2.5. EBRT in A,B,fA,fB,⊆ on (ELG,INF).

In this section, we use the tree methodology described in section 2.1 to analyze EBRT in A,B,fA,fB,⊆ on (ELG,INF) and (EVSD,INF). We handle both BRT settings at once, as they behave the same way for EBRT in A,B,fA,fB,⊆. In particular, we show that they are RCA_0 secure (see Definition 1.1.43).

Some of this treatment is the same as for EBRT in A,B,fA,fB,⊆ on (SD,INF) given in section 2.4. However, many new features appear that makes this section considerably more involved than section 2.4.

A key difference between EBRT in A,B,fA,fB,⊆ on (SD,INF) and on (ELG,INF) is that the Compelmentation Theorem holds on (SD,INF), yet fails on (ELG,INF). E.g., it fails for $f(x) = 2x$.

Let $f:N^k \to N$ be partial. Define the following series of sets by induction $i \geq 1$.

$$S_1 = N.$$
$$S_{i+1} = N\backslash fS_i.$$

LEMMA 2.5.1. $S_2 \subseteq S_4 \subseteq S_6 \subseteq \ldots \subseteq \ldots \subseteq S_5 \subseteq S_3 \subseteq S_1$. I.e., for all $i \geq 1$, $S_{2i} \subseteq S_{2i+2} \subseteq S_{2i+1} \subseteq S_{2i-1}$.

Proof: We argue by induction on $i \geq 1$. The basis case is

$$S_2 \subseteq S_4 \subseteq S_3 \subseteq S_1.$$

To see this, clearly

$S_3 \subseteq S_1$.
$N\backslash S_1 \subseteq N\backslash S_3$.
$S_2 \subseteq S_4$.
$S_2 \subseteq S_1$.
$N\backslash S_1 \subseteq N\backslash S_2$.
$S_2 \subseteq S_3$.
$fS_2 \subseteq fS_3$.
$N\backslash fS_3 \subseteq N\backslash fS_2$.
$S_4 \subseteq S_3$.

Now assume the induction hypothesis

$$S_{2i} \subseteq S_{2i+2} \subseteq S_{2i+1} \subseteq S_{2i-1}.$$

Then

$$fS_{2i} \subseteq fS_{2i+2} \subseteq fS_{2i+1} \subseteq fS_{2i-1}.$$
$$N\backslash fS_{2i-1} \subseteq N\backslash fS_{2i+1} \subseteq N\backslash fS_{2i+2} \subseteq N\backslash fS_{2i}.$$
$$S_{2i} \subseteq S_{2i+2} \subseteq S_{2i+3} \subseteq S_{2i+1}.$$
$$fS_{2i} \subseteq fS_{2i+2} \subseteq fS_{2i+3} \subseteq fS_{2i+1}.$$
$$N\backslash fS_{2i+1} \subseteq N\backslash fS_{2i+3} \subseteq N\backslash fS_{2i+2} \subseteq N\backslash fS_{2i}.$$
$$S_{2i+2} \subseteq S_{2i+4} \subseteq S_{2i+3} \subseteq S_{2i+1}.$$

QED

LEMMA 2.5.2. Let $f:N^k \rightarrow N$ be partial, where each $f^{-1}(n)$ is finite. Let $A = S_2 \cup S_4 \cup ...$, and $B = S_1 \cap S_3 \cap ...$. Then $A \subseteq B$, $A = N\backslash fB$, $B = N\backslash fA$.

Proof: Let A,B be as given. By Lemma 2.5.1, $A \subseteq B$.

Fix $i \geq 1$. S2i = N\fS2i-1, S2i ∩ fS2i-1 = ∅, S2i ∩ fB = ∅. Since $i \geq 1$ is arbitrary, $A \cap fB = \emptyset$. I.e., $A \subseteq N\backslash fB$.

Since $S_{2i+1} = N\backslash fS_{2i}$, we see that for all $j \geq i$, $S_{2i+1} \cap fS_{2j} = \emptyset$. Hence $S_{2i+1} \cap fA = \emptyset$. Since $i \geq 1$ is arbitrary, $B \cap fA = \emptyset$. I.e., $B \subseteq N\backslash fA$.

Now let $n \in N\backslash fB$. We claim that for some $j \geq 0$, $n \notin fS_{2j+1}$. Suppose that for all $j \geq 0$, $n \in fS_{2j+1}$. Since $f^{-1}(n)$ is finite, there exists $x \in f^{-1}(n)$ which lies in infinitely many S_{2j+1}. Hence there exists $x \in f^{-1}(n)$ such that $x \in B$. Therefore $n \in fB$. This establishes the claim. Fix $j \geq 0$ such that $n \notin fS_{2j+1}$. Then $n \in S_{2j+2}$, and so $n \in A$. This establishes that $A = N\backslash fB$.

