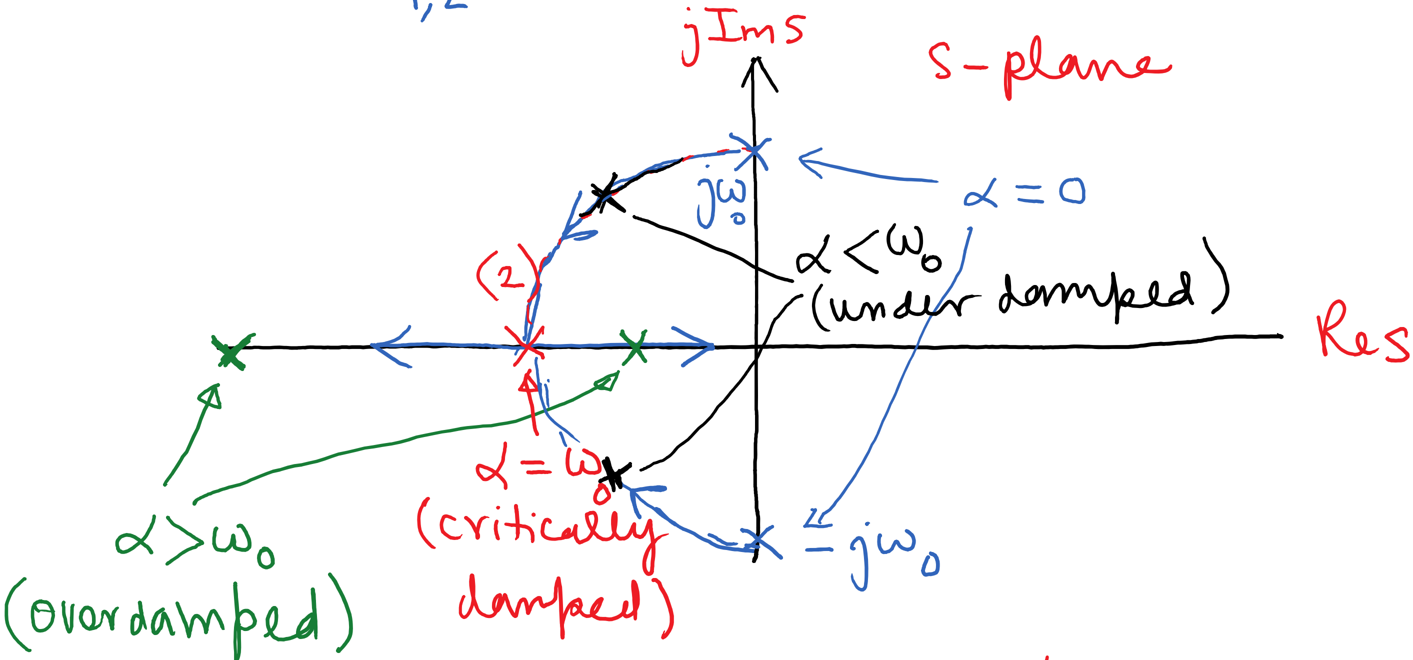
ω_0 : Resonant frequency

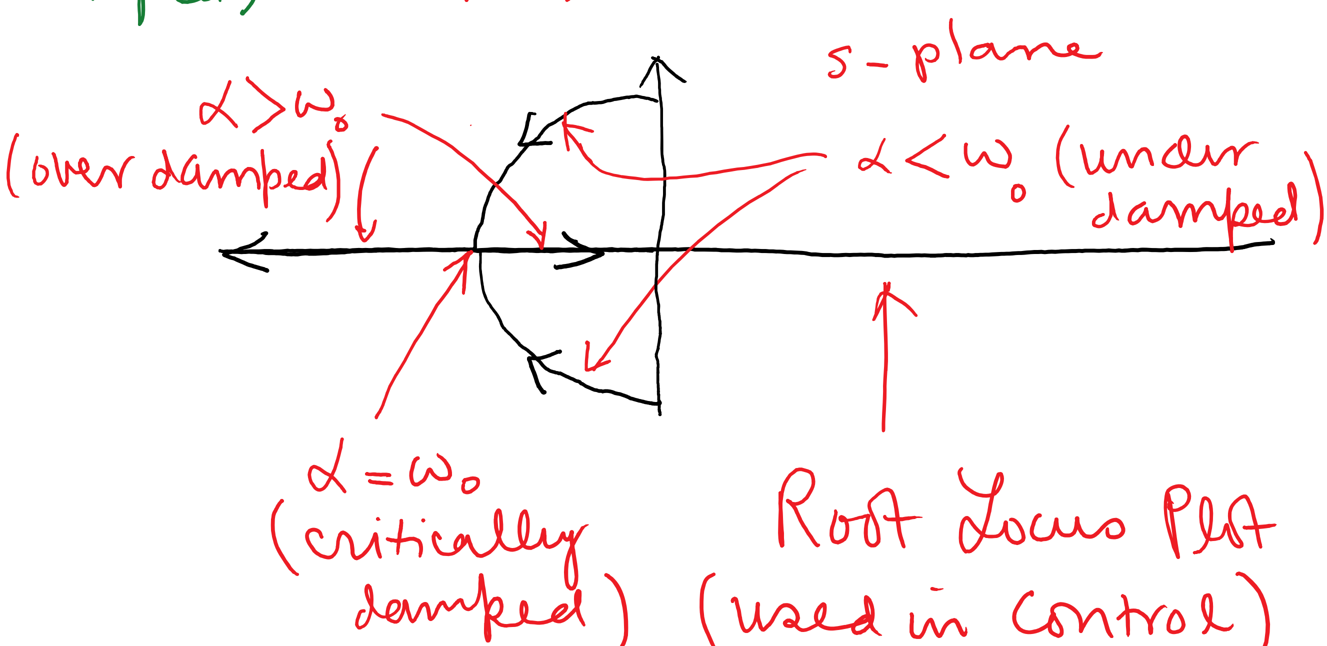
$$H(s) = \frac{\omega_0^2}{s^2 + 2\alpha s + \omega_0^2}$$

poles

$$s_{1,2} = \frac{-2\alpha \pm \sqrt{4\alpha^2 - 4\omega_0^2}}{2}$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$





