2.4. EBRT in A,B,fA,fB,⊆ on (SD,INF).

In this section, we use the tree methodology described in section 2.1 to classify EBRT in A,B,fA,fB,⊆ on (SD,INF) and (ELG ∩ SD,INF). We handle both BRT settings at once, as they behave the same way for EBRT in A,B,fA,fB,⊆. In particular, we show that they are RCA_0 secure (see Definition 1.1.43).

We begin with a list of five Lemmas that we will need for documenting the classification.

LEMMA 2.4.1. Let f ∈ SD. There exist infinite A ⊆ B ⊆ N such that B ∪. fA = N and A = B ∩ fB.

Proof: By the BRT Fixed Point Theorem, section 1.3, let A be the unique A ⊆ N such that A = N\fA ∩ f(N\fA). Let B = N\fA. Clearly A ⊆ B and B ∪. fA = N. Also B ∩ fB = N\fA ∩ f(N\fA) = A.

Suppose A is finite. Then N\fA is cofinite and f(N\fA) is infinite. Hence their intersection is infinite, and so A is infinite. So we conclude that A is infinite. QED

LEMMA 2.4.2. Let f ∈ SD. There exist infinite A ⊆ B ⊆ N such that A ∪. fB = N, fA ⊆ B, and B ∩ fB ⊆ fA.

Proof: By the BRT Fixed Point Theorem, section 1.3, let B be the unique B ⊆ N such that B = N\fB ∪ f(N\fB). Let A = N\fB. Then A ⊆ B, fA ⊆ B. Now B ∩ fB = (N\fB ∪ f(N\fB)) ∩ fB = f(N\fB) ∩ fB ⊆ fA. Suppose A is finite. Then B = A ∪ fA is finite. Hence N\fB = A is infinite, which is a contradiction. Hence A is infinite. Therefore fA,B are infinite. QED

The following is a sharpening of the Complementation Theorem.

LEMMA 2.4.3. Let f ∈ SD and X ⊆ N. There exists a unique A such that A ⊆ X ⊆ A ∪. fA.

Proof: We will give a direct argument from scratch. Let f,X be as given. Define membership in A inductively as follows. Suppose membership in A for 0,...,n-1 has been defined. Define n ∈ A if and only if n ∈ X and n ∉ fA thus far. The construction is unique. QED

LEMMA 2.4.4. The following is false. For all $f \in ELG \cap SD$ there exist infinite $A \subseteq B \subseteq N$ such that $A \cap fB = \emptyset$ and $fB \subseteq B$.

Proof: Let f be given by Lemma 3.2.1, and let $A \subseteq B \subseteq N$, $A \cap fB = \emptyset$, and $fB \subseteq B$, where A is infinite. Just using $fB \subseteq B$, $B \neq \emptyset$, we see that fB is cofinite, and hence A is finite. This is the desired contradiction. QED

LEMMA 2.4.5. Let $f \in SD$. There is no nonempty $A \subseteq N$ such that $A \subseteq fA$.

Proof: Let n be the least element of A. Then $n \notin fA$. QED

Note that in the proofs of Lemmas 2.4.1, 2.4.2, 2.4.3, 2.4.5, we never used the fact that f is everywhere defined. Hence these Lemmas hold even for partially defined f. We will use Lemma 2.4.2 for partial f in section 2.5.

The 16 A,B,fA,fB pre elementary inclusions are as follows (see Definition 1.1.35).

$A \cap B \cap fA \cap fB = \emptyset$.
$A \cup B \cup fA \cup fB = N$.
$A \subseteq B \cup fA \cup fB$.
$B \subseteq A \cup fA \cup fB$.
$fA \subseteq A \cup B \cup fB$.
$fB \subseteq A \cup B \cup fA$.
$A \cap B \subseteq fA \cup fB$.
$A \cap fA \subseteq B \cup fB$.
$A \cap fB \subseteq B \cup fA$.
$B \cap fA \subseteq A \cup fB$.
$B \cap fB \subseteq A \cup fA$.
$fA \cap fB \subseteq A \cup B$.
$A \cap B \cap fA \subseteq fB$.
$A \cap B \cap fB \subseteq fA$.
$A \cap fA \cap fB \subseteq B$.
$B \cap fA \cap fB \subseteq A$.

The 9 A,B,fA,fB,$\subseteq$ elementary inclusions are as follows (see Definition 1.1.37).

$A \cap fA = \emptyset$.
$B \cup fB = N$.
$B \subseteq A \cup fB$.
$fB \subseteq B \cup fA$.
$A \subseteq fB$.

