3.9. ABAB.

Recall the following reduced table for AB from section 3.5.

REDUCED AB

1. A ∪. fA ⊆ B ∪. gA. INF. AL. ALF. FIN. NON.
2. A ∪. fA ⊆ B ∪. gB. INF. AL. ALF. FIN. NON.
3. A ∪. fA ⊆ B ∪. gC. INF. AL. ALF. FIN. NON.
4. C ∪. fA ⊆ B ∪. gA. INF. AL. ALF. FIN. NON.
5. C ∪. fA ⊆ B ∪. gB. INF. AL. ALF. FIN. NON.
6. C ∪. fA ⊆ B ∪. gC. INF. AL. ALF. FIN. NON.

The duplicate pairs were treated in section 3.3. We now treat the 15 ordered pairs from this table, where the first clause is earlier in the list than the second clause. We determine the status of INF, AL, ALF, FIN, NON for each such ordered pair.

1,2. A ∪. fA ⊆ B ∪. gA, A ∪. fA ⊆ B ∪. gB. ¬INF. ¬AL. ¬ALF. FIN. NON.
1,3. A ∪. fA ⊆ B ∪. gA, A ∪. fA ⊆ B ∪. gC. INF. AL. ALF. FIN. NON.
1,4. A ∪. fA ⊆ B ∪. gA, C ∪. fA ⊆ B ∪. gA. INF. AL. ALF. FIN. NON.
1,5. A ∪. fA ⊆ B ∪. gA, C ∪. fA ⊆ B ∪. gB. ¬INF. ¬AL. ¬ALF. FIN. NON.
1,6. A ∪. fA ⊆ B ∪. gA, C ∪. fA ⊆ B ∪. gC. INF. AL. ALF. FIN. NON.
2,3. A ∪. fA ⊆ B ∪. gB, A ∪. fA ⊆ B ∪. gC. INF. AL. ALF. FIN. NON.
2,4. A ∪. fA ⊆ B ∪. gB, C ∪. fA ⊆ B ∪. gA. ¬INF. ¬AL. ¬ALF. FIN, NON.
2,5. A ∪. fA ⊆ B ∪. gB, C ∪. fA ⊆ B ∪. gB. INF. AL. ALF. FIN. NON.
2,6. A ∪. fA ⊆ B ∪. gB, C ∪. fA ⊆ B ∪. gC. ¬INF. ¬AL. ¬ALF. FIN. NON.
3,4. A ∪. fA ⊆ B ∪. gC, C ∪. fA ⊆ B ∪. gA. INF. AL. ALF. FIN. NON.
3,5. A ∪. fA ⊆ B ∪. gC, C ∪. fA ⊆ B ∪. gB. ¬INF. ¬AL. ¬ALF. FIN. NON.
3,6. A ∪. fA ⊆ B ∪. gC, C ∪. fA ⊆ B ∪. gC. INF. AL. ALF. FIN. NON.
4,5. C ∪. fA ⊆ B ∪. gA, C ∪. fA ⊆ B ∪. gB. ¬INF. AL. ¬ALF. FIN. NON.
4,6. C ∪. fA ⊆ B ∪. gA, C ∪. fA ⊆ B ∪. gC. INF. AL. ALF. FIN. NON.

5,6. C ∪. fA ⊆ B ∪. gB, C ∪. fA ⊆ B ∪. gC. ¬INF. ¬AL. ¬ALF. FIN, NON.

LEMMA 3.9.1. (1,3), (1,4), (1,6), (2,3), (2,5), (3,4), (3,6), (4,6) have INF, ALF, even for EVSD.

Proof: Note that A ∪. fA ⊆ B ∪. gA has INF, ALF, and A ∪. fA ⊆ B ∪. gB has INF, ALF, even for EVSD, by the AB table in section 3.3. Now set C = A in all of the above ordered pairs except (2,3). For (2,3), set C = B. QED

The following pertains to (1,2).

LEMMA 3.9.2. A ∪. fA ⊆ B ∪. gA, A ∪. fA ⊆ B ∪. gB has FIN.

Proof: Let f,g ∈ ELG. Let A = {n}, where n is sufficiently large.

case 1. f(n,...,n) = g(n,...,n). Set A = B = {n}.

case 2. f(n,...,n) ≠ g(n,...,n). Set A = {n}, B = {n,f(n,...,n)}. Note that A ⊆ B.

In case 1, fA = gA = gB, A ∩ fA = B ∩ gA = B ∩ gB = ∅.

In case 2, note that A ⊆ B, A ∩ fA = B ∩ gA = ∅, fA ⊆ B.