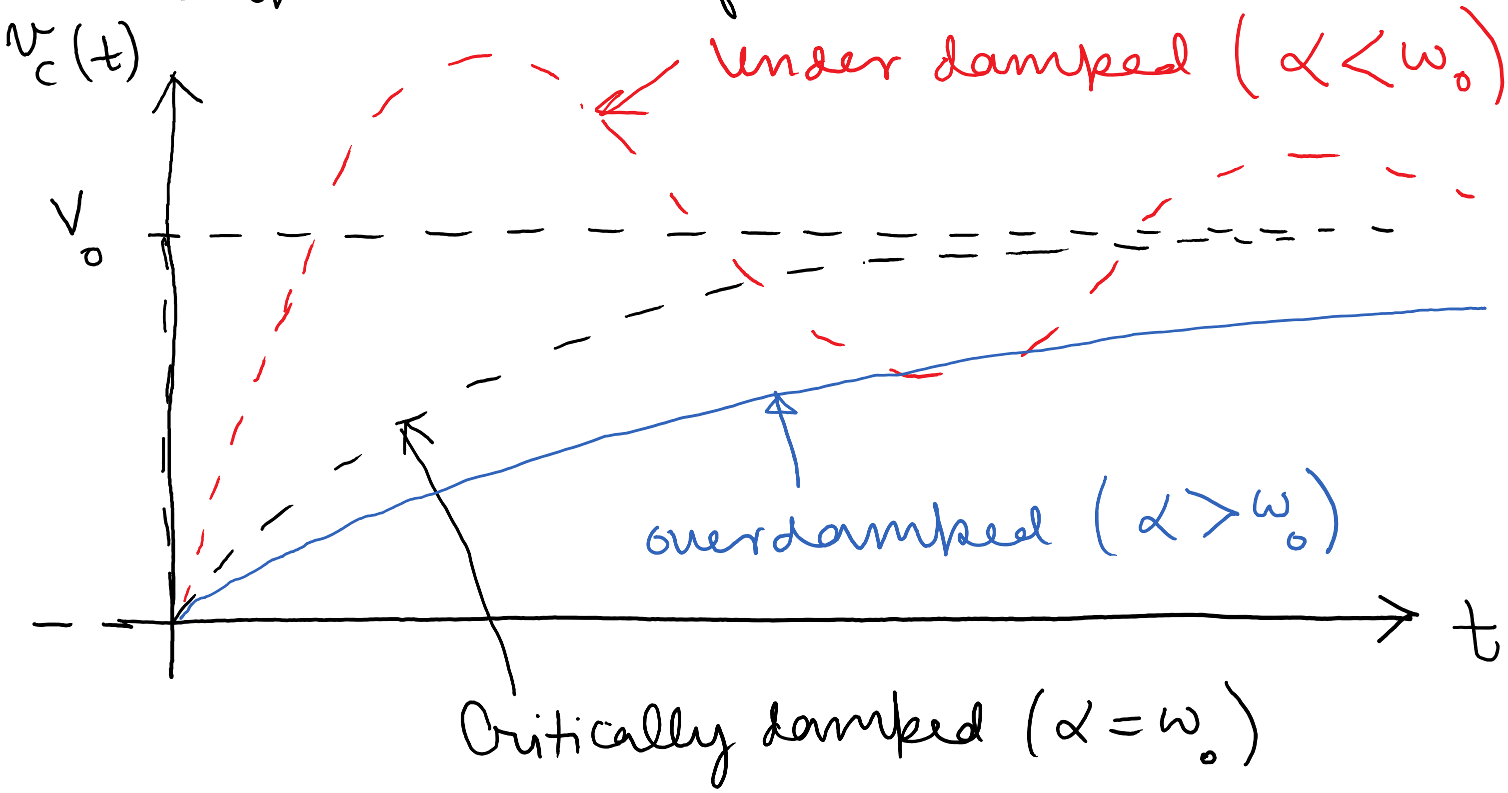
Root Locus Plot (used in Control)

$$V_c = \frac{\omega_0^2}{s^2 + 2\alpha s + \omega_0^2} \frac{V_o}{s}$$

poles: $s = 0$, $s = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$

$$V_c = V_o \omega_0^2 \left[\frac{A}{s} + \frac{B}{s + \alpha + \sqrt{\alpha^2 - \omega_0^2}} + \frac{C}{s + \alpha - \sqrt{\alpha^2 - \omega_0^2}} \right]$$

solve for A, B & C then take inverse Laplace transform



MATLAB example
$\alpha = \frac{\omega_0}{2}$
(underdamped)