B ∩ fB ⊆ A ∪ fA.
fA ⊆ B.
A ∩ fB ⊆ fA.
B ∩ fA ⊆ A.

Our classification provides a determination of the subsets S of the above nine inclusions for which

(∀f ∈ SD)(∃A ⊆ B from INF)(S)
(∀f ∈ ELG ∩ SD)(∃A ⊆ B from INF)(S)

holds, where S is interpreted conjunctively.

We now build an RCA_0 classification for α (see Definition 2.1.9), where α is the BRT fragment: EBRT in A,B,fA,fB,⊆ on (SD,INF).

Recall that RCA_0 classifications for α are trees whose vertices are labeled with worklists. Our presentation of such trees in text, presents each vertex with a numerical label and the worklist label. (There are two special exceptions to this - see two paragraphs down).

The numerical label consists of finite sequences of small positive integers, in lexicographic order, reflecting the tree structure. The worklist label is presented as a list of elementary inclusions, where the items in the first part of the worklist end with colons, and the items in the second part of the worklist end with periods.

We begin with the presentation of the root of the classification tree, which does not have a numerical label, but instead has a label stating the BRT fragment(s) we are classifying. Its worklist label is a list of the elementary inclusions. It is immediately followed by the unique son of the root, with the same non numerical label appending with *, and its worklist label is a permutation of the list of the elementary inclusions. Note that these elementary inclusions end with periods because the first part of the worklist is empty.

If a presented vertex is terminal, then it must be documented that it is entirely α,T correct, in the sense that the format obtained by ignoring the colons of the worklist is α,T correct.

If a worklist has numerical label $n_1.n_2. \ldots n_k.$, then either this worklist is terminal (no sons), or it has a unique son labeled $n_1.n_2. \ldots n_k.*$. In the latter case, there is a documented α,RCA_0 reduction from the former's worklist to the latter's worklist (see Definition 2.1.5).

If a worklist is labeled $n_1.n_2. \ldots n_k.*$, then it is either terminal, or has one or more sons, none of which end with *. The worklist of the last son is terminal.

The symbols # k that appear right under the label of a vertex with a starred label indicates the number of sons. These # k are placed under the numerical label.

We begin with the root worklist. It consists of the 9 A,B,fA,fB,⊆ elementary inclusions above.

The root worklist is followed by an α,RCA_0 reduction, which permutes the entries in a perhaps strategic way. This starred worklist has five sons, as indicated by # 5.

EBRT in A,B,fA,fB,⊆ on (SD,INF), (ELG ∩ SD,INF).

A ∩ fA = ∅.
B ∪ fB = N.
B ⊆ A ∪ fB.
fB ⊆ B ∪ fA.
A ⊆ fB.
B ∩ fB ⊆ A ∪ fA.
fA ⊆ B.
A ∩ fB ⊆ fA.
B ∩ fA ⊆ A.

EBRT in A,B,fA,fB,⊆ on (SD,INF), (ELG ∩ SD,INF).*
5

A ∩ fA = ∅.
B ∪ fB = N.
fA ⊆ B.
A ⊆ fB.
B ⊆ A ∪ fB.
fB ⊆ B ∪ fA.
A ∩ fB ⊆ fA.
B ∩ fA ⊆ A.
B ∩ fB ⊆ A ∪ fA.

LIST 1.

A ∩ fA = ∅:
B ∪ fB = N.
fA ⊆ B.
A ⊆ fB.
B ⊆ A ∪ fB.
fB ⊆ B ∪ fA.
A ∩ fB ⊆ fA. A ∩ fB = ∅.
B ∩ fA ⊆ A. B ∩ fA = ∅.
B ∩ fB ⊆ A ∪ fA.

LIST 1*.
5

A ∩ fA = ∅:
B ∩ fA = ∅.
A ∩ fB = ∅.
fA ⊆ B.
A ⊆ fB.
B ∪ fB = N.
B ⊆ A ∪ fB.
fB ⊆ B ∪ fA.
B ∩ fB ⊆ A ∪ fA.

LIST 1.1.

A ∩ fA = ∅:
B ∩ fA = ∅:
A ∩ fB = ∅.
fA ⊆ B. B ∩ fA = fA = ∅. No.
A ⊆ fB.
B ∪ fB = N.
B ⊆ A ∪ fB.
fB ⊆ B ∪ fA.
B ∩ fB ⊆ A ∪ fA. B ∩ fB ⊆ A.