We claim that B ∩ gB = ∅. To see this, first note that n ∉ gB since n is sufficiently large. Also note that f(n,...,n) ∉ gB, since f(n,...,n) ≠ g(n,...,n), and f(n,...,n) ≠ g(...,f(n,...,n)...). QED

LEMMA 3.9.3. (1,5), (2,4), (2,6), (3,5), (4,5), (5,6) have FIN.

Proof: From Lemma 3.9.2, by setting C = A in the cited ordered pairs. QED

LEMMA 3.9.4. fA ⊆ B ∪. gA, A ∩ fA = B ∩ gB = ∅ has ¬AL.

Proof: Define f,g ∈ ELG as follows. For all n < m, let f(n,n) = 2n, f(n,m) = 4m, f(m,n) = 8m, g(n) = 2n. Let fA ⊆ B ∪. gA, A ∩ fA = B ∩ gB = ∅, where A,B have at least two elements. Let n < m be from A.

Clearly 2m,4m,8m ∈ fA. Hence 2m,4m,8m ∉ A. So 4m,8m ∉ gA. Hence 4m,8m ∈ B, 8m ∈ fB. This contradicts B ∩ fB = ∅. QED

LEMMA 3.9.5. (1,2), (1,5) have ¬AL.

Proof: By Lemma 3.9.4. QED

The following pertains to (2,4).

LEMMA 3.9.6. A ∪. fA ⊆ B ∪. gB, C ∪. fA ⊆ B ∪. gA has ¬AL.

Proof: Define f,g ∈ ELG as follows. For all n < m, let f(n,n) = 2n, f(n,m) = f(m,n) = 4m+1, g(n) = 2n+1. Let A ∪. fA ⊆ B ∪. gB, C ∪. fA ⊆ B ∪. gA, where A,B have at least two elements. Let n < m be from A.

Clearly 2m ∈ fA, 2m ∈ B, 2m ∉ A, 4m+1 ∉ gA, 4m+1 ∈ fA, 4m+1 ∈ B, 4m+1 ∈ gB. This contradicts B ∩ gB = ∅. QED

LEMMA 3.9.7. fA ⊆ B ∪. gB, fA ⊆ B ∪. gC, C ∩ fA = ∅ has ¬AL.

Proof: Define f,g ∈ ELG as follows. For all n < m, let f(n,n) = 2n, f(n,m) = f(m,n) = 4m+1, g(n) = 2n+1. Let fA ⊆ B ∪. gB, fA ⊆ B ∪. gC, C ∩ fA = ∅, where A,B,C have at least two elements. Let n < m be from A.

Clearly 2m ∈ fA, 2m ∈ B, 2m ∉ C, 4m+1 ∉ gC, 4m+1 ∈ fA, 4m+1 ∈ B, 4m+1 ∈ gB. This contradicts B ∩ gB = ∅. QED

LEMMA 3.9.8. (2,6), (3,5), (5,6) have ¬AL.

Proof: By Lemma 3.9.7. QED

The following pertains to (4,5).

LEMMA 3.9.9. C ∪. fA ⊆ B ∪. gA, C ∪. fA ⊆ B ∪. gB has AL.

Proof: Note that C ∪. fA ⊆ A ∪. gA has AL by the AA table of section 3.3. Replace B by A in the cited pair. QED

The following pertains to (4,5).

LEMMA 3.9.10. C ∪. fA ⊆ B ∪. gA, C ∪. fA ⊆ B ∪. gB has ¬INF, ¬ALF.

Proof: Let f be as given by Lemma 3.2.1. Let f' ∈ ELG be given by f'(a,b,c,d) = f(a,b,c) if c = d; 2f(a,b,c)+1 if c > d; 2|a,b,c,d|+2 if c < d. Let g ∈ ELG be given by g(n) =

2n+1. Let C ∪. f’A ⊆ B ∪. gA, C ∪. f’A ⊆ B ∪. gB, where A,B,C have at least two elements. Let A’ = A\{min(A)}.

Note that fA’ ⊆ fA ⊆ f’A. To see this, let a,b,c ∈ A. Then f(a,b,c) = f’(a,b,c,c).

Let n ∈ fA’ ∩ 2N. Write n = f(a,b,c), a,b,c ∈ A'. Then 2n+1 = f'(a,b,c,min(A)), 2n+1 ∈ f'A. Also n ∈ f'A. Hence n ∈ B, 2n+1 ∈ gB, 2n+1 ∉ B, 2n+1 ∈ gA, n ∈ A, n > min(A), n ∈ A'. Thus we have shown that fA’ ∩ 2N ⊆ A’. Hence by Lemma 3.2.1, fA’ is cofinite.

It is now clear that A’ is infinite, and therefore A is infinite. This establishes ¬ALF.

We also see that C is finite, since f’A is cofinite and C ∩ f’A = ∅. This establishes ¬INF. QED