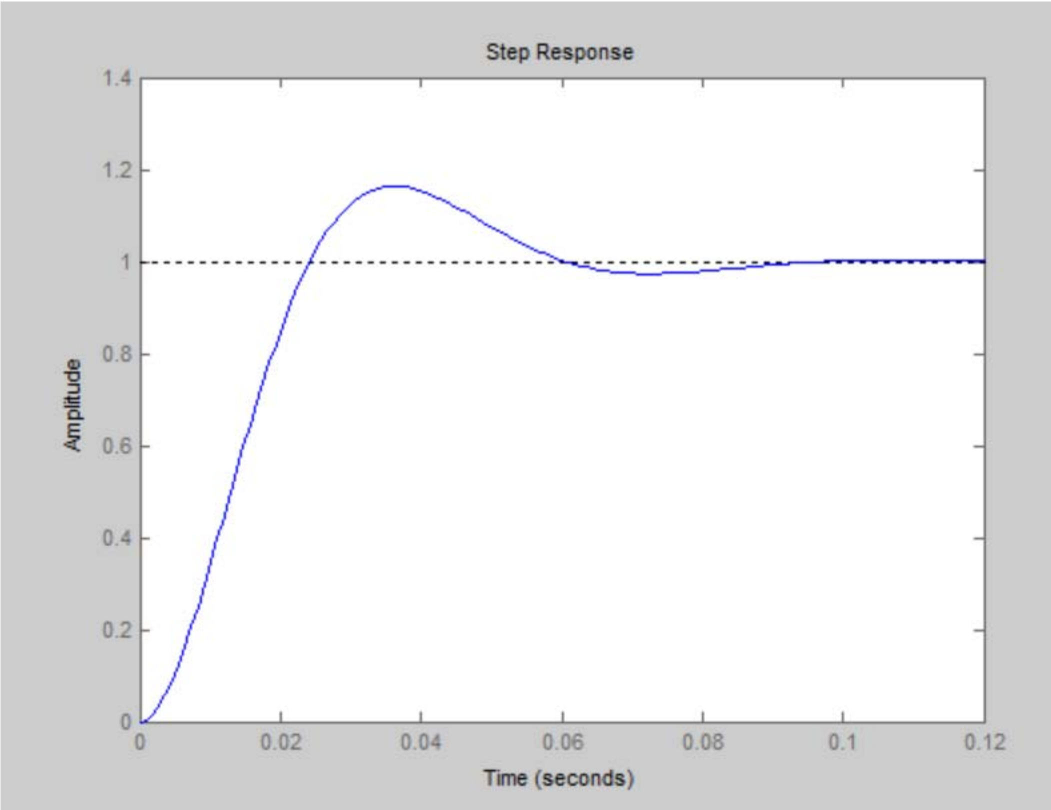
```
>> w0=100;
>> alpha=w0/2;
>> num=[0, 0, w0*w0];
>> den=[1, 2*alpha, w0*w0];
>> H=tf(num, den)

H =

       10000
  -------------------
  s^2 + 100 s + 10000

Continuous-time transfer function.

>> step(H)
>>
```



MATLAB Example

$\alpha = \omega_0$

(critically damped)

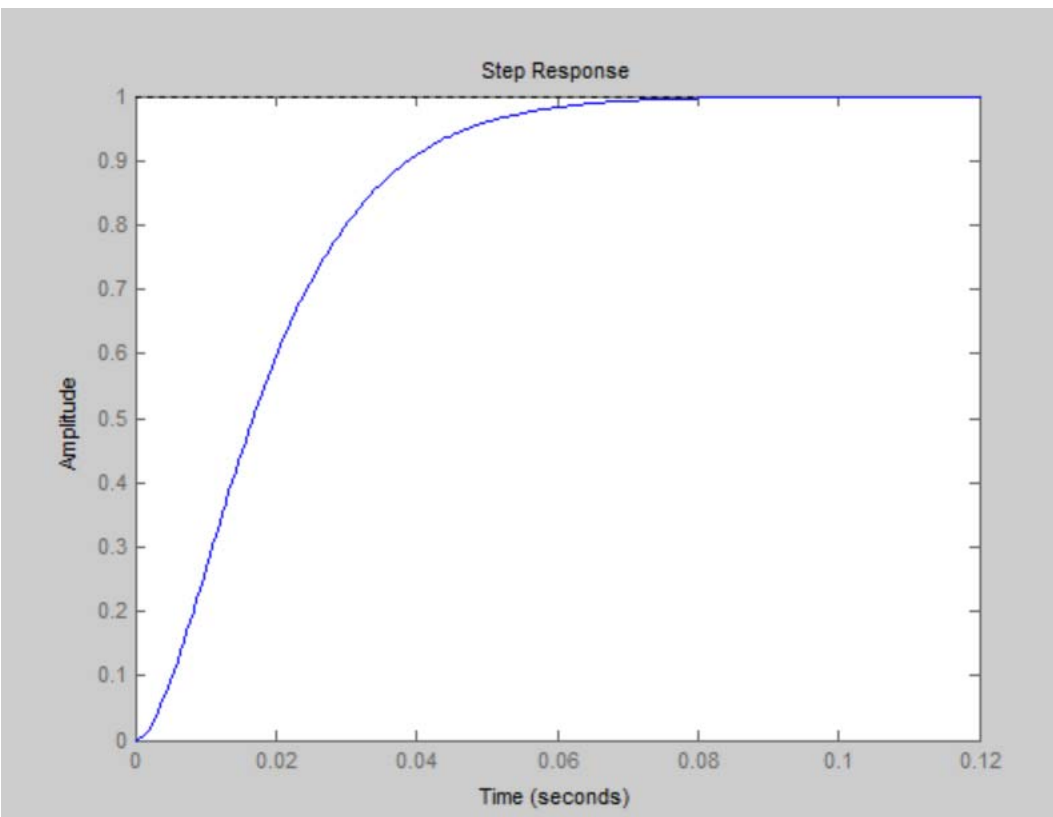
```
>> w0=100;
>> alpha=w0;
>> num=[0, 0, w0*w0];
>> den=[1, 2*alpha, w0*w0];
>> H=tf(num, den)

H =

       10000
  ------------------
  s^2 + 200 s + 10000

Continuous-time transfer function.

>> step(H)
>>
```