LIST 1.1.*
3

A ∩ fA = ∅:
B ∩ fA = ∅:
A ∩ fB = ∅.
A ⊆ fB.
B ∪ fB = N.
B ⊆ A ∪ fB.
fB ⊆ B ∪ fA.
B ∩ fB ⊆ A.

LIST 1.1.1.

A ∩ fA = ∅:
B ∩ fA = ∅:
A ∩ fB = ∅:
A ⊆ fB. No.
B ∪ fB = N.
B ⊆ A ∪ fB.
fB ⊆ B ∪ fA.
B ∩ fB ⊆ A. B ∩ fB = ∅.

LIST 1.1.1.*
0

A ∩ fA = ∅:
B ∩ fA = ∅:
A ∩ fB = ∅:
B ∪ fB = N.
B ⊆ A ∪ fB.
fB ⊆ B ∪ fA.
B ∩ fB = ∅.

Entirely RCA_0 correct. By the Complementation Theorem, let A ∪. fA = N. Set B = A.

LIST 1.1.2.

A ∩ fA = ∅:
B ∩ fA = ∅:
A ⊆ fB:
B ∪ fB = N.
B ⊆ A ∪ fB. B ⊆ fB. No. Lemma 2.4.5.
fB ⊆ B ∪ fA.
B ∩ fB ⊆ A.

LIST 1.1.2.*
0

A ∩ fA = ∅:
B ∩ fA = ∅:
A ⊆ fB:
B ∪ fB = N.
fB ⊆ B ∪ fA.
B ∩ fB ⊆ A.

Entirely RCA_0 correct. By Lemma 2.4.1, let A ⊆ B ⊆ N, B ∪. fA = N, A = B ∩ fB.

LIST 1.1.3.

A ∩ fA = ∅:
B ∩ fA = ∅:
B ∪ fB = N:
B ⊆ A ∪ fB.
fB ⊆ B ∪ fA.
B ∩ fB ⊆ A.

Entirely RCA_0 correct. By the Complementation Theorem, let A ∪. fA = N. Set B = A.

LIST 1.2.

A ∩ fA = ∅:
A ∩ fB = ∅:
fA ⊆ B.
A ⊆ fB. No.
B ∪ fB = N.
B ⊆ A ∪ fB.
fB ⊆ B ∪ fA.
B ∩ fB ⊆ A ∪ fA. B ∩ fB ⊆ fA.

LIST 1.2.*
2

A ∩ fA = ∅:
A ∩ fB = ∅:
fA ⊆ B.
B ∪ fB = N.
B ⊆ A ∪ fB.
fB ⊆ B ∪ fA.
B ∩ fB ⊆ fA.

LIST 1.2.1.

A ∩ fA = ∅:
A ∩ fB = ∅:
fA ⊆ B:
B ∪ fB = N.
B ⊆ A ∪ fB.
fB ⊆ B ∪ fA. fB ⊆ B. No. Lemma 2.4.4.
B ∩ fB ⊆ fA.

LIST 1.2.1.*
0

A ∩ fA = ∅:
A ∩ fB = ∅:
fA ⊆ B:
B ∪ fB = N.
B ⊆ A ∪ fB.
B ∩ fB ⊆ fA.

Entirely RCA_0 correct. By Lemma 2.4.2, let A ⊆ B ⊆ N, A ∪. fB = N, fA ⊆ B, B ∩ fB ⊆ fA.

LIST 1.2.2.

A ∩ fA = ∅:
A ∩ fB = ∅:
B ∪ fB = N:
B ⊆ A ∪ fB.
fB ⊆ B ∪ fA.
B ∩ fB ⊆ fA.

Entirely RCA_0 correct. By the Complementation Theorem, let A ∪. fA = N. Set B = A.

LIST 1.3.

A ∩ fA = ∅:
fA ⊆ B:
A ⊆ fB.
B ∪ fB = N.
B ⊆ A ∪ fB.
fB ⊆ B ∪ fA. fB ⊆ B.
B ∩ fB ⊆ A ∪ fA.

LIST 1.3.*
2

A ∩ fA = ∅:
fA ⊆ B:
A ⊆ fB.
B ∪ fB = N.
B ⊆ A ∪ fB.
fB ⊆ B.
B ∩ fB ⊆ A ∪ fA.

LIST 1.3.1.

A ∩ fA = ∅:
fA ⊆ B:
A ⊆ fB:
B ∪ fB = N.
B ⊆ A ∪ fB. B ⊆ fB. No. Lemma 2.4.5.
fB ⊆ B.
B ∩ fB ⊆ A ∪ fA.