Finally, let $n \in N\backslash fA$. Then for all i, $n \notin fS_{2i}$. Hence for all j, $n \in S_{2j+1}$. Therefore $n \in B$. This estabslihes that $B = N\backslash fA$. QED

LEMMA 2.5.3. Let $f:[0,n]^k \rightarrow [0,n]$ be partial, $n \geq 0$. There exist $A \subseteq B \subseteq [0,n]$ such that $A = [0,n]\backslash fB$ and $B = [0,n]\backslash fA$.

Proof: Let n,f be as given. Obviously f:Nk → N is partial, and each f-1(n) is finite. By Lemma 2.5.2, let $A = S_2 \cup S_4 \cup ...$, and $B = S_1 \cap S_3 \cap ...$. Then $A \subseteq B$, $A = N\backslash fB$, $B = N\backslash fA$. Note that $A \cap [0,n] \subseteq B \cap [0,n]$, $A \cap [0,n] = [0,n]\backslash fB$, $B \cap [0,n]\backslash fA$. QED

LEMMA 2.5.4. For all $f \in EVSD$ there exist infinite $A \subseteq B \subseteq N$ such that $B \cup. fA = A \cup fB = N$.

Proof: Let $f \in EVSD$. Let $n \geq 1$ be such that $|x| \geq n \rightarrow f(x) > |x|$. Let f' be the restriction of f to those elements of $[0,n-1]^k$ whose value lies in $[0,n-1]$. Then $f':[0,n-1]^k \rightarrow [0,n-1]$ is partial.

By Lemma 2.5.3, let $A' \subseteq B' \subseteq [0,n-1]$, where $A' = [0,n-1]\backslash f'B'$ and $B' = [0,n-1]\backslash f'A'$.

We now define the required A,B by induction. Membership in A,B for $m < n$ is just membership in A',B'. Thus for all $m < n$,

$$m \in B \leftrightarrow m \in B' \leftrightarrow m \notin f'A' \leftrightarrow m \notin fA.$$
$$m \in A \leftrightarrow m \in A' \leftrightarrow m \notin f'B' \leftrightarrow m \notin fB.$$

Now suppose membership in A,B has been defined for all $0 \leq i < m$, where $m \geq n$, and we have $A \subseteq B$ thus far.

case 1. $m \notin fA$ thus far. Put $m \in A,B$.
case 2. $m \in fA$ thus far. Put $m \notin A,B$.

This defines membership of m in A,B. NOte that we still have $A \subseteq B$.

Now let A,B be the result of this inductive construction. Note that by the choice of n, all of the "thus far" remain true of the actual A,B, where $m \geq n$. Thus we have for all $m \geq n$,

$$A \subseteq B.$$
$$m \notin fA \leftrightarrow m \in A \leftrightarrow m \in B.$$
$$m \notin A \rightarrow m \in fA \rightarrow m \in fB.$$

Hence for all $m \geq n$, $m \in B \cup. fA$ and $m \in A \cup fB$. Since this also holds for $m < n$, this holds for all $m \in N$.

Finally, suppose A is finite. Then fA is finite, and so eventually all m are placed in A. Thus A is infinite. Hence A is infinite. QED

LEMMA 2.5.5. For all $f \in EVSD$ there exist infinite $A \subseteq B \subseteq N$ such that $A \cup. fB = N$ and $B \cap fA = \emptyset$.

Proof: Let $f \in EVSD$. Let n,A',B' be as in the first paragraph of the proof of Lemma 2.5.4.

We now define the required A,B by induction. Membership in A,B for $m < n$ is just membership in A',B'. Thus for all $m < n$,

$$m \in B \leftrightarrow m \in B' \leftrightarrow m \notin f'A' \leftrightarrow m \notin fA.$$
$$m \in A \leftrightarrow m \in A' \leftrightarrow m \notin f'B' \leftrightarrow m \notin fB.$$

Now suppose membership in A,B has been defined for all $i < m$, where $m \geq n$, and we have $A \subseteq B$ thus far.

case 1. $m \notin fB$ thus far. Put $m \in A,B$.
case 2. $m \in fB$ thus far. Put $m \notin A,B$.

This defines membership of m in A,B. Note that we still have $A \subseteq B$.

Now let A,B be the result of this inductive construction. Note that by the choice of n, all of the "thus far" remain true of the actual A,B, where $m \geq n$. Thus we have for all $m \geq n$,

$$A \subseteq B.$$
$$m \notin fB \leftrightarrow m \in A \leftrightarrow m \in B.$$
$$m \in B \rightarrow m \notin fB \rightarrow m \notin fA.$$

Hence for all $m \geq n$, $m \in A \cup. fB$ and $m \notin B \cap fA$. Since this also holds for $m < n$, this holds for all $m \in N$.