MATLAB Example
over damped
$\alpha = 2\omega_0$

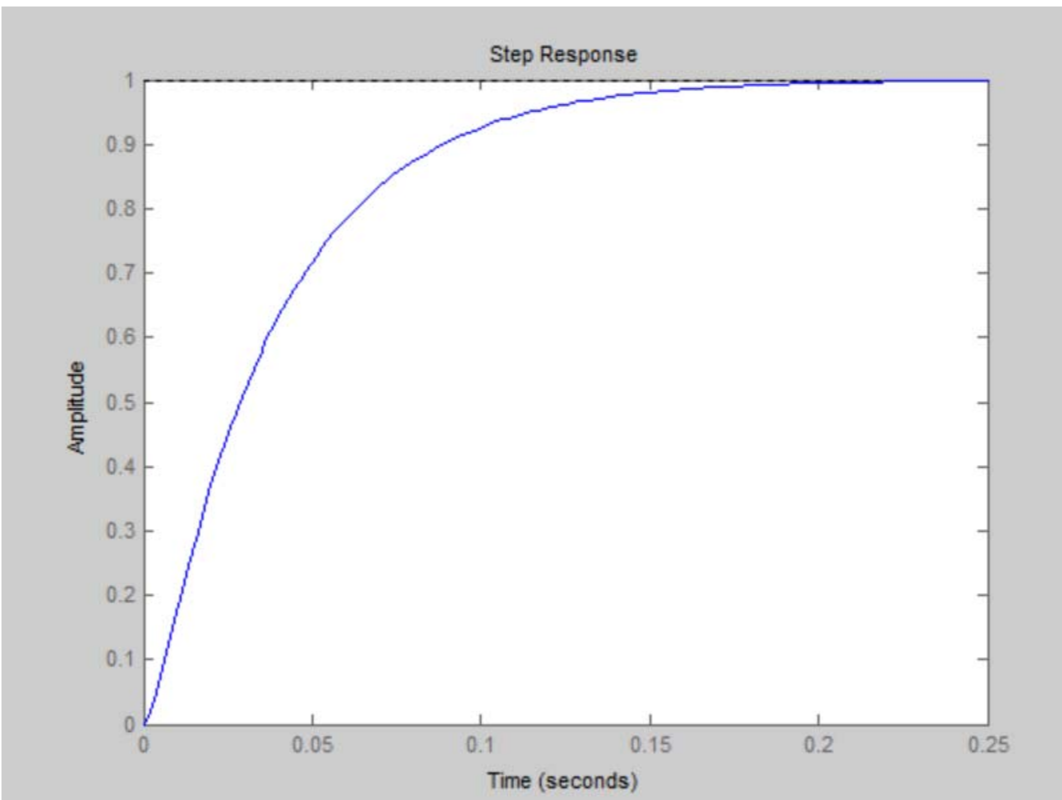
```
>> w0=100;
>> alpha=2*w0;
>> num=[0, 0, w0*w0];
>> den=[1, 2*alpha, w0*w0];
>> H=tf(num, den)

H =

       10000
  ------------------
  s^2 + 400 s + 10000

Continuous-time transfer function.

>> step(H)
>>
```



Screen clipping taken: 4/16/2014 11:06 AM poles of transfer function

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

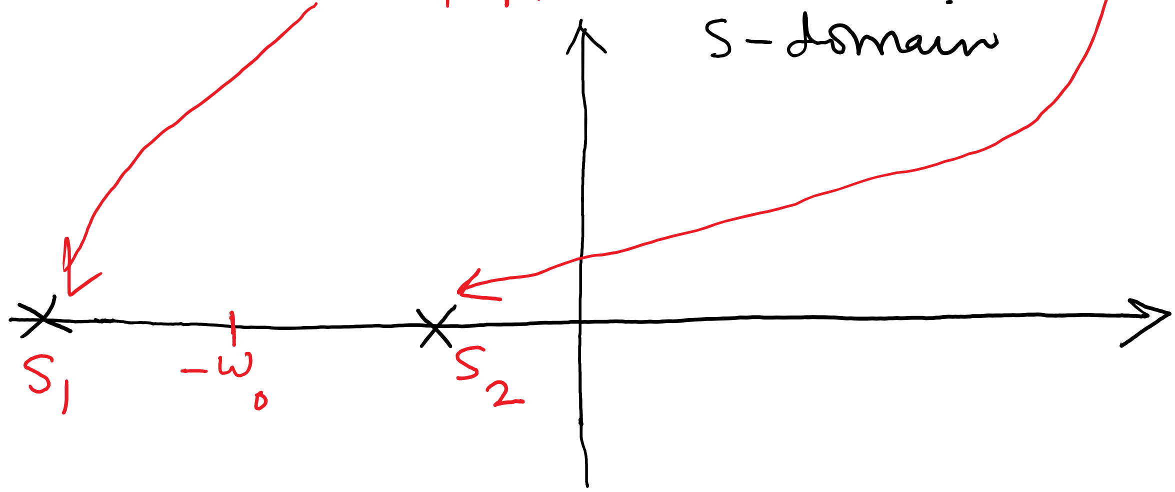
over damped. $(\alpha > \omega_0)$

two distinct real poles

$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$ $\qquad$ $s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$

exponential decay with time constant $\frac{1}{|s_1|}$

exponential decay with time constant $\frac{1}{|s_2|}$



Underdamped ($\alpha < \omega_0$)

