3.8. AABC.

Recall the reduced AA table from section 3.4.

REDUCED AA

1. B ∪. fA ⊆ A ∪. gA. ¬INF. AL. ¬ALF. ¬FIN. NON.
2. B ∪. fA ⊆ A ∪. gB. ¬INF. AL. ¬ALF. ¬FIN. NON.
3. B ∪. fA ⊆ A ∪. gC. ¬INF. AL. ¬ALF. ¬FIN. NON.
4. C ∪. fA ⊆ A ∪. gA. ¬INF. AL. ¬ALF. ¬FIN. NON.
5. C ∪. fA ⊆ A ∪. gB. ¬INF. AL. ¬ALF. ¬FIN. NON.
6. C ∪. fA ⊆ A ∪. gC. ¬INF. AL. ¬ALF. ¬FIN. NON.

Recall the reduced AB table from section 3.5.

REDUCED AB

1. A ∪. fA ⊆ B ∪. gA. INF. AL. ALF. FIN. NON.
2. A ∪. fA ⊆ B ∪. gB. INF. AL. ALF. FIN. NON.
3. A ∪. fA ⊆ B ∪. gC. INF. AL. ALF. FIN. NON.
4. C ∪. fA ⊆ B ∪. gA. INF. AL. ALF. FIN. NON.
5. C ∪. fA ⊆ B ∪. gB. INF. AL. ALF. FIN. NON.
6. C ∪. fA ⊆ B ∪. gC. INF. AL. ALF. FIN. NON.

The reduced BC table is obtained from the reduced AB table via the permutation sending A to B, B to C, C to A. We use 1'-6' to avoid any confusion.

REDUCED BC

1′. B ∪. fB ⊆ C ∪. gB. INF. AL. ALF. FIN. NON.
2′. B ∪. fB ⊆ C ∪. gC. INF. AL. ALF. FIN. NON.
3′. B ∪. fB ⊆ C ∪. gA. INF. AL. ALF. FIN. NON.
4′. A ∪. fB ⊆ C ∪. gB. INF. AL. ALF. FIN. NON.
5′. A ∪. fB ⊆ C ∪. gC. INF. AL. ALF. FIN. NON.
6′. A ∪. fB ⊆ C ∪. gA. INF. AL. ALF. FIN. NON.

All attributes are determined from the reduced AA table, except for AL and NON. So we merely have to determine the status of AL and NON.

part 1. B ∪. fA ⊆ A ∪. gA.

1,1′. B ∪. fA ⊆ A ∪. gA, B ∪. fB ⊆ C ∪. gB. ¬INF. AL. ¬ALF. ¬FIN. NON.
1,2′. B ∪. fA ⊆ A ∪. gA, B ∪. fB ⊆ C ∪. gC. ¬INF. AL. ¬ALF. ¬FIN. NON.

1,3′. B ∪. fA ⊆ A ∪. gA, B ∪. fB ⊆ C ∪. gA. ¬INF. AL. ¬ALF. ¬FIN. NON.
1,4′. B ∪. fA ⊆ A ∪. gA, A ∪. fB ⊆ C ∪. gB. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
1,5′. B ∪. fA ⊆ A ∪. gA, A ∪. fB ⊆ C ∪. gC. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
1,6′. B ∪. fA ⊆ A ∪. gA, A ∪. fB ⊆ C ∪. gA. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.

The following pertains to 1,1′, 1,3′.

LEMMA 3.8.1. B ∪. fA ⊆ A ∪. gA, B ∪. fB ⊆ C ∪. gX has AL, provided X ∈ {A,B}.

Proof: Let f,g ∈ ELG(N) and p > 0. Let B = [n,n+p], where n is sufficiently large. By Lemma 3.3.3, let A be unique such that A ⊆ [n,∞) ⊆ A ∪. gA. Let C = [n,∞)\gX.

Note that B ∩ fA = B ∩ fB = B ∩ gA = A ∩ gA = B ∩ gB = C ∩ gX = ∅. Hence B ⊆ A,C. Also B ∪ fA ⊆ [n,∞) = A ∪ gA, and B ∪ fB ⊆ [n,∞) ⊆ C ∪ gX. QED

The following pertains to 1,2′.

LEMMA 3.8.2. B ∪. fA ⊆ A ∪. gA, B ∪. fB ⊆ C ∪. gC has AL.