Finally, suppose A is finite. Then eventually all m are placed in fB. Hence eventually all m are placed outside B. Hence B is finite. So fB is finite. Then eventually all m are put in A,B. This is a contradiction. QED

LEMMA 2.5.6. There exists $f \in ELG$ such that $f^{-1}(0) = \{(0,\ldots,0)\}$, $f(N\backslash\{0\}) \subseteq 2N+1$, and for all $A \subseteq N$ containing 0, $fA \cap 2N \subseteq A \rightarrow fA$ is cofinite.

Proof: Let $g \in ELG \cap SD$ be given by Lemma 3.2.1. We define 4-ary $f \in ELG$ as follows. $f(0,0,0,0) = 0$. $f(0,n,m,r) = g(n,m,r)$ if $(n,m,r) \neq (0,0,0)$. $f(t,n,m,r) = 2|t,n,m,r|+1$ if $t \neq 0$. Obviously $f \in ELG \cap SD$, $f(N\backslash\{0\}) \subseteq 2N+1$, and $f^{-1}(0) = \{(0,0,0,0)\}$.

Now let $A \subseteq N$, $0 \in A$, where $fA \cap 2N \subseteq A$. Since $gA \subseteq fA$, we have $gA \cap 2N \subseteq A$, and so by Lemma 3.2.1, gA is cofinite. Hence fA is cofinite. QED

LEMMA 2.5.7. The following is false. For all $f \in ELG$ there exist infinite $A \subseteq B \subseteq N$ such that $A \cap fB = \emptyset$, $B \cup fB = N$, and $fB \subseteq B \cup fA$.

Proof: Let $f \in ELG$ be given by Lemma 2.5.6. Let $A \cap fB = \emptyset$, $B \cup fB = N$, and $fB \subseteq B \cup fA$, where A is infinite. Now $0 \in B \vee 0 \in fB$. Since $f^{-1}(0) = \{(0,0,0,0)\}$, we have $0 \in B$, $0 \in fB$, $0 \notin A$. Therefore $fA \subseteq 2N+1$. Since $fB \subseteq B \cup fA$, we have $fB \cap 2N \subseteq B$. Therefore fB is cofinite. This contradicts $A \cap fB = \emptyset$. QED

LEMMA 2.5.8. The following is false. For all $f \in ELG$ there exist infinite $A \subseteq B \subseteq N$ such that $B \cup. fA = N$ and $A \cap fB = \emptyset$.

Proof: Let f be as given by Lemma 2.5.6. Let $A \subseteq B \subseteq N$, $B \cup. fA = N$, $A \cap fB = \emptyset$, where A is infnite. Since $0 \in B \cup. fA$, we have $0 \in B \vee 0 \in fA$. If $0 \in fA$ then $0 \in A,B$, because $f^{-1}(0) = \{(0,0,0,0)\}$. Hence $0 \notin fA$, $0 \notin A$. Therefore $fA \subseteq 2N+1$. Since $B \cup fA = N$, we have $2N \subseteq B$. By Lemma 3.2.1, fB is cofinite. By $A \cap fB = \emptyset$, A is finite. But A is infinite. QED

LEMMA 2.5.9. For all $f \in EVSD$ there exist infinite $A \subseteq B \subseteq N$ such that $B \cup. fA = N$ and $A \subseteq fB$.

Proof: Let n be such that $|x| \geq n \rightarrow f(x) > |x|$. We can use Lemma 2.4.1 with N replaced by $[n,\infty)$. Let $A,B \subseteq [n,\infty)$, $A \subseteq B$, $B \cup. fA = [n,\infty)$ and $A = B \cap fB$, where A is infinite. Then $B \cup. fA = [n,\infty)$, $A \subseteq fB$. Replace B with $B \cup [0,n-1]$. QED

LEMMA 2.5.10. The following is false. For all $f \in ELG$ there exist infinite $A \subseteq B \subseteq N$ such that $A \cap fA = \emptyset$, $B \cup fB = N$, $B \cap fB \subseteq A \cup fA$.

Proof: Let f be as given by Lemma 2.5.6. Let $A \subseteq B \subseteq N$ such that $A \cap fA = \emptyset$, $B \cup fB = N$, $B \cap fB \subseteq A \cup fA$, where A,B are infinite. Then $0 \in B \cup fB$, and so $0 \in B \cap fB$. Hence $0 \in A \cup fA$, in which case $0 \in A \cap fA$. QED

LEMMA 2.5.11. For all $f \in EVSD$ there exist infinite $A \subseteq B \subseteq N$ such that $A \cup. fB = N$ and $fA \subseteq B$.