poles of transfer function

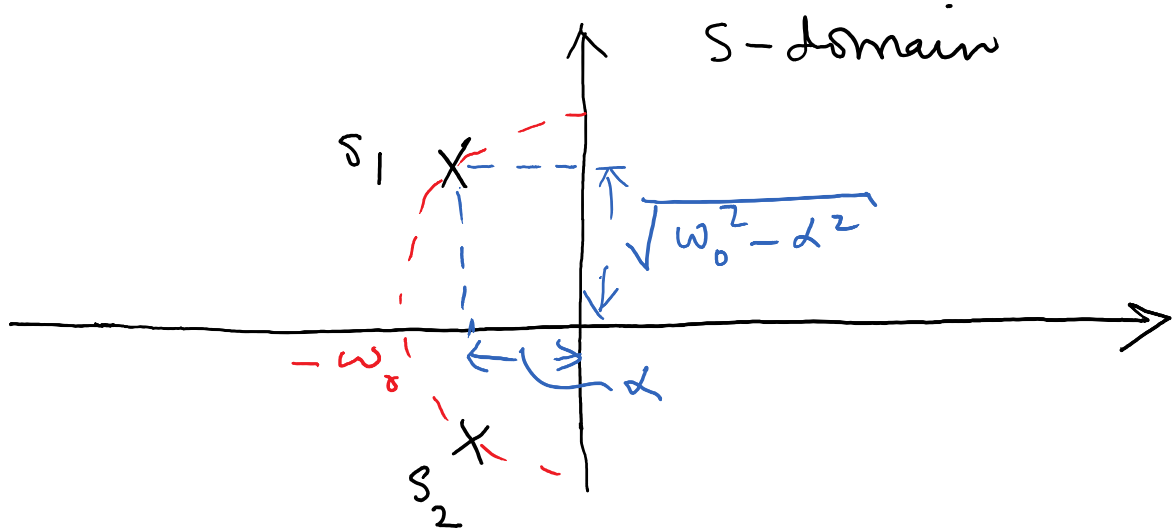
$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -\alpha \pm j\underbrace{\sqrt{\omega_0^2 - \alpha^2}}_{\text{real}}$$

over damped. ($\alpha > \omega_0$)

two distinct complex conjugate poles

$$s_1 = -\alpha + j\sqrt{\omega_0^2 - \alpha^2} \qquad s_2 = -\alpha - j\sqrt{\omega_0^2 - \alpha^2}$$

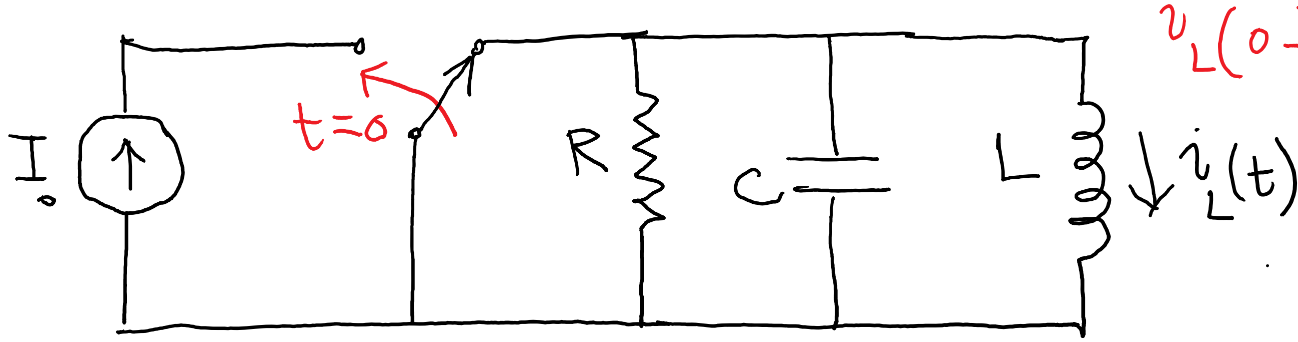
Sinusoidal frequency $\sqrt{\omega_0^2 - \alpha^2}$ with exponential decay $e^{-\alpha t}$ with time constant $\frac{1}{\alpha}$