LIST 1.3.1.*
0

A ∩ fA = ∅:
fA ⊆ B:
A ⊆ fB:
B ∪ fB = N.
fB ⊆ B.
B ∩ fB ⊆ A ∪ fA.

Entirely RCA_0 correct. By Lemma 2.4.3, let A ⊆ fN ⊆ A ∪. fA. Set B = N.

LIST 1.3.2.

A ∩ fA = ∅:
fA ⊆ B:
B ∪ fB = N:
B ⊆ A ∪ fB.
fB ⊆ B.
B ∩ fB ⊆ A ∪ fA.

Entirely RCA_0 correct. By the Complementation Theorem, let A ∪. fA = N. Set B = N.

LIST 1.4.

A ∩ fA = ∅:
A ⊆ fB:
B ∪ fB = N.
B ⊆ A ∪ fB. B ⊆ fB. No. Lemma 2.4.5.
fB ⊆ B ∪ fA.
B ∩ fB ⊆ A ∪ fA.

LIST 1.4.*
0

A ∩ fA = ∅:

A ⊆ fB:
B ∪ fB = N.
fB ⊆ B ∪ fA.
B ∩ fB ⊆ A ∪ fA.

Entirely RCA_0 correct. By Lemma 2.4.3, let A ⊆ fN ⊆ A ∪. fA. Set B = N.

LIST 1.5.

A ∩ fA = ∅:
B ∪ fB = N:
B ⊆ A ∪ fB.
fB ⊆ B ∪ fA.
B ∩ fB ⊆ A ∪ fA.

Entirely RCA_0 correct. By the Complementation Theorem, let A ∪. fA = N. Set B = A.

LIST 2.

B ∪ fB = N:
fA ⊆ B.
A ⊆ fB.
B ⊆ A ∪ fB. A ∪ fB = N.
fB ⊆ B ∪ fA. B ∪ fA = N.
A ∩ fB ⊆ fA.
B ∩ fA ⊆ A.
B ∩ fB ⊆ A ∪ fA.

LIST 2.*
2

B ∪ fB = N:
A ⊆ fB.
fA ⊆ B.
A ∪ fB = N.
B ∪ fA = N.
A ∩ fB ⊆ fA.
B ∩ fA ⊆ A.
B ∩ fB ⊆ A ∪ fA.

LIST 2.1.

B ∪ fB = N:
A ⊆ fB:
fA ⊆ B.

A ∪ fB = N. fB = N. No. Lemma 2.4.5.
B ∪ fA = N.
A ∩ fB ⊆ fA. A ⊆ fA. No. Lemma 2.4.5.
B ∩ fA ⊆ A.
B ∩ fB ⊆ A ∪ fA.

LIST 2.1.*
2

B ∪ fB = N:
A ⊆ fB:
fA ⊆ B.
B ∪ fA = N.
B ∩ fA ⊆ A.
B ∩ fB ⊆ A ∪ fA.

LIST 2.1.1.

B ∪ fB = N:
A ⊆ fB:
fA ⊆ B:
B ∪ fA = N. B = N.
B ∩ fA ⊆ A. fA ⊆ A.
B ∩ fB ⊆ A ∪ fA.

LIST 2.1.1.*
0

B ∪ fB = N:
A ⊆ fB:
fA ⊆ B:
B = N.
fA ⊆ A.
B ∩ fB ⊆ A ∪ fA.

Entirely RCA_0 correct. Set A = fN, B = N.

LIST 2.1.2.

B ∪ fB = N:
A ⊆ fB:
B ∪ fA = N:
B ∩ fA ⊆ A.
B ∩ fB ⊆ A ∪ fA.

Entirely RCA_0 correct. Let B = N, A = fN.

LIST 2.2.

B ∪ fB = N:
fA ⊆ B:
A ∪ fB = N.
B ∪ fA = N.
A ∩ fB ⊆ fA.
B ∩ fA ⊆ A.
B ∩ fB ⊆ A ∪ fA.

Entirely RCA_0 correct. Set A = B = N.

LIST 3.

fA ⊆ B:
A ⊆ fB.
B ⊆ A ∪ fB.
fB ⊆ B ∪ fA. fB ⊆ B.
A ∩ fB ⊆ fA.
B ∩ fA ⊆ A. fA ⊆ A.
B ∩ fB ⊆ A ∪ fA.

LIST 3*.
3

fA ⊆ B:
fA ⊆ A.
A ⊆ fB.
B ⊆ A ∪ fB.
fB ⊆ B.
A ∩ fB ⊆ fA.
B ∩ fB ⊆ A ∪ fA.