Proof: Let f′ be the restriction of f to $\{x: f(x) > |x|\}$. Then f′ is defined at all but finitely many elements of dom(f). As remarked right after Lemma 2.4.5, Lemma 2.4.2 holds even for partial functions, and so in particular for f′. Let $A \subseteq B \subseteq N$, where $A \cup. f'B = N$ and $f'A \subseteq B$ and A is infinite. Let $A' = N\backslash fB \subseteq A$. Since f′B contains all but finitely many elements of fB, we see that A′ remains infinite. Then A′,B are as required. QED

LEMMA 2.5.12. Let $f \in EVSD$. There exist infinite $A \subseteq B \subseteq N$ such that $fB \subseteq B \cup. fA$ and $A = B \cap fB$.

Proof: Let n be such that $|x| \geq n \rightarrow f(x) > |x|$. We can use Lemma 2.4.1 with N replaced by $[n,\infty)$. Let $A,B \subseteq [n,\infty)$, $A \subseteq B$, $B \cup. fA = [n,\infty)$, and $A = B \cap fB$, where A is infinite. Since $fB \subseteq [n,\infty)$, the proof is complete. QED

LEMMA 2.5.13. Let $f \in EVSD$. There exist infinite $A \subseteq N$ such that $A \cap f(A \cup fA) = \emptyset$.

Proof: Let n be such that $|x| \geq n \rightarrow f(x) > |x|$. Define $n_0 < n_1 < \ldots$ by induction as follows. Let $n_0 = n$. Suppose n_i has been defined, $i \geq 0$. Let n_{i+1} be greater than all elements of $f(A \cup fA)$, thus far. Finally, let $A = \{n_0,n_1,\ldots\}$. QED

LEMMA 2.5.14. Let $f \in EVSD$ and let $X \subseteq N$, where min(X) is sufficiently large. There exists a unique A such that $A \subseteq X \subseteq A \cup. fA$. If X is infinite then A is infinite.

Proof: Let f,X be as given. Then $|x| \geq \min(X) \rightarrow f(x) > |x|$. We can use Lemma 2.4.3 with N replaced by $[\min(X),\infty)$. Let $A \subseteq X \cap [\min(X),\infty) \subseteq A \cup. fA$.

For uniqueness, suppose $A \subseteq X \subseteq A \cup. fA$, $A' \subseteq X \subseteq A' \cup. fA'$, and let $n = \min(A \Delta A')$. Since $f \in SD$, clearly $n \in fA \leftrightarrow n \in fA'$. This is a contradiction. QED

As in section 2.4, we start with the 9 elementary inclusions in $A,B,fA,fB,\subseteq$.

EBRT in $A,B,fA,fB,\subseteq$ on (ELG,INF), (EVSD,INF).

$A \cap fA = \emptyset$.
$B \cup fB = N$.

B ⊆ A ∪ fB.
fB ⊆ B ∪ fA.
A ⊆ fB.
B ∩ fB ⊆ A ∪ fA.
fA ⊆ B.
A ∩ fB ⊆ fA.
B ∩ fA ⊆ A.

Our classification amounts to a determination of the subsets S of the above nine inclusions for which

(∀f ∈ ELG)(∃A ⊆ B from INF)(S)
(∀f ∈ EVSD)(∃A ⊆ B from INF)(S)

holds, where S is interpreted conjunctively.

EBRT in A,B,fA,fB,⊆ on (ELG,INF), (EGS ∩ SD,INF).*
5

A ∩ fA = ∅.
B ∪ fB = N.
fA ⊆ B.
A ⊆ fB.
B ⊆ A ∪ fB.
fB ⊆ B ∪ fA.
A ∩ fB ⊆ fA.
B ∩ fA ⊆ A.
B ∩ fB ⊆ A ∪ fA.

LIST 1.

A ∩ fA = ∅:
B ∪ fB = N.
fA ⊆ B.
A ⊆ fB.
B ⊆ A ∪ fB.
fB ⊆ B ∪ fA.
A ∩ fB ⊆ fA. A ∩ fB = ∅.
B ∩ fA ⊆ A. B ∩ fA = ∅.
B ∩ fB ⊆ A ∪ fA.

LIST 1*.
6

A ∩ fA = ∅:
B ∩ fA = ∅.
A ∩ fB = ∅.

fA ⊆ B.
A ⊆ fB.
B ∪ fB = N.
B ⊆ A ∪ fB.
fB ⊆ B ∪ fA.
B ∩ fB ⊆ A ∪ fA.