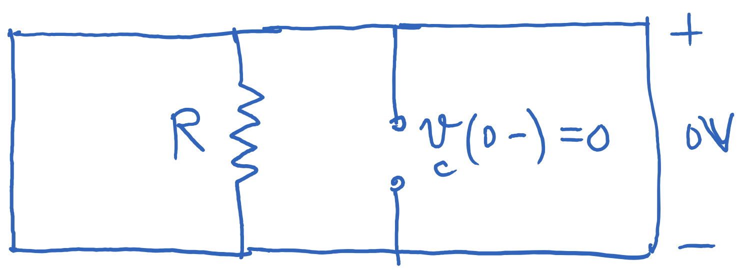
Parallel RLC Circuit

given

$i_L(0-) = 0$

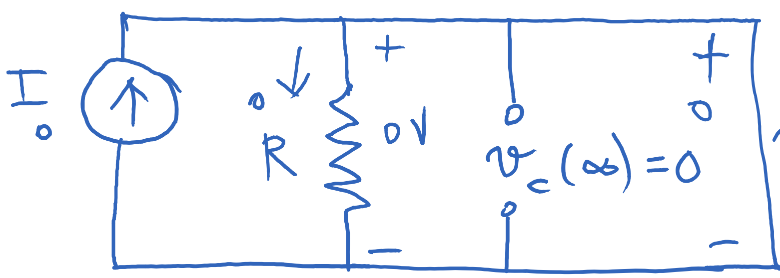


$t = 0-$ dc-conditions



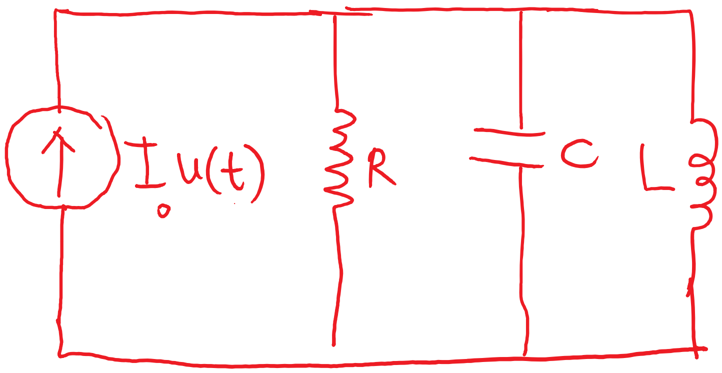
$v_c(0-) = 0$

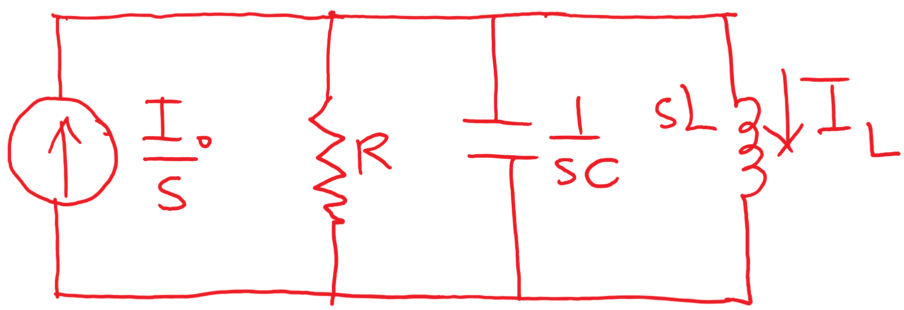
$t \to \infty$ dc-conditions



$i_L(\infty) = I_0$

$t > 0$





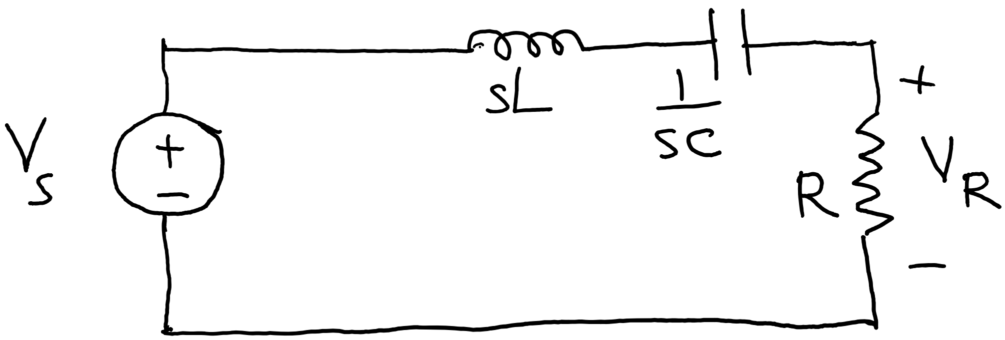
$$I_L = \frac{\frac{1}{sL}}{\frac{1}{R} + sC + \frac{1}{sL}} \frac{I_0}{s}$$ current division

$$= \frac{1}{CL} \frac{1}{s^2 + s\frac{1}{RL} + \frac{1}{CL}} \frac{I_0}{s}$$

define $\alpha = \frac{1}{2RL}$ $\quad \omega_0 = \frac{1}{\sqrt{CL}}$

$$I_L = \frac{\omega_0^2}{\underbrace{s^2 + 2\alpha s + \omega_0^2}_{\text{second order system}}} \frac{I_0}{s}$$

Frequency Response of RLC second order Circuits



zero initial conditions

$$V_R = \frac{R}{sL + \frac{1}{sC} + R} V_s$$

$$H(s) = \frac{V_R}{V_s} = \frac{R}{sL + \frac{1}{sC} + R}$$

$$H(s) = \frac{R}{L} \frac{s}{s^2 + s\frac{R}{L} + \frac{1}{LC}}$$

$$\alpha = \frac{R}{2L} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

Damping coefficient Resonant frequency

$$H(s) = \frac{2\alpha s}{s^2 + 2\alpha s + \omega_0^2}$$

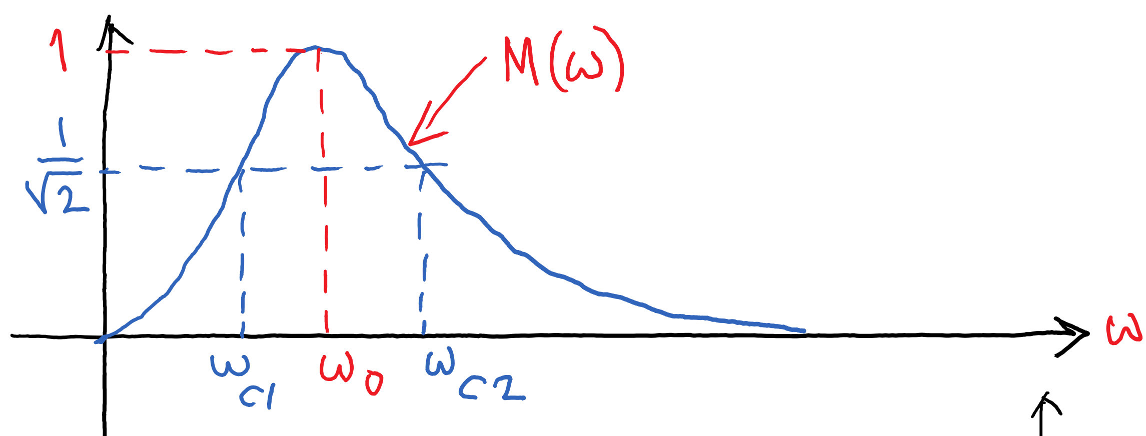
to frequency domain

$$H(j\omega) = \frac{2\alpha j\omega}{-\omega^2 + 2\alpha j\omega + \omega_0^2}$$