Proof: Let f,g ∈ ELG and p > 0. Let B = [n,n+p], where n is sufficiently large. By Lemma 3.3.3, let A be unique such that A ⊆ [n,∞) ⊆ A ∪. gA. By Lemma 3.3.3, let C be unique such that C ⊆ B ∪ fB ⊆ C ∪. gC.

Note that B ∩ fA = B ∩ fB = B ∩ gC = B ∩ gA = A ∩ gA = C ∩ gC = ∅. Hence B ⊆ A,C. Also B ∪ fA ⊆ [n,∞) = A ∪ gA. QED

The following pertains to 1,4′, 1,5′, 1,6′.

LEMMA 3.8.3. B ∪. fA ⊆ A ∪. gA, A ∩ fB = ∅ has ¬NON.

Proof: Define f,g ∈ ELG as follows. For all n < m, let f(n,n) = 2n+2, f(n,m) = f(m,n) = 2m+1, g(n) = 4n+5. Let B ∪. fA ⊆ A ∪. gA, A ∩ fB = ∅, where A,B,C are nonempty.

We claim that gA ⊆ fA. I.e., n ∈ A → 4n+5 ∈ fA. To see this, let n ∈ A. Then 2n+2 ∈ fA, 2n+2 ∈ A. Since n < 2n+2 are from A, we have 4n+5 ∈ fA.

We claim that B ⊆ A. To see this, let n ∈ B\A. Then n ∈ A ∪ gA, n ∈ gA, n ∈ fA. This contradicts B ∩ fA = ∅.

Now let n ∈ B. Then n ∈ A, 2n+2 ∈ fA, 2n+2 ∈ A, 2n+2 ∈ fB. This contradicts A ∩ fB = ∅. QED

part 2. B ∪. fA ⊆ A ∪. gB.

2,1′. B ∪. fA ⊆ A ∪. gB, B ∪. fB ⊆ C ∪. gB. ¬INF. AL. ¬ALF. ¬FIN. NON.
2,2′. B ∪. fA ⊆ A ∪. gB, B ∪. fB ⊆ C ∪. gC. ¬INF. AL. ¬ALF. ¬FIN. NON.
2,3′. B ∪. fA ⊆ A ∪. gB, B ∪. fB ⊆ C ∪. gA. ¬INF. AL. ¬ALF. ¬FIN. NON.
2,4′. B ∪. fA ⊆ A ∪. gB, A ∪. fB ⊆ C ∪. gB. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
2,5′. B ∪. fA ⊆ A ∪. gB, A ∪. fB ⊆ C ∪. gC. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
2,6′. B ∪. fA ⊆ A ∪. gB, A ∪. fB ⊆ C ∪. gA. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.

The following pertains to 2,1′, 2,3′.

LEMMA 3.8.4. B ∪. fA ⊆ A ∪. gB, B ∪. fB ⊆ C ∪. gX has AL, provided X ∈ {A,B}.

Proof: Let f,g ∈ ELG and p > 0. Let B = [n,n+p], where n is sufficiently large. Let A = [n,∞)\gB. Let C = [n,∞)\gX.

Note that B ∩ fA = B ∩ fB = B ∩ gB = B ∩ gA = A ∩ gB = C ∩ gX = ∅. Hence B ⊆ A,C. Also B ∪ fA ⊆ [n,∞) = A ∪ gB, and B ∪ fB ⊆ [n,∞) = C ∪ gX. QED

The following pertains to 2,2′.

LEMMA 3.8.5. B ∪. fA ⊆ A ∪. gB, B ∪. fB ⊆ C ∪. gC has AL.

Proof: Let f,g ∈ ELG and p > 0. Let B = [n,n+p], where n is sufficiently large. Let A = [n,∞)\gB. Let C ⊆ [n,∞) ⊆ C ∪. gC.

Note that B ∩ fA = B ∩ fB = B ∩ gC = B ∩ gB = A ∩ gB = C ∩ gC = ∅. Hence B ⊆ A,C. Also B ∪ fA ⊆ [n,∞) = A ∪ gB, and B ∪ fB ⊆ [n,∞) = C ∪ gC. QED

The following pertains to 2,4′ - 2,6′.