LIST 1.1.

A ∩ fA = ∅: Redundant.
B ∩ fA = ∅:
A ∩ fB = ∅.
fA ⊆ B. No.
A ⊆ fB.
B ∪ fB = N.
B ⊆ A ∪ fB.
fB ⊆ B ∪ fA.
B ∩ fB ⊆ A ∪ fA. B ∩ fB ⊆ A.

LIST 1.1.*
4

B ∩ fA = ∅:
A ∩ fB = ∅.
A ⊆ fB.
B ∪ fB = N.
B ⊆ A ∪ fB.
fB ⊆ B ∪ fA.
B ∩ fB ⊆ A.

LIST 1.1.1.

B ∩ fA = ∅:
A ∩ fB = ∅:
A ⊆ fB. No.
B ∪ fB = N.
B ⊆ A ∪ fB.
fB ⊆ B ∪ fA.
B ∩ fB ⊆ A. B ∩ fB = ∅.

LIST 1.1.1.*
2

B ∩ fA = ∅:
A ∩ fB = ∅:
B ∪ fB = N.
B ⊆ A ∪ fB.

fB ⊆ B ∪ fA.
B ∩ fB = ∅.

LIST 1.1.1.1.

B ∩ fA = ∅:
A ∩ fB = ∅:
B ∪ fB = N:
B ⊆ A ∪ fB. A ∪ fB = N.
fB ⊆ B ∪ fA. B ∪ fA = N. No. Lemma 2.5.8.
B ∩ fB = ∅. No. Lemma 2.5.10.

LIST 1.1.1.1.*
0

B ∩ fA = ∅:
A ∩ fB = ∅:
B ∪ fB = N:
A ∪ fB = N.

Entirely RCA_0 correct. Lemma 2.5.5.

LIST 1.1.1.2.

B ∩ fA = ∅:
A ∩ fB = ∅:
B ⊆ A ∪ fB:
fB ⊆ B ∪ fA.
B ∩ fB = ∅.

Entirely RCA_0 correct. Set A ∩ fA = ∅, B = A.

LIST 1.1.2.

B ∩ fA = ∅:
A ⊆ fB:
B ∪ fB = N.
B ⊆ A ∪ fB. B ⊆ fB. No.
fB ⊆ B ∪ fA.
B ∩ fB ⊆ A.

LIST 1.1.2.*
2

B ∩ fA = ∅:
A ⊆ fB:
B ∪ fB = N.

fB ⊆ B ∪ fA.
B ∩ fB ⊆ A.

LIST 1.1.2.1.

B ∩ fA = ∅:
A ⊆ fB:
B ∪ fB = N:
fB ⊆ B ∪ fA. B ∪ fA = N.
B ∩ fB ⊆ A. No. Lemma 2.5.10.

LIST 1.1.2.1.*
0

B ∩ fA = ∅:
A ⊆ fB:
B ∪ fB = N:
B ∪ fA = N.

Entirely RCA_0 correct. Lemma 2.5.9.

LIST 1.1.2.2.

B ∩ fA = ∅:
A ⊆ fB:
fB ⊆ B ∪ fA:
B ∩ fB ⊆ A.

Entirely RCA_0 correct. Lemma 2.5.12.

LIST 1.1.3.

B ∩ fA = ∅:
B ∪ fB = N:
B ⊆ A ∪ fB. A ∪ fB = N.
fB ⊆ B ∪ fA. B ∪ fA = N.
B ∩ fB ⊆ A. No. Lemma 2.5.10.

LIST 1.1.3.*
0

B ∩ fA = ∅:
B ∪ fB = N:
A ∪ fB = N.
B ∪ fA = N.

Entirely RCA_0 correct. Lemma 2.5.4.

LIST 1.1.4.

$B \cap fA = \emptyset$:
$B \subseteq A \cup fB$:
$fB \subseteq B \cup fA$.
$B \cap fB \subseteq A$.

Entirely RCA_0 correct. Set $A \cap fA = \emptyset$, $B = A$.

LIST 1.2.

$A \cap fA = \emptyset$: Redundant.
$A \cap fB = \emptyset$:
$fA \subseteq B$.
$A \subseteq fB$. No.
$B \cup fB = N$.
$B \subseteq A \cup fB$.
$fB \subseteq B \cup fA$.
$B \cap fB \subseteq A \cup fA$. $B \cap fB \subseteq fA$.

LIST 1.2.*
3

$A \cap fB = \emptyset$:
$fA \subseteq B$.
$B \cup fB = N$.
$B \subseteq A \cup fB$.
$fB \subseteq B \cup fA$.
$B \cap fB \subseteq fA$.

LIST 1.2.1.

