3.5. AAAB.

Recall the reduced AA table from section 3.4.

REDUCED AA

1. B ∪. fA ⊆ A ∪. gA. ¬INF. AL. ¬ALF. ¬FIN. NON.
2. B ∪. fA ⊆ A ∪. gB. ¬INF. AL. ¬ALF. ¬FIN. NON.
3. B ∪. fA ⊆ A ∪. gC. ¬INF. AL. ¬ALF. ¬FIN. NON.
4. C ∪. fA ⊆ A ∪. gA. ¬INF. AL. ¬ALF. ¬FIN. NON.
5. C ∪. fA ⊆ A ∪. gB. ¬INF. AL. ¬ALF. ¬FIN. NON.
6. C ∪. fA ⊆ A ∪. gC. ¬INF. AL. ¬ALF. ¬FIN. NON.

Recall the AB table from section 3.3.

AB

1. A ∪. fA ⊆ B ∪. gA. INF. AL. ALF. FIN. NON.
2. A ∪. fA ⊆ B ∪. gB. INF. AL. ALF. FIN. NON.
3. A ∪. fA ⊆ B ∪. gC. INF. AL. ALF. FIN. NON.
4. B ∪. fA ⊆ B ∪. gA. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
5. B ∪. fA ⊆ B ∪. gB. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
6. B ∪. fA ⊆ B ∪. gC. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
7. C ∪. fA ⊆ B ∪. gA. INF. AL. ALF. FIN. NON.
8. C ∪. fA ⊆ B ∪. gB. INF. AL. ALF. FIN. NON.
9. C ∪. fA ⊆ B ∪. gC. INF. AL. ALF. FIN. NON.

Here is the reduced AB table, renumbered with ′ to distinguish it from the reduced AA table above.

REDUCED AB

1′. A ∪. fA ⊆ B ∪. gA. INF. AL. ALF. FIN. NON.
2′. A ∪. fA ⊆ B ∪. gB. INF. AL. ALF. FIN. NON.
3′. A ∪. fA ⊆ B ∪. gC. INF. AL. ALF. FIN. NON.
4′. C ∪. fA ⊆ B ∪. gA. INF. AL. ALF. FIN. NON.
5′. C ∪. fA ⊆ B ∪. gB. INF. AL. ALF. FIN. NON.
6′. C ∪. fA ⊆ B ∪. gC. INF. AL. ALF. FIN. NON.

We consider all 36 ordered pairs, arranged in cases according to the first clause of the ordered pair.

As before, we need only obtain the status of AL and NON for the ordered pairs, because of the reduced AA table.

part 1. B ∪. fA ⊆ A ∪. gA.

1,1′. B ∪. fA ⊆ A ∪. gA, A ∪. fA ⊆ B ∪. gA. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
1,2′. B ∪. fA ⊆ A ∪. gA, A ∪. fA ⊆ B ∪. gB. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
1,3′. B ∪. fA ⊆ A ∪. gA, A ∪. fA ⊆ B ∪. gC. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
1,4′. B ∪. fA ⊆ A ∪. gA, C ∪. fA ⊆ B ∪. gA. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
1,5′. B ∪. fA ⊆ A ∪. gA, C ∪. fA ⊆ B ∪. gB. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
1,6′. B ∪. fA ⊆ A ∪. gA, C ∪. fA ⊆ B ∪. gC. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.

The following pertains to 1,1′ - 1,6′.

LEMMA 3.5.1. fA ⊆ A ∪. gX, fA ⊆ B ∪. gY, Z ∩ fA = ∅ has ¬NON, where X,Y ∈ {A,B,C} and Z ∈ {A,B}.

Proof: Let f be as given by Lemma 3.2.1. Let g ∈ ELG be defined by g(n) = 2n+1. Let fA ⊆ A ∪. gX, fA ⊆ B ∪. gY, Z ∩ fA = ∅, where A,B,C are nonempty. Let n ∈ fA ∩ 2N. Then n ∈ A. Hence fA ∩ 2N ⊆ A. So fA is cofinite. Hence A is infinite. Also fA ∩ 2N ⊆ B, and so B is infinite. Hence Z is infinite. This contradicts Z ∩ fA = ∅. QED

part 2. B ∪. fA ⊆ A ∪. gB.