LEMMA 3.8.6. $B \cup.\ fA \subseteq A \cup.\ gB$, $A \cap fB = \emptyset$ has ¬NON.

Proof: Let $f,g \in ELG$ be defined as follows. For all n, $f(n) = 2n$, $g(n) = 2n+1$. Let $B \cup.\ fA \subseteq A \cup.\ gB$, $A \cap fB = \emptyset$, where A,B are nonempty.

Let $n = \min(B)$. Then $n \in B$, $n \notin gB$, $n \in A$, $2n \in fA$, $2n \in A$, $2n \in fB$. This contradicts $A \cap fB = \emptyset$. QED

part 3. $B \cup.\ fA \subseteq A \cup.\ gC$.

3,1'. $B \cup.\ fA \subseteq A \cup.\ gC$, $B \cup.\ fB \subseteq C \cup.\ gB$. ¬INF. AL. ¬ALF. ¬FIN. NON.
3,2'. $B \cup.\ fA \subseteq A \cup.\ gC$, $B \cup.\ fB \subseteq C \cup.\ gC$. ¬INF. AL. ¬ALF. ¬FIN. NON.
3,3'. $B \cup.\ fA \subseteq A \cup.\ gC$, $B \cup.\ fB \subseteq C \cup.\ gA$. ¬INF. AL. ¬ALF. ¬FIN. NON.
3,4'. $B \cup.\ fA \subseteq A \cup.\ gC$, $A \cup.\ fB \subseteq C \cup.\ gB$. ¬INF. AL. ¬ALF. ¬FIN. NON.
3,5'. $B \cup.\ fA \subseteq A \cup.\ gC$, $A \cup.\ fB \subseteq C \cup.\ gC$. ¬INF. AL. ¬ALF. ¬FIN. NON.
3,6'. $B \cup.\ fA \subseteq A \cup.\ gC$, $A \cup.\ fB \subseteq C \cup.\ gA$. ¬INF. AL. ¬ALF. ¬FIN. NON.

The following pertains to 3,1'.

LEMMA 3.8.7. $B \cup.\ fA \subseteq A \cup.\ gC$, $B \cup.\ fB \subseteq C \cup.\ gB$ has AL.

Proof: Let $f,g \in ELG$ and $p > 0$. Let $B = [n,n+p]$, where n is sufficiently large. Let $C = [n,\infty)\backslash gB$, $A = [n,\infty)\backslash gC$.

Note that $B \cap fA = B \cap fB = A \cap gC = C \cap gB = B \cap gB = B \cap gC = \emptyset$. Hence $B \subseteq A,C$. Also $B \cup fA \subseteq [n,\infty) = A \cup gC$ and $B \cup fB \subseteq [n,\infty) \subseteq C \cup gB$. QED

The following pertains to 3,2'.

LEMMA 3.8.8. $B \cup.\ fA \subseteq A \cup.\ gC$, $B \cup.\ fB \subseteq C \cup.\ gC$ has AL.

Proof: Let $f,g \in ELG$ and $p > 0$. Let $B = [n,n+p]$, where n is sufficiently large. By Lemma 3.3.3, let C be unique such that $C \subseteq B \cup fB \subseteq C \cup.\ gC$. Let $A = [n,\infty)\backslash gC$.

Note that $B \cap fA = B \cap fB = A \cap gC = C \cap gC = B \cap gB = B \cap gC = \emptyset$. Hence $B \subseteq A,C$. Also $B \cup fA \subseteq [n,\infty) = A \cup gC$ and $B \cup fB \subseteq C \cup gC$. QED

The following pertains to 3,3'.

LEMMA 3.8.9. B ∪. fA ⊆ A ∪. gC, B ∪. fB ⊆ C ∪. gA has AL.

Proof: Let f,g ∈ ELG and p > 0. Let B = [n,n+p], where n is sufficiently large. We define A,C inductively. Suppose membership in A,C have been defined for all elements of [n,k), where k ≥ n. We define membership of k in A,C as follows.

If k is already in B ∪ fA but not yet in gC, put k in A. if k is already in B ∪ fB but not yet in gA, put k in C. Obviously A,C ⊆ [n,∞).