$A \cap fB = \emptyset$:
$fA \subseteq B$:
$B \cup fB = N$.
$B \subseteq A \cup fB$.
$fB \subseteq B \cup fA$. $fB \subseteq B$. No. Lemma 2.4.4.
$B \cap fB \subseteq fA$.

LIST 1.2.1.*
2

$A \cap fB = \emptyset$:
$fA \subseteq B$:
$B \cup fB = N$.
$B \subseteq A \cup fB$.

B ∩ fB ⊆ fA.

LIST 1.2.1.1.

A ∩ fB = ∅:
fA ⊆ B:
B ∪ fB = N:
B ⊆ A ∪ fB.
B ∩ fB ⊆ fA. No. Lemma 2.5.10.

LIST 1.2.1.1.*
0

A ∩ fB = ∅:
fA ⊆ B:
B ∪ fB = N:
B ⊆ A ∪ fB.

Entirely RCA_0 correct. See Lemma 2.5.11.

LIST 1.2.1.2.
0

A ∩ fB = ∅:
fA ⊆ B:
B ⊆ A ∪ fB:
B ∩ fB ⊆ fA.

Entirely RCA_0 correct. Le A be given by Lemma 2.5.13. Set B = A ∪ fA.

LIST 1.2.2.

A ∩ fB = ∅:
B ∪ fB = N:
B ⊆ A ∪ fB. A ∪ fB = N.
fB ⊆ B ∪ fA. No. Lemma 2.5.7.
B ∩ fB ⊆ fA. No. Lemma 2.5.10.

LIST 1.2.2.*
0

A ∩ fB = ∅:
B ∪ fB = N:
A ∪ fB = N.

Entirely RCA_0 correct. Lemma 2.5.5.

LIST 1.2.3.

$A \cap fB = \emptyset$:
$B \subseteq A \cup fB$:
$fB \subseteq B \cup fA$.
$B \cap fB \subseteq fA$.

Entirely RCA_0 correct. Set $A \cap fA = \emptyset$, $B = A$.

LIST 1.3.

$A \cap fA = \emptyset$:
$fA \subseteq B$:
$A \subseteq fB$.
$B \cup fB = N$.
$B \subseteq A \cup fB$.
$fB \subseteq B \cup fA$.
$B \cap fB \subseteq A \cup fA$.

LIST 1.3.*
3

$A \cap fA = \emptyset$:
$fA \subseteq B$:
$A \subseteq fB$.
$B \cup fB = N$.
$B \subseteq A \cup fB$.
$fB \subseteq B$.
$B \cap fB \subseteq A \cup fA$.

LIST 1.3.1.

$A \cap fA = \emptyset$:
$fA \subseteq B$:
$A \subseteq fB$:
$B \cup fB = N$.
$B \subseteq A \cup fB$. $B \subseteq fB$. No. Lemma 2.4.5.
$fB \subseteq B$.
$B \cap fB \subseteq A \cup fA$.

LIST 1.3.1.*
2

$A \cap fA = \emptyset$:
$fA \subseteq B$:
$A \subseteq fB$:

B ∪ fB = N.
fB ⊆ B.
B ∩ fB ⊆ A ∪ fA.

LIST 1.3.1.1.

A ∩ fA = ∅:
fA ⊆ B:
A ⊆ fB:
B ∪ fB = N:
fB ⊆ B.
B ∩ fB ⊆ A ∪ fA. No. Lemma 2.5.10.

LIST 1.3.1.1.*
0

A ∩ fA = ∅:
fA ⊆ B:
A ⊆ fB:
B ∪ fB = N:
fB ⊆ B.

Entirely RCA_0 correct. Let A be given by Lemma 2.4.3 with A ⊆ fN ⊆ A ∪. fA. Set B = N.

LIST 1.3.1.2.

A ∩ fA = ∅:
fA ⊆ B:
A ⊆ fB:
fB ⊆ B:
B ∩ fB ⊆ A ∪ fA.

Entirely RCA_0 correct. Let B = [n,∞), n sufficiently large. By Lemma 2.5.14, let A ⊆ fB ⊆ A ∪. fA.

LIST 1.3.2.

A ∩ fA = ∅:
fA ⊆ B:
B ∪ fB = N:
B ⊆ A ∪ fB. A ∪ fB = N.
fB ⊆ B. B = N.
B ∩ fB ⊆ A ∪ fA. No. Lemma 2.5.10.

LIST 1.3.2.*
0

A ∩ fA = ∅:
fA ⊆ B:
B ∪ fB = N:
A ∪ fB = N.
B = N.

Entirely RCA_0 correct. Set A = N\fN, B = N.

LIST 1.3.3.