2,1′. B ∪. fA ⊆ A ∪. gB, A ∪. fA ⊆ B ∪. gA. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
2,2′. B ∪. fA ⊆ A ∪. gB, A ∪. fA ⊆ B ∪. gB. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
2,3′. B ∪. fA ⊆ A ∪. gB, A ∪. fA ⊆ B ∪. gC. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
2,4′. B ∪. fA ⊆ A ∪. gB, C ∪. fA ⊆ B ∪. gA. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
2,5′. B ∪. fA ⊆ A ∪. gB, C ∪. fA ⊆ B ∪. gB. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
2,6′. B ∪. fA ⊆ A ∪. gB, C ∪. fA ⊆ B ∪. gC. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.

LEMMA 3.5.2. 2,1′ - 2,6′ have ¬NON.

Proof: By Lemma 3.5.1. QED

part 3. B ∪. fA ⊆ A ∪. gC.

3,1′. B ∪. fA ⊆ A ∪. gC, A ∪. fA ⊆ B ∪. gA. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
3,2′. B ∪. fA ⊆ A ∪. gC, A ∪. fA ⊆ B ∪. gB. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
3,3′. B ∪. fA ⊆ A ∪. gC, A ∪. fA ⊆ B ∪. gC. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
3,4′. B ∪. fA ⊆ A ∪. gC, C ∪. fA ⊆ B ∪. gA. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
3,5′. B ∪. fA ⊆ A ∪. gC, C ∪. fA ⊆ B ∪. gB. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
3,6′. B ∪. fA ⊆ A ∪. gC, C ∪. fA ⊆ B ∪. gC. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.

LEMMA 3.5.3. 3,1′ - 3,6′ have ¬NON.

Proof: By Lemma 3.5.1. QED

part 4. C ∪. fA ⊆ A ∪. gA.

4,1′. C ∪. fA ⊆ A ∪. gA, A ∪. fA ⊆ B ∪. gA. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
4,2′. C ∪. fA ⊆ A ∪. gA, A ∪. fA ⊆ B ∪. gB. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
4,3′. C ∪. fA ⊆ A ∪. gA, A ∪. fA ⊆ B ∪. gC. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
4,4′. C ∪. fA ⊆ A ∪. gA, C ∪. fA ⊆ B ∪. gA. ¬INF. AL. ¬ALF. ¬FIN. NON.
4,5′. C ∪. fA ⊆ A ∪. gA, C ∪. fA ⊆ B ∪. gB. ¬INF. AL. ¬ALF. ¬FIN. NON.
4,6′. C ∪. fA ⊆ A ∪. gA, C ∪. fA ⊆ B ∪. gC. ¬INF. AL. ¬ALF. ¬FIN. NON.

LEMMA 3.5.4. 4,1′ - 4.3′ have ¬NON.

Proof: By Lemma 3.5.1. QED

LEMMA 3.5.5. 4,4′, 4,5′ have AL.

Proof: From the reduced AA table, C ∪. fA ⊆ A ∪. gA has AL. In the cited ordered pairs, replace B by A. QED

The following pertains to 4,6′.

LEMMA 3.5.6. C ∪. fA ⊆ A ∪. gA, C ∪. fA ⊆ B ∪. gC has AL.

Proof: Let f,g ∈ ELG and p > 0. Let C = [n,n+p], where n is sufficiently large. By Lemma 3.3.3, let A be unique such that A ⊆ [n,∞) ⊆ A ∪. gA. Let B = (C ∪ fA)\gC.

Note that fA ⊆ [n,∞), and so C ∪ fA ⊆ A ∪ gA. Also C ∩ fA = C ∩ gA = C ∩ gC = ∅. Hence C ⊆ A,B, and so A,B,C have at least p elements. QED

part 5. C ∪. fA ⊆ A ∪. gB.