Clearly B ∩ fA = B ∩ gA = B ∩ fB = B ∩ gC = A ∩ gC = C ∩ gA = ∅. Hence we have put every element of B in A, and every element of B in C. Also fA ⊆ A ∪ gC, fB ⊆ C ∪ gA. QED

LEMMA 3.8.10. Let g ∈ ELG and p > 0. There exist finite D such that D,gD,ggD are pairwise disjoint and each have at least p elements.

Proof: Let g,p be as given, and n be sufficiently large. Let $n = b_1 < \ldots < b_p$, where for all $1 \leq i \leq p$, $b_{i+1} > b_i^n$. Let $D = \{b_1,\ldots,b_p\}$. QED

The following pertains to 3,4'.

LEMMA 3.8.11. B ∪. fA ⊆ A ∪. gC, A ∪. fB ⊆ C ∪. gB has AL.

Proof: Let f,g ∈ ELG and p > 0. Let D be as given by Lemma 3.8.10. Let B = gD.

Let n be sufficiently large. By an obvious generalization of Lemma 3.3.3, let A be unique such that A ⊆ [n,∞) ⊆ A ∪. g(A ∪ D ∪ (fB\gB)). Let C = A ∪ D ∪ (fB\gB). Then [n,∞) ⊆ A ∪. gC.

Obviously B,D are finite and A,C are infinite. Since n is sufficiently large, we have B ∩ fA = A ∩ fB = A ∩ gB = D ∩ gB = ∅. Hence C ∩ gB = ∅.

Since B = gD ⊆ gC and fA ⊆ [n,∞) ⊆ A ∪ gC, we have B ∪ fA ⊆ A ∪ gC.

Since A ⊆ C and fB\gB ⊆ C, we have A ∪ fB ⊆ C ∪ gB. QED

The following pertains to 3,6′.

LEMMA 3.8.12. B ∪. fA ⊆ A ∪. gC, A ∪. fB ⊆ C ∪. gA has AL.

Proof: Let f,g ∈ ELG and p > 0. Let D be as given by Lemma 3.8.10. Let B = gD.

Let n be sufficiently large. Let A ⊆ [n,∞) ⊆ A ∪. g(A ∪ D ∪ fB). Let C = A ∪ D ∪ fB. Then [n,∞) ⊆ A ∪. gC.

Obviously D,B are finite and A,C are infinite. Since n is sufficiently large, we have B ∩ fA = A ∩ fB = fB ∩ gA = ∅. Also A ∩ gA ⊆ A ∩ gC = ∅, and D ∩ gA = ∅. Hence C ∩ gA = ∅.

Since B = gD ⊆ gC and fA ⊆ [n,∞) ⊆ A ∪ gC, we have B ∪ fA ⊆ A ∪ gC. Also A ∪ fB ⊆ C. QED

The following pertains to 3,5′.

LEMMA 3.8.13. B ∪. fA ⊆ A ∪. gC, A ∪. fB ⊆ C ∪. gC has AL.

Proof: Let f,g ∈ ELG and p > 0. Let n be sufficiently large. Let C ⊆ [n,∞) ⊆ C ∪. gC.

Clearly C is infinite. Let B ⊆ gC have cardinality p. Let m be sufficiently large relative to p,n,max(B). Let A = C ∩ [m,∞). Then A,C are infinite.

Clearly B ∩ fA = A ∩ gC = A ∩ fB = C ∩ gC = ∅.

We claim that fA ⊆ A ∪ gC. To see this, let r ∈ fA. Then r > m > n, and so r ∈ C ∪ gC. If r ∈ gC then we are done. If r ∈ C, then r ∈ A.

Finally, A ∪ fB ⊆ A ∪ fgC ⊆ [n,∞) ⊆ C ∪. gC. QED

part 4. C ∪. fA ⊆ A ∪. gA.

4,1′. C ∪. fA ⊆ A ∪. gA, B ∪. fB ⊆ C ∪. gB. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
4,2′. C ∪. fA ⊆ A ∪. gA, B ∪. fB ⊆ C ∪. gC. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
4,3′. C ∪. fA ⊆ A ∪. gA, B ∪. fB ⊆ C ∪. gA. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
4,4′. C ∪. fA ⊆ A ∪. gA, A ∪. fB ⊆ C ∪. gB. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.

4,5’. C ∪. fA ⊆ A ∪. gA, A ∪. fB ⊆ C ∪. gC. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
4,6’. C ∪. fA ⊆ A ∪. gA, A ∪. fB ⊆ C ∪. gA. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.