A ∩ fA = ∅:
fA ⊆ B:
B ⊆ A ∪ fB:
fB ⊆ B.
B ∩ fB ⊆ A ∪ fA.

Entirely RCA_0 correct. Let B = [n,∞) for n sufficiently large. Let A ⊆ B ⊆ A ∪. fA, by Lemma 2.5.14.

LIST 1.4.

A ∩ fA = ∅:
A ⊆ fB:
B ∪ fB = N.
B ⊆ A ∪ fB. B ⊆ fB. No. Lemma 2.4.5.
fB ⊆ B ∪ fA.
B ∩ fB ⊆ A ∪ fA.

LIST 1.4.*
2

A ∩ fA = ∅:
A ⊆ fB:
B ∪ fB = N.
fB ⊆ B ∪ fA.
B ∩ fB ⊆ A ∪ fA.

LIST 1.4.1.

A ∩ fA = ∅:
A ⊆ fB:
B ∪ fB = N:
fB ⊆ B ∪ fA.
B ∩ fB ⊆ A ∪ fA. No. Lemma 2.5.10.

LIST 1.4.1.*

0

A ∩ fA = ∅:
A ⊆ fB:
B ∪ fB = N:
fB ⊆ B ∪ fA.

Entirely RCA_0 correct. Let A ⊆ fN ⊆ A ∪. fA be given by Lemma 2.4.3. Set B = N.

LIST 1.4.2.

A ∩ fA = ∅:
A ⊆ fB:
fB ⊆ B ∪ fA.
B ∩ fB ⊆ A ∪ fA.

Entirely RCA_0 correct. Lemma 2.5.12.

LIST 1.5.

A ∩ fA = ∅:
B ∪ fB = N:
B ⊆ A ∪ fB.
fB ⊆ B ∪ fA.
B ∩ fB ⊆ A ∪ fA. No. Lemma 2.5.10.

LIST 1.5.*
0

A ∩ fA = ∅:
B ∪ fB = N:
B ⊆ A ∪ fB.
fB ⊆ B ∪ fA.

Entirely RCA_0 correct. Lemma 2.5.4.

LIST 1.6.

A ∩ fA = ∅:
B ⊆ A ∪ fB:
fB ⊆ B ∪ fA.
B ∩ fB ⊆ A ∪ fA.

Entirely RCA_0 correct. Let A ∩ fA = ∅, B = A.

LIST 2.

B ∪ fB = N:
fA ⊆ B.
A ⊆ fB.
B ⊆ A ∪ fB. A ∪ fB = N.
fB ⊆ B ∪ fA. B ∪ fA = N.
A ∩ fB ⊆ fA.
B ∩ fA ⊆ A.
B ∩ fB ⊆ A ∪ fA.

LIST 2.*
3

B ∪ fB = N:
fA ⊆ B.
A ⊆ fB.
A ∪ fB = N.
B ∪ fA = N.
A ∩ fB ⊆ fA.
B ∩ fA ⊆ A.
B ∩ fB ⊆ A ∪ fA.

LIST 2.1.

B ∪ fB = N:
fA ⊆ B:
A ⊆ fB.
A ∪ fB = N.
B ∪ fA = N. B = N.
A ∩ fB ⊆ fA.
B ∩ fA ⊆ A. fA ⊆ A.
B ∩ fB ⊆ A ∪ fA.

LIST 2.1.*
2

B ∪ fB = N:
fA ⊆ B:
A ⊆ fB.
A ∪ fB = N.
B = N.
A ∩ fB ⊆ fA.
fA ⊆ A.
B ∩ fB ⊆ A ∪ fA.

LIST 2.1.1.

B ∪ fB = N:
fA ⊆ B:
A ⊆ fB:
A ∪ fB = N. fB = N. No. Lemma 2.4.5.
B = N.
A ∩ fB ⊆ fA. A ⊆ fA. No. Lemma 2.4.5.
fA ⊆ A.
B ∩ fB ⊆ A ∪ fA.

LIST 2.1.1.*
0

B ∪ fB = N:
fA ⊆ B:
A ⊆ fB:
B = N.
fA ⊆ A.
B ∩ fB ⊆ A ∪ fA.

Entirely RCA_0 correct. Set A = fN, B = N.

LIST 2.1.2.

B ∪ fB = N:
fA ⊆ B:
A ∪ fB = N.
B = N.
A ∩ fB ⊆ fA.
B ∩ fA ⊆ A.
B ∩ fB ⊆ A ∪ fA.

Entirely RCA_0 correct. Set A = B = N.

LIST 2.2.

