CHAPTER 4.
PROOF OF PRINCIPAL EXOTIC CASE

4.1. Strongly Mahlo Cardinals of Finite Order.
4.2. Proof using Strongly Mahlo Cardinals.
4.3. Some Existential Sentences.
4.4. Proof using 1-consistency.

4.1. Strongly Mahlo Cardinals of Finite Order.

The large cardinal properties used in this book are the strongly Mahlo cardinals of order \( n \), where \( n \in \omega \). These are defined inductively as follows.

**DEFINITION 4.1.1.** The strongly 0-Mahlo cardinals are the strongly inaccessible cardinals (uncountable regular strong limit cardinals). The strongly \( n+1 \)-Mahlo cardinals are the infinite cardinals all of whose closed unbounded subsets contain a strongly \( n \)-Mahlo cardinal.

It is easy to prove by induction on \( n \) that for all \( n < m < \omega \), every strongly \( m \)-Mahlo cardinal is a strongly \( n \)-Mahlo cardinal.

There is a closely related notion: \( n \)-Mahlo cardinal.

**DEFINITION 4.1.2.** The 0-Mahlo cardinals are the weakly inaccessible cardinals (uncountable regular limit cardinals). The \( n+1 \)-Mahlo cardinals are the infinite cardinals all of whose closed unbounded subsets contain an \( n \)-Mahlo cardinal.

Again, for all \( n < m < \omega \), every \( m \)-Mahlo cardinal is an \( n \)-Mahlo cardinal.

**NOTE:** Sometimes (strongly) \( n \)-Mahlo cardinals are called (strongly) Mahlo cardinals of order \( \leq n \). Also, sometimes what we call \( n \)-Mahlo cardinals are called weakly \( n \)-Mahlo cardinals.
The well known relationship between n-Mahlo cardinals and strongly n-Mahlo cardinals is given as follows.

**THEOREM 4.1.1.** The following is provable in ZFC. Let \( n < \omega \).
A cardinal is strongly n-Mahlo if and only if it is n-Mahlo and strongly inaccessible. Under the GCH, a cardinal is strongly n-Mahlo if and only if it is n-Mahlo.

**Proof:** For the first claim, note that it is obvious for \( n = 0 \). Assume that every strongly inaccessible n-Mahlo cardinal is strongly n-Mahlo. Let \( \kappa \) be a strongly inaccessible n+1-Mahlo cardinal. Let \( A \subseteq \kappa \) be closed and unbounded. Since \( \kappa \) is strongly inaccessible, the set \( B \subseteq \kappa \) consisting of the strong limit cardinals in \( A \) is closed and unbounded. Let \( \lambda \in B \) be an n-Mahlo cardinal. As previously remarked, \( \lambda \) is an inaccessible cardinal. Since \( \lambda \) is a strong limit cardinal, \( \lambda \) is a strongly inaccessible cardinal. By the induction hypothesis, \( \lambda \) is a strongly n-Mahlo cardinal. We have thus shown that every closed unbounded \( A \subseteq \kappa \) contains a strongly n-Mahlo element. Hence \( \kappa \) is strongly n+1-Mahlo.

For the final claim, assume the GCH. By an obvious induction, every strongly n-Mahlo cardinal is an n-Mahlo cardinal. For the converse, let \( \kappa \) be an n-Mahlo cardinal. As previously remarked, \( \kappa \) is a weakly inaccessible cardinal. Hence \( \kappa \) is a strongly inaccessible cardinal (by GCH). By the first claim, \( \kappa \) is a strongly n-Mahlo cardinal. QED

We now develop the essential combinatorics of strongly Mahlo cardinals of finite order used in this Chapter.

**DEFINITION 4.1.3.** Let \( [A]^n \) be the set of all n element subsets of \( A \). Sometimes we write \( x \in [A]^n \) in the form \( \{x_1,...,x_n\} \) to indicate that the \( x_i \) are strictly increasing. Let \( A \) be a set of ordinals. We say that \( f:[A]^n \rightarrow \omega \) is regressive if and only if for all \( x \in [A \setminus \{0\}]^n \), \( f(x) < \min(x) \).

**DEFINITION 4.1.4.** We say that \( E \) is min homogenous for \( f:[A]^n \rightarrow \omega \) if and only if \( E \subseteq A \) and for all \( x,y \in [E]^n \), \( \min(x) = \min(y) \rightarrow f(x) = f(y) \).