The following pertains to 4,1’.

LEMMA 3.8.14. C ∪. fA ⊆ A ∪. gA, B ∪. fB ⊆ C ∪. gB has ¬NON.

Proof: Define f,g ∈ ELG as follows. For all n < m, let f(n,n) = 2n+2, f(n,m) = 2m+1, f(m,n) = 4m+6, g(n) = 4n+5. Let C ∪. fA ⊆ A ∪. gA, B ∪. fB ⊆ C ∪. gB, where A,B,C are nonempty.

Let m ∈ B. Then 2m+2 ∈ fB, 2m+2 ∈ C, 2m+2 ∈ A, 4m+6 ∈ fA, 2m+2 ∉ fA, m ∉ A, m ∈ C ∪ gB.

case 1. m ∈ C. Then m ∈ A ∪ gA, m ∈ gA. Let m = 4n+5, n ∈ A. Then 2n+2 ∈ fA, 2n+2 ∈ A. Since n < 2n+2 are from A, we have 4n+5 ∈ fA. This contradicts C ∩ fA = ∅.

case 2. m ∈ gB. Let m = 4n+5, n ∈ B. Since n < m are from B, we have 4m+6 ∈ fB, 4m+6 ∈ C. Since 4m+6 ∈ fA, this contradicts C ∩ fA = ∅. QED

The following pertains to 4,2’.

LEMMA 3.8.15. C ∪. fA ⊆ A ∪. gA, B ∪. fB ⊆ C ∪. gC has ¬NON.

Proof: Define f,g ∈ ELG as follows. For all n < m, let f(n,n) = 2n+2, f(n,m) = 2m+1, f(m,n) = 2m, g(n) = 4n+5. Let C ∪. fA ⊆ A ∪. gA, B ∪. fB ⊆ C ∪. gC, where A,B,C are nonempty.

Let m ∈ B. Then 2m+2 ∈ fB, 2m+2 ∈ C, 2m+2 ∉ fA, m ∉ A, m ∈ C ∪ gC.

case 1. m ∈ C. Then m ∈ A ∪ gA, m ∈ gA. Let m = 4n+5, n ∈ A. Hence 2n+2 ∈ fA, 2n+2 ∈ A. Since n < 2n+2 are from A, we have 4n+5 = m ∈ fA. This contradicts C ∩ fA = ∅.

case 2. m ∈ gC. Let m = 4n+5, n ∈ C. Hence n ∈ A ∪ gA.

case 2a. n ∈ A. Then 2n+2 ∈ fA, 2n+2 ∈ A, 4n+6 ∈ fA, 4n+6 ∈ A, 8n+12 ∈ fA. Since m ∈ B, we have 2m+2 = 8n+12 ∈ fB, 8n+12 ∈ C. This contradicts C ∩ fA = ∅.

case 2b. n ∈ gA. Let n = 4r+5, r ∈ A. Then 2r+2 ∈ fA, 2r+2 ∈ A, 4r+6 ∈ fA, 4r+6 ∈ A, 8r+12 ∈ fA, 8r+12 ∈ A, 16r+26 ∈ fA, 16r+26 ∈ A, 32r+52 ∈ fA.

Since m ∈ B, we have 2m+2 = 8n+12 = 32r+52 ∈ fB, and so 32r+52 ∈ C. This contradicts C ∩ fA = ∅. QED

The following pertains to 4,3’.

LEMMA 3.8.16. C ∪. fA ⊆ A ∪. gA, B ∪. fB ⊆ C ∪. gA has ¬NON.

Proof: Define f,g ∈ ELG as follows. For all n < m, let f(n,n) = 2n+2, f(n,m) = 2m+1, f(m,n) = 2m, g(n) = 4n+5. Let C ∪. fA ⊆ A ∪. gA, B ∪. fB ⊆ C ∪. gA, where A,B,C are nonempty.