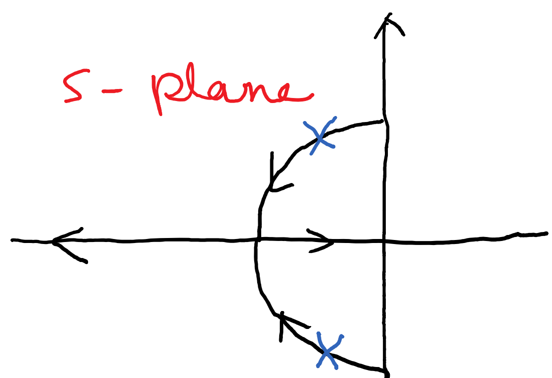
$$= \frac{j2\alpha\omega}{(\omega_0^2 - \omega^2) + j2\alpha\omega}$$

$$M(\omega) = |H(j\omega)| = \frac{2\alpha\omega}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\alpha^2\omega^2}}$$

Magnitude Frequency Response



$$BW = \omega_{c2} - \omega_{c1}$$



root locus plot

$$BW = \omega_{c2} - \omega_{c1}$$

$$\frac{1}{\sqrt{2}} = \frac{2\alpha\omega_c}{\sqrt{(\omega_o^2 - \omega_c^2) + 4\alpha^2\omega_c^2}}$$

$$\frac{1}{2} = \frac{4\alpha^2\omega_c^2}{(\omega_o^2 - \omega_c^2) + 4\alpha^2\omega_c^2}$$

$$(\omega_o^2 - \omega_c^2) + 4\alpha^2\omega_c^2 = 8\alpha^2\omega_c^2$$

$$(\omega_o^2 - \omega_c^2) = 4\alpha^2\omega_c^2$$

$$\omega_{c1} = -\alpha + \sqrt{\alpha^2 + \omega_o^2}$$

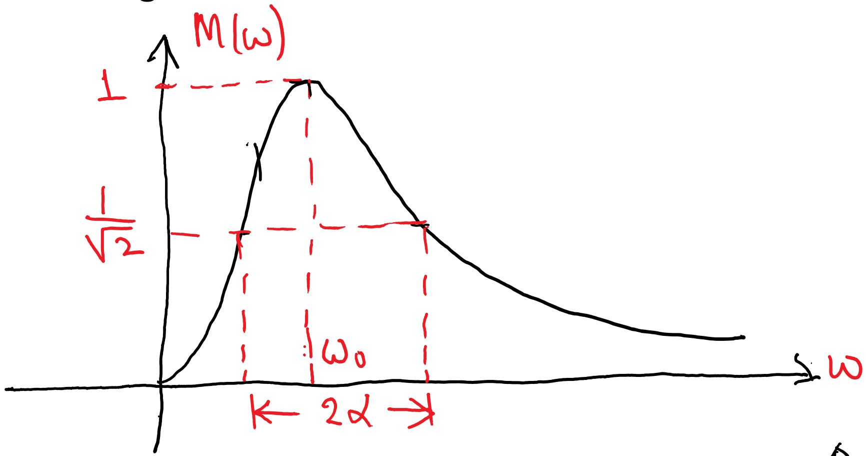
$$\omega_{c2} = \alpha + \sqrt{\alpha^2 + \omega_o^2}$$

$$BW = 2\alpha$$

$$\omega_{c1}\omega_{c2} = \left(\sqrt{\alpha^2 + \omega_o^2}\right)^2 - \alpha^2$$

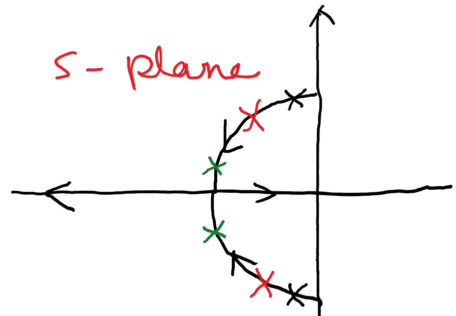
$$= \alpha^2 + \omega_o^2 - \alpha^2 = \omega_o^2$$

$\omega_o = \sqrt{\omega_{c1}\omega_{c2}}$ ← geometric mean



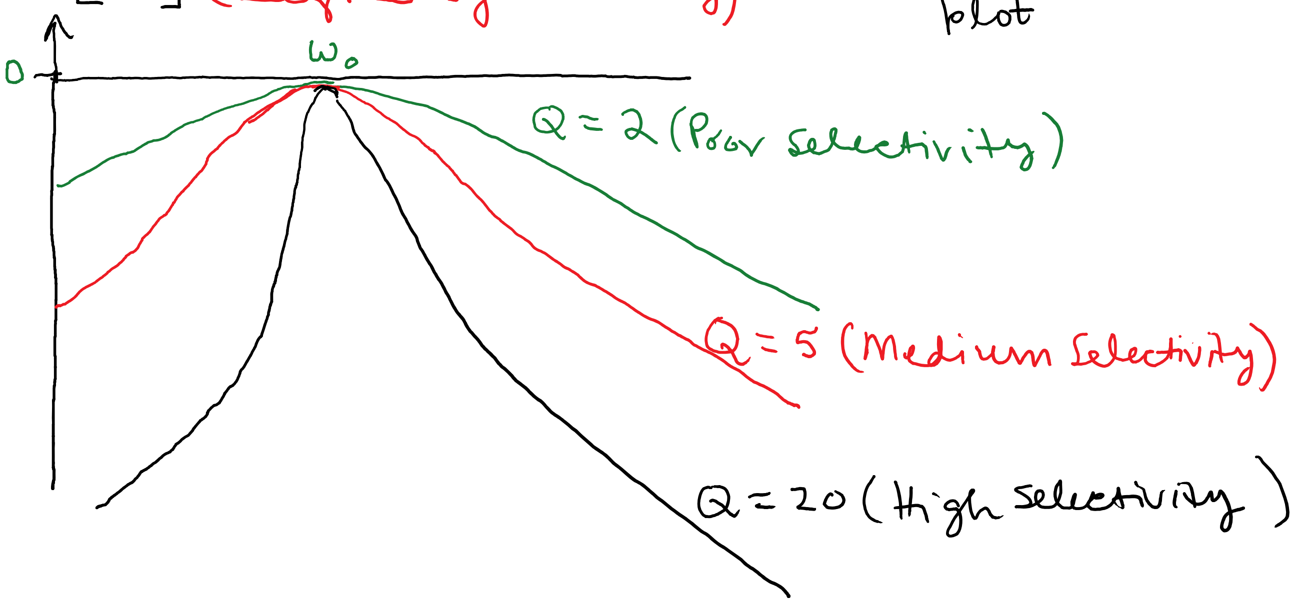
$$Q = \frac{\omega_0}{BW} = \frac{\omega_0}{2\alpha}$$