5,1′. C ∪. fA ⊆ A ∪. gB, A ∪. fA ⊆ B ∪. gA. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
5,2′. C ∪. fA ⊆ A ∪. gB, A ∪. fA ⊆ B ∪. gB. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
5,3′. C ∪. fA ⊆ A ∪. gB, A ∪. fA ⊆ B ∪. gC. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
5,4′. C ∪. fA ⊆ A ∪. gB, C ∪. fA ⊆ B ∪. gA. ¬INF. AL. ¬ALF. ¬FIN. NON.
5,5′. C ∪. fA ⊆ A ∪. gB, C ∪. fA ⊆ B ∪. gB. ¬INF. AL. ¬ALF. ¬FIN. NON.
5,6′. C ∪. fA ⊆ A ∪. gB, C ∪. fA ⊆ B ∪. gC. ¬INF. AL. ¬ALF. ¬FIN. NON.

LEMMA 3.5.7. 5,1′ - 5,3′ have ¬NON.

Proof: By Lemma 3.5.1. QED

LEMMA 3.5.8. 5,4′, 5,5′ have AL.

Proof: From the reduced AA table, C ∪. fA ⊆ A ∪. gA has AL. In the cited ordered pairs, replace B by A. QED

The following pertains to 5,6′.

LEMMA 3.5.9. C ∪. fA ⊆ A ∪. gX, C ∪. fA ⊆ B ∪. gY has AL, provided X ∈ {B,C} and Y ∈ {A,C}.

Proof: Let f,g ∈ ELG and p > 0. Let C = [n,n+p], where n is sufficiently large. Let A = [n,∞)\gX. Let B = [n,∞)\gY. These are ordinary explicit definitions provided X ≠ A and Y ≠ B.

Clearly C ∩ fA = C ∩ gA = C ∩ gB = C ∩ gC = ∅. Hence C ⊆ A,B, and so |A|,|B| ≥ p. Since C ∪ fA ⊆ [n,∞), we have C ∪ fA ⊆ A ∪. gX, C ∪ fA ⊆ B ∪ gY. QED

part 6. C ∪. fA ⊆ A ∪. gC.

6,1′. C ∪. fA ⊆ A ∪. gC, A ∪. fA ⊆ B ∪. gA. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
6,2′. C ∪. fA ⊆ A ∪. gC, A ∪. fA ⊆ B ∪. gB. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
6,3′. C ∪. fA ⊆ A ∪. gC, A ∪. fA ⊆ B ∪. gC. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
6,4′. C ∪. fA ⊆ A ∪. gC, C ∪. fA ⊆ B ∪. gA. ¬INF, AL, ¬ALF, ¬FIN. NON.
6,5′. C ∪. fA ⊆ A ∪. gC, C ∪. fA ⊆ B ∪. gB. ¬INF, AL, ¬ALF, ¬FIN. NON.
6,6′. C ∪. fA ⊆ A ∪. gC, C ∪. fA ⊆ B ∪. gC. ¬INF, AL, ¬ALF, ¬FIN. NON.

LEMMA 3.5.10. 6,1′ - 6,3′ have ¬NON.

Proof: By Lemma 3.5.1. QED

The following pertains to 6,6′.

LEMMA 3.5.11. C ∪ fA ⊆ A ∪. gC, C ∪. fA ⊆ B ∪. gC has AL.

Proof: From the reduced AA table, C ∪. fA ⊆ A ∪. gC has AL. In the cited ordered pair, replace B by A. QED

The following pertains to 6,4′.

LEMMA 3.5.12. C ∪. fA ⊆ A ∪. gC, C ∪. fA ⊆ B ∪. gA has AL.

Proof: By Lemma 3.5.9. QED

The following pertains to 6,5′.

LEMMA 3.5.13. C ∪. fA ⊆ A ∪. gC, C ∪. fA ⊆ B ∪. gB has AL.

Proof: Let f,g ∈ ELG and p > 0. Let C = [n,n+p], where n is sufficiently large. Let A = [n,∞)\gC. By Lemma 3.3.3, let B be unique such that B ⊆ C ∪ fA ⊆ B ∪. gB.

Clearly C ∩ fA = C ∩ gB = C ∩ gC = ∅ and C ⊆ A,B. Hence |A|,|B| ≥ p. Also C ∪ fA ⊆ B ∪ gB, and C ∪ fA ⊆ [n,∞) ⊆ A ∪ gC. QED