B ∪ fB = N:
A ⊆ fB:
A ∪ fB = N. Yes.
B ∪ fA = N.
A ∩ fB ⊆ fA. A ⊆ fA. No. Lemma 2.4.5.
B ∩ fA ⊆ A.
B ∩ fB ⊆ A ∪ fA.

LIST 2.2.*
0

B ∪ fB = N:

A ⊆ fB:
fB ⊆ B ∪ fA. B ∪ fA = N.
B ∩ fA ⊆ A.
B ∩ fB ⊆ A ∪ fA.

Entirely RCA_0 correct. Set A = fN, B = N.

LIST 2.3.

B ∪ fB = N:
A ∪ fB = N.
B ∪ fA = N.
A ∩ fB ⊆ fA.
B ∩ fA ⊆ A.
B ∩ fB ⊆ A ∪ fA.

Entirely RCA_0 correct. Set A = B = N.

LIST 3.

fA ⊆ B:
A ⊆ fB.
B ⊆ A ∪ fB.
fB ⊆ B ∪ fA.
A ∩ fB ⊆ fA.
B ∩ fA ⊆ A.
B ∩ fB ⊆ A ∪ fA.

LIST 3*.
2

fA ⊆ B:
A ⊆ fB.
B ⊆ A ∪ fB.
fB ⊆ B ∪ fA.
A ∩ fB ⊆ fA.
B ∩ fA ⊆ A.
B ∩ fB ⊆ A ∪ fA.

LIST 3.1.

fA ⊆ B:
A ⊆ fB:
B ⊆ A ∪ fB. B ⊆ fB. No. Lemma 2.4.5.
fB ⊆ B ∪ fA.
A ∩ fB ⊆ fA. A ⊆ fA. No. Lemma 2.4.5.
B ∩ fA ⊆ A.

B ∩ fB ⊆ A ∪ fA.

LIST 3.1.*
0

fA ⊆ B:
A ⊆ fB:
fB ⊆ B ∪ fA.
B ∩ fA ⊆ A.
B ∩ fB ⊆ A ∪ fA.

Entirely RCA_0 correct. Set A = fN, B = N.

LIST 3.2.

fA ⊆ B:
B ⊆ A ∪ fB:
fB ⊆ B ∪ fA.
A ∩ fB ⊆ fA.
B ∩ fA ⊆ A.
B ∩ fB ⊆ A ∪ fA.

Entirely RCA_0 correct. Set A = B = fN.

LIST 4.

A ⊆ fB:
B ⊆ A ∪ fB. B ⊆ fB. No. Lemma 2.4.5.
fB ⊆ B ∪ fA.
A ∩ fB ⊆ fA. A ⊆ fA. No. Lemma 2.4.5.
B ∩ fA ⊆ A.
B ∩ fB ⊆ A ∪ fA.

LIST 4.*
0

A ⊆ fB:
fB ⊆ B ∪ fA.
B ∩ fA ⊆ A.
B ∩ fB ⊆ A ∪ fA.

Entirely RCA_0 correct. Set A = fN, B = N.

LIST 5.

B ⊆ A ∪ fB:
fB ⊆ B ∪ fA.

A ∩ fB ⊆ fA.
B ∩ fA ⊆ A.
B ∩ fB ⊆ A ∪ fA.

Entirely RCA_0 correct. Set A = B = N.

THEOREM 2.5.15. EBRT in A,B,fA,fB,⊆ on (ELG,INF), (EVSD,INF) have the same correct formats. EBRT in A,B,fA,fB,⊆ on (ELG,INF) and (EVSD,INF) are RCA_0 secure.

Proof: We have presented an RCA_0 classification of EBRT in A,B,fA,fB,⊆ on (ELG,INF), (EVSD,INF) in the sense of the tree methodology of section 2.1. All of the documentation works equally well on (ELG,INF) and (EVSD,INF). We have stayed within RCA_0. QED

THEOREM 2.5.16. There are at most 26 maximal α correct α formats, where α is EBRT in A,B,fA,fB,⊆ on (ELG,INF), (EVSD,INF).

Proof: Here is the list of numerical labels of terminal vertices in the RCA_0 classification of EBRT in A,B,fA,fB,⊆ on (ELG,INF), (EVSD,INF) given above:
1.1.1.1.*
1.1.1.2.
1.1.2.1.*
1.1.2.2.
1.1.3.*
1.1.3.
1.2.1.1.*
1.2.1.2.
1.2.2.*
1.2.3.
1.3.1.1.*
1.3.1.2.
1.3.2.*
1.3.3.
1.4.1.*
1.4.2.
1.5.*
1.6.
2.1.1.*
2.1.2.
2.2.*
2.3.
3.1.*
3.2.

4.*
5.
The count is 26. Apply Theorem 2.1.5. QED