Quality Factor



root locus plot

M [dB] (Degree of selectivity)



Bode Plot of M

```
>> w0=400;
>> alpha=10;
>> num = [0, 2*alpha, 0];
>> den = [1, 2*alpha, w0*w0];
>> H = tf(num, den)

H =

        20 s
  --------------------
  s^2 + 20 s + 160000

Continuous-time transfer function.

>> bode(H)
```

MATLAB example

$Q = 20$

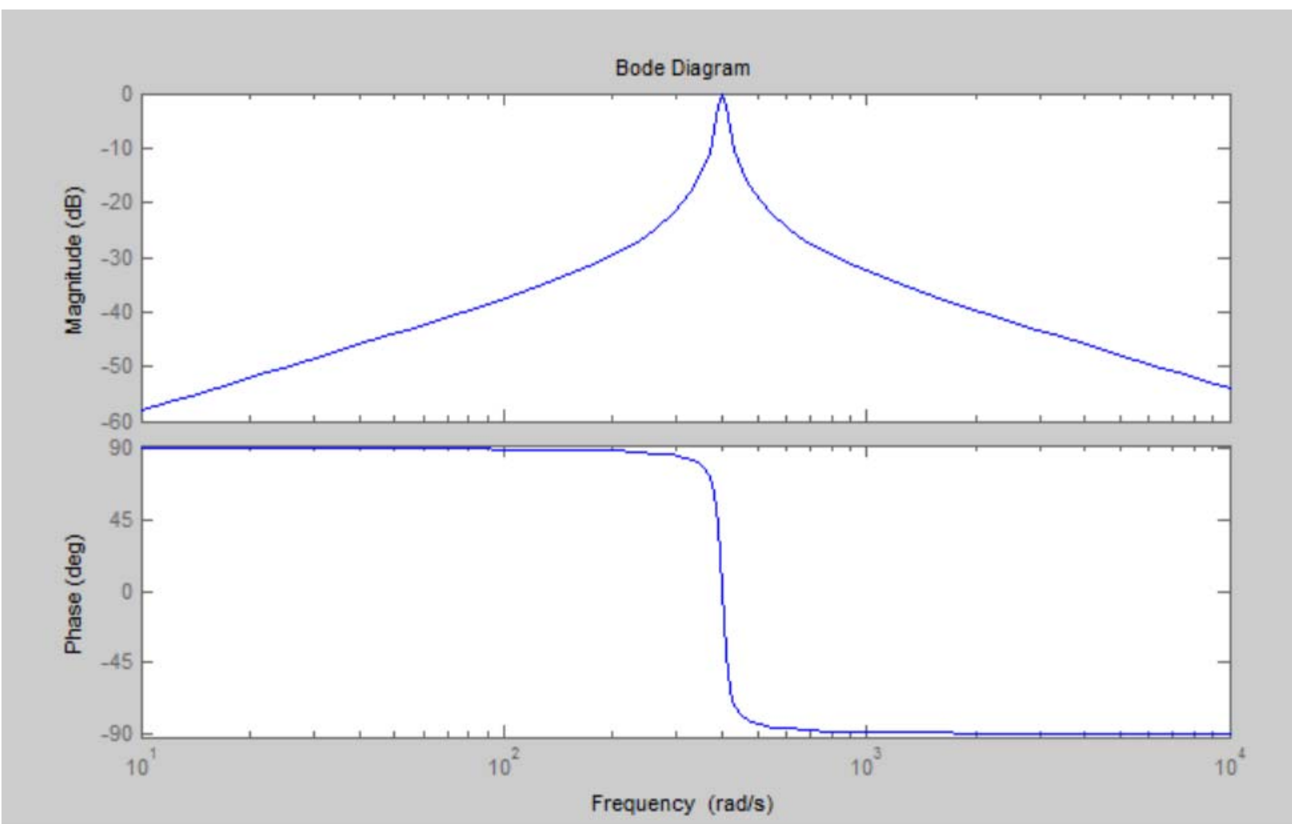
$\frac{\omega_0}{2\alpha} = 20$

$\alpha = \frac{\omega_0}{40}$

choose

$\omega_0 = 400$

$\alpha = \frac{400}{40} = 10$



MATLAB example

$Q = 0.25$

$\frac{\omega_0}{2\alpha} = 0.25$

$\alpha = \frac{\omega_0}{2 \times 0.25} = 2\omega_0$

choose

$\omega_0 = 400$

$\alpha = 800$

```
>> w0=400;
>> alpha=800;
>> num = [0, 2*alpha, 0];
>> den = [1, 2*alpha, w0*w0];
>> H = tf(num, den)

H =

        1600 s
  ---------------------
  s^2 + 1600 s + 160000

Continuous-time transfer function.

>> bode(H)
>>
```