Let m ∈ B. Then 2m+2 ∈ fB, 2m+2 ∈ C, 2m+2 ∉ fA, m ∉ A, m ∈ C ∪ gA.

case 1. m ∈ C. Then m ∈ A ∪ gA, m ∈ gA. Let m = 4n+5, n ∈ A. Hence 2n+2 ∈ fA, 2n+2 ∈ A. Since n < 2n+2 are from A, we have m = 4n+5 ∈ fA. This contradicts C ∩ fA = ∅.

case 2. m ∈ gA. Let m = 4n+5, n ∈ A. Hence 2n+2 ∈ fA, 2n+2 ∈ A, 4n+6 ∈ fA, 4n+6 ∈ A, 8n+12 = 2m+2 ∈ fA. Since 2m+2 ∈ C, this contradicts C ∩ fA = ∅. QED

The following pertains to 4,4’, 4,5’, 4,6’.

LEMMA 3.8.17. C ∪. fA ⊆ A ∪. gX, A ∪. fB ⊆ C ∪. gY has ¬NON.

Proof: Define f,g ∈ ELG as follows. For all n < m, let f(n,n) = 2n+2, f(n,m) = f(m,n) = 2m+2, g(n) = 2n+1. Let C ∪. fA ⊆ A ∪. gX, B ∪. fB ⊆ C ∪. gY, where A,B,C are nonempty.

Let m ∈ A. Then 2m+2 ∈ fA, 2m+2 ∈ A, 2m+2 ∈ C. This contradicts C ∩ fA = ∅. QED

part 5. C ∪. fA ⊆ A ∪. gB.

5,1′. C ∪. fA ⊆ A ∪. gB, B ∪. fB ⊆ C ∪. gB. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
5,2′. C ∪. fA ⊆ A ∪. gB, B ∪. fB ⊆ C ∪. gC. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
5,3′. C ∪. fA ⊆ A ∪. gB, B ∪. fB ⊆ C ∪. gA. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
5,4′. C ∪. fA ⊆ A ∪. gB, A ∪. fB ⊆ C ∪. gB. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
5,5′. C ∪. fA ⊆ A ∪. gB, A ∪. fB ⊆ C ∪. gC. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
5,6′. C ∪. fA ⊆ A ∪. gB, A ∪. fB ⊆ C ∪. gA. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.

The following pertains to 5,1′.

LEMMA 3.8.18. C ∪. fA ⊆ A ∪. gB, B ∪. fB ⊆ C ∪. gB has ¬NON.

Proof: Define f,g ∈ ELG as follows. For all n < m, let f(n,n) = 2n+2, f(n,m) = f(m,n) = 4m+6, g(n) = 2n+1. Let C ∪. fA ⊆ A ∪. gB, B ∪. fB ⊆ C ∪. gB, where A,B,C are nonempty.

Let m ∈ B. Then 2m+2 ∈ fB, 2m+2 ∈ C, 2m+2 ∈ A, 4m+6 ∈ fA, 2m+2 ∉ fA, m ∉ A, m ∈ C ∪ gB.

case 1. m ∈ C. Then m ∈ A ∪ gB, m ∈ gB. This contradicts C ∩ gB = ∅.

case 2. m ∈ gB. Let m = 2n+1, n ∈ B. Since n < m are from B, we have 4m+6 ∈ fB, 4m+6 ∈ C. Since 4m+6 ∈ fA, this contradicts C ∩ fA = ∅.

QED

The following pertains to 5,2′.

LEMMA 3.8.19. C ∪. fA ⊆ A ∪. gB, B ∪. fB ⊆ C ∪. gC has ¬NON.

Proof: Define f,g ∈ ELG as follows. For all n < m, let f(n,n) = 2n+2, f(n,m) = 2m+8, f(m,n) = 2m+4, g(n) = 2n+3. Let C ∪. fA ⊆ A ∪. gB, B ∪. fB ⊆ C ∪. gC, where A,B,C are nonempty.

Let m ∈ B. Then 2m+2 ∈ fB, 2m+2 ∈ C, 2m+2 ∉ fA, m ∉ A, m ∈ C ∪ gC.

case 1. m ∈ C. Then m ∈ A ∪ gB. Hence m ∈ gB. This contradicts B ∩ gB = ∅.

case 2. m ∈ gC. Let m = 2n+3, n ∈ C. Hence n ∈ A ∪ gB.

case 2a. n ∈ A. Then 2n+2 ∈ fA, 2n+2 ∈ A. Since n < 2n+2 are from A, we have 4n+8 = 2m+2 ∈ fA. But 2m+2 ∉ fA.

case 2b. n ∈ gB. Let n = 2r+3, r ∈ B. Now m = 2n+3 = 4r+9 ∈ B. So 2m+2 = 8r+20 ∈ fB, 2m+2 = 8r+20 ∈ C. Note that 2r+2 ∈ fB, 2r+2 ∈ C, 2r+2 ∈ A, 4r+6 ∈ fA, 4r+6 ∈ A. Since 2r+2 < 4r+6 are from A, we have 8r+20 ∈ fA. This contradicts C ∩ fA = ∅. QED

The following pertains to 5,3'.