LIST 3.1.

fA ⊆ B:
fA ⊆ A:
A ⊆ fB.
B ⊆ A ∪ fB.
fB ⊆ B.
A ∩ fB ⊆ fA.
B ∩ fB ⊆ A ∪ fA. B ∩ fB ⊆ A.

LIST 3.1.*
2

fA ⊆ B:

fA ⊆ A:
A ⊆ fB.
B ⊆ A ∪ fB.
fB ⊆ B.
A ∩ fB ⊆ fA.
B ∩ fB ⊆ A.

LIST 3.1.1.

fA ⊆ B:
fA ⊆ A:
A ⊆ fB:
B ⊆ A ∪ fB. B ⊆ fB. No. Lemma 2.4.5.
fB ⊆ B.
A ∩ fB ⊆ fA. A ⊆ fA. No. Lemma 2.4.5.
B ∩ fB ⊆ A.

LIST 3.1.1.*
0

fA ⊆ B:
fA ⊆ A:
A ⊆ fB:
fB ⊆ B.
B ∩ fB ⊆ A.

Entirely RCA_0 correct. Set A = fN, B = N.

LIST 3.1.2.

fA ⊆ B:
fA ⊆ A:
B ⊆ A ∪ fB:
fB ⊆ B.
A ∩ fB ⊆ fA.
B ∩ fB ⊆ A.

Entirely RCA_0 correct. Set A = B = N.

LIST 3.2.

fA ⊆ B:
A ⊆ fB:
B ⊆ A ∪ fB. B ⊆ fB. No. Lemma 2.4.5.
fB ⊆ B.
A ∩ fB ⊆ fA. A ⊆ fA. No. Lemma 2.4.5.
B ∩ fB ⊆ A ∪ fA.

LIST 3.2.*
0

fA ⊆ B:
A ⊆ fB:
fB ⊆ B.
B ∩ fB ⊆ A ∪ fA.

Entirely RCA_0 correct. Set A = fN, B = N.

LIST 3.3.

fA ⊆ B:
B ⊆ A ∪ fB:
fB ⊆ B.
A ∩ fB ⊆ fA.
B ∩ fB ⊆ A ∪ fA.

Entirely RCA_0 correct. Set A = B = N.

LIST 4.

A ⊆ fB:
B ⊆ A ∪ fB. B ⊆ fB. No. Lemma 2.4.5.
fB ⊆ B ∪ fA.
A ∩ fB ⊆ fA. A ⊆ fA. No. Lemma 2.4.5.
B ∩ fA ⊆ A.
B ∩ fB ⊆ A ∪ fA.

LIST 4.*
0

A ⊆ fB:
fB ⊆ B ∪ fA.
B ∩ fA ⊆ A.
B ∩ fB ⊆ A ∪ fA.

Entirely RCA_0 correct. Set A = fN, B = N.

LIST 5.

B ⊆ A ∪ fB:
fB ⊆ B ∪ fA.
A ∩ fB ⊆ fA.
B ∩ fA ⊆ A.
B ∩ fB ⊆ A ∪ fA.

Entirely RCA_0 correct. Set A = B = N.

THEOREM 2.4.6. EBRT in A,B,fA,fB,⊆ on (SD,INF) and (ELG ∩ SD,INF) have the same correct formats. EBRT in A,B,fA,fB,⊆ on (SD,INF) and (ELG ∩ SD,INF) are RCA_0 secure.

Proof: We have presented an RCA_0 classification of EBRT in A,B,fA,fB,⊆ on (SD,INF), (ELG ∩ SD,INF) in the sense of the tree methodology of section 2.1. All of the documentation works equally well on (SD,INF) and (ELG ∩ SD,INF), and we have remained within RCA_0. QED

THEOREM 2.4.7. There are at most 18 maximally α correct α formats, where α is EBRT in A,B,fA,fB,⊆ on (SD,INF), (ELG ∩ SD,INF).

Proof: Here is the list of numerical labels of terminal vertices in the RCA_0 classification of EBRT in A,B,fA,fB,⊆ on (SD,INF), (ELG ∩ SD,INF) given above:
1.1.1.*
1.1.2.*
1.1.3.
1.2.1.*
1.2.2.
1.3.1.*
1.3.2.
1.4.*
1.5.
2.1.1.*
2.1.2.
2.2.
3.1.1.*
3.1.2.
3.2.*
3.3.
4.*
5.
The count is 18. Apply Theorem 2.1.5. QED