LEMMA 3.8.20. C ∪. fA ⊆ A ∪. gB, B ∪. fB ⊆ C ∪. gA has ¬NON.

Proof: Define f,g ∈ ELG as follows. For all n < m, let f(n,n) = 2n+2, f(n,m) = 4m+6, f(m,n) = 2m, g(n) = 4n+5. Let C ∪. fA ⊆ A ∪. gB, B ∪. fB ⊆ C ∪. gA, where A,B,C are nonempty.

Let m ∈ B. Then 2m+2 ∈ fB, 2m+2 ∈ C, 2m+2 ∈ A, 4m+6 ∈ fA, 2m+2 ∉ fA, m ∉ A, m ∈ C ∪ gA.

case 1. m ∈ C. Then m ∈ A ∪ gB, m ∈ gB. Let m = 4n+5, n ∈ B. Since n < m are from B, we have 4m+6 ∈ fB, 4m+6 ∈ C. This contradicts C ∩ fA = ∅.

case 2. m ∈ gA. Let m = 4n+5, n ∈ A. Then 2n+2 ∈ fA, 2n+2 ∈ A, 4n+6 ∈ fA, 4n+6 ∈ A. Since 2n+2 < 4n+6 are from A, we have 8n+12 = 2m+2 ∈ fA. Since 2m+2 ∈ C, this contradicts C ∩ fA = ∅.

QED

LEMMA 3.8.21. X ∪. fA ⊆ A ∪. gY, A ∪. fZ ⊆ X ∪. gW has ¬NON.

Proof: Let f be as given by Lemma 3.2.1. Let g ∈ ELG be defined by g(n) = 2n+1. Let X ∪. fA ⊆ A ∪. gY, A ∪. fZ ⊆ X ∪. gW, where X,A,Y,Z,W are nonempty.

Let n ∈ fA ∩ 2N. Then n ∈ A. Hence fA ∩ 2N ⊆ A. By Lemma 3.2.1, fA is cofinite. Hence A contains almost all of 2N. Therefore X contains almost all of 2N. This contradicts X ∩ fA = ∅. QED

LEMMA 3.8.22. 5,4′, 5,5′, 5,6′ have ¬NON.

Proof: By Lemma 3.8.21. QED

part 6. C ∪. fA ⊆ A ∪. gC.

6,1′. C ∪. fA ⊆ A ∪. gC, B ∪. fB ⊆ C ∪. gB. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
6,2′. C ∪. fA ⊆ A ∪. gC, B ∪. fB ⊆ C ∪. gC. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
6,3′. C ∪. fA ⊆ A ∪. gC, B ∪. fB ⊆ C ∪. gA. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
6,4′. C ∪. fA ⊆ A ∪. gC, A ∪. fB ⊆ C ∪. gB. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
6,5′. C ∪. fA ⊆ A ∪. gC, A ∪. fB ⊆ C ∪. gC. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
6,6′. C ∪. fA ⊆ A ∪. gC, A ∪. fB ⊆ C ∪. gA. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.

The following pertains to 6,1′.

LEMMA 3.8.23. C ∪. fA ⊆ A ∪. gC, B ∪. fB ⊆ C ∪. gB has ¬NON.

Proof: Define f,g ∈ ELG as follows. For all n < m, let f(n,n) = 2n+2, f(n,m) = f(m,n) = 2m, g(n) = 2n+1. Let C ∪. fA ⊆ A ∪. gC, B ∪. fB ⊆ C ∪. gB, where A,B,C are nonempty.

We claim that for all t ∈ A and p ≥ 0, $2g^p(t)+2$ ∈ A ∩ fA. To see this, fix t ∈ A and argue by induction on p ≥ 0. Obviously $2g^0(t)+2 = 2t+2$ ∈ fA, and so $2g^0(t)+2 = 2t+2$ ∈ A ∩ fA. Suppose $2g^p(t)+2$ ∈ A ∩ fA. Note that $2g^{p+1}(t)+2 = 2(2g^p(t)+1)+2 = 2(2g^p(t)+2)$ ∈ fA, since $t < 2g^p(t)+2$ are from A. Hence $2g^{p+1}(t)+2$ ∈ A ∩ fA.

Let m = min(B). Then 2m+2 ∈ fB, 2m+2 ∈ C, 2m+2 ∉ fA, m ∉ A, m ∈ C ∪ gB, m ∉ gB, m ∈ C, m ∈ A ∪ gC, m ∈ gC, $g^{-1}(m)$ ∈ C.

Let p be greatest such that

$g^{-1}(m),\ldots,g^{-p}(m)$ ∈ C.

Then $p \geq 1$ and $g^{-p}(m) \in C \backslash gC$. Hence $g^{-p}(m) \in A$.

By the claim, $2g^p(g^{-p}(m))+2 \in A \cap fA$. Hence $2m+2 \in A \cap fA$. Since $2m+2 \in C$, this contradicts $C \cap fA = \emptyset$. QED

The following pertains to 6,2'.

LEMMA 3.8.24. $C \cup. fA \subseteq A \cup. gC$, $B \cup. fB \subseteq C \cup. gC$ has ¬NON.

Proof: Define $f,g \in ELG$ as follows. For all $n < m$, let $f(n,n) = 2n+2$, $f(n,m) = f(m,n) = 2m$, $g(n) = 2n+1$. Let $C \cup. fA \subseteq A \cup. gC$, $B \cup. fB \subseteq C \cup. gC$, where A,B,C are nonempty.

Let $m \in B$. Then $m \in C \cup gC$.

case 1. $m \in C$. Then $m \in A \cup gC$, $m \notin gC$, $m \in A$, $2m+2 \in fA$, $2m+2 \in A$, $2m+2 \in fB$, $2m+2 \in C$. This contradicts $C \cap fA = \emptyset$.

case 2. $m \in gC$. Let $m = 2n+1$, $n \in C$. Then $n \notin gC$, $n \in A$, $2n+2 \in fA$, $2n+2 \in A$. Since $n < 2n+2$ are from A, we have $4n+4 = 2m+2 \in fA$, $2m+2 \in fB$, $2m+2 \in C$. This contradicts $C \cap fA = \emptyset$. QED

The following pertains to 6,3'.

LEMMA 3.8.25. $C \cup. fA \subseteq A \cup. gC$, $B \cup. fB \subseteq C \cup. gA$ has ¬NON.

Proof: Define $f,g \in ELG$ as follows. For all $n < m$, let $f(n,n) = 2n+2$, $f(n,m) = f(m,n) = 2m$, $g(n) = 2n+1$. Let $C \cup. fA \subseteq A \cup. gC$, $B \cup. fB \subseteq C \cup. gA$, where A,B,C are nonempty.

As in the proof of Lemma 3.8.23, for all $t \in A$ and $p \geq 0$, $2g^p(t)+2 \in A \cap fA$.

Let $m \in B$. Then $2m+2 \in fB$, $2m+2 \in C$, $2m+2 \notin fA$, $m \notin A$, $m \in C \cup gA$.

case 1. $m \in C$. Then $m \in A \cup gC$, $m \in gC$, $g^{-1}(m) \in C$.

Let p be greatest such that $g^{-1}(m),\ldots,g^{-p}(m) \in C$.

Then $p \geq 1$ and $g^{-p}(m) \in C \backslash gC$. Hence $g^{-p}(m) \in A$.

By the claim, $2g^p(g^{-p}(m))+2 \in A \cap fA$. Hence $2m+2 \in A \cap fA$.

Since 2m+2 ∈ C, this contradicts C ∩ fA = ∅.

case 2. m ∈ gA. Let m = 2n+1, n ∈ A. Then 2n+2 ∈ fA, 2n+2 ∈ A. Since n < 2n+2 are from A, we have 4n+4 = 2m+2 ∈ fA. Since 2m+2 ∈ C, this contradicts C ∩ fA = ∅.

QED

LEMMA 3.8.26. 6,4′, 6,5′, 6,6′ have ¬NON.

Proof: By Lemma 3.8.21. QED