**LEMMA 4.1.2.** Let \( n \geq 0 \), \( \kappa \) a strongly n-Mahlo cardinal, \( A \subseteq \kappa \) unbounded, and \( f:[A]^{n+2} \rightarrow \kappa \) be regressive. For all \( \alpha < \kappa \),
there exists $E \subseteq A$ of order type $\alpha$ which is min homogenous for $f$.

Proof: This result originally appeared in [Sc74], in somewhat sharper form, using different notation. We present the proof in [HKS87], p. 147, using Erdös-Rado trees.

DEFINITION 4.1.5. Let $A$ be a set of ordinals with at least two elements. An $A$-tree is an irreflexive transitive relation $T$ with field $A$ such that

i. $\alpha \ T \ \beta \rightarrow \alpha < \beta$.

ii. $\left\{ \beta : \beta \ T \ \alpha \right\}$ is linearly (and hence well) ordered by $T$.

DEFINITION 4.1.6. Let $m \geq 2$, $A$ be a nonempty set of ordinals, and $f : [A]^n \rightarrow \text{On}$ be regressive. The Erdős-Rado tree $E^R(f)$ is the unique $A$-tree $T$ with field $A$ such that for all $\alpha, \beta \in A, \alpha \ T \beta$ if and only if

i. $\alpha < \beta$.

ii. For all $\gamma_1, \ldots, \gamma_{m-1} \ T \alpha$ with $\gamma_1 < \ldots < \gamma_{m-1}, f(\{\gamma_1, \ldots, \gamma_{m-1}, \alpha\}) = f(\{\gamma_1, \ldots, \gamma_{m-1}, \beta\})$.

To see that there is such a unique $T$, build $E^R(f, \alpha), \alpha \in A$, by transfinite recursion on $\alpha \in A$. Here $E^R(f, \alpha)$ is $E^R(f)$ restricted to $A \cap \alpha$. The details are left to the reader.

DEFINITION 4.1.7. For $\alpha \in A$, the height of $\alpha$ in $E^R(f)$ is the order type of $\left\{ \beta : \beta \ E^R(f) \ \alpha \right\}$. We say that $\alpha, \beta \in A$ are siblings in $E^R(f)$ if and only if they are distinct, and have the same strict predecessors in $E^R(f)$. For ordinals $\gamma$, let $E^R(f)[<\gamma]$ be the restriction of $E^R(f)$ to the elements of $A$ (vertices) of height $< \gamma$.

We now assume that $f : [A]^{n+2} \rightarrow \text{On}$ is regressive and $\text{sup}(A)$ is a strongly inaccessible cardinal $\kappa$. Observe that for all $\alpha \in A$, the number of siblings of $\alpha$ in $E^R(f)$ is at most the number of functions from $\alpha^{n+1}$ into $\alpha$, which is at most $2^{\alpha^{n+1}} + \omega$. Next observe that by transfinite induction on $\alpha < \kappa$, $E^R(f)[<\alpha]$ has $< \k \alpha$ vertices. Hence for all $\alpha < \kappa$, $E^R(f)$ has a vertex of height $\alpha$. By the construction of $E^R(f)$, every vertex has height $< \kappa$.

Now observe that if $n = 0$ then the set of strict predecessors of every element of $E^R(f)$ is min homogeneous for $f$. This establishes the Lemma for the basis case $n = 0$. 
Suppose that the Lemma holds for a fixed \( n \geq 0 \). Let \( \kappa \) be a strongly \( n+1 \)-Mahlo cardinal, \( A \subseteq \kappa \) be unbounded, \( \alpha < \kappa \), and \( f:[A]^{n+3} \to \kappa \) be regressive. We use the Erdős-Rado tree \( \text{ERT}(f) \).

Since \( \kappa \) is strongly inaccessible, \( C = \{ \lambda < \kappa : \lambda \) is a limit ordinal \( \) and \( \text{ERT}(f)[<\lambda] \) is an \( A \cap \lambda \)-tree and \( A \cap \lambda \) is unbounded in \( \lambda \} \) is a closed and unbounded subset of \( \kappa \). Since \( \kappa \) is a strongly \( n+1 \)-Mahlo cardinal, fix \( \lambda < \kappa \) to be a strongly \( n \)-Mahlo cardinal \( > \alpha \) such that \( \text{ERT}(f)[<\lambda] \) is an \( A \cap \lambda \)-tree and \( A \cap \lambda \) is unbounded in \( \lambda \).

Let \( v \) be a vertex of \( \text{ERT}(f) \) of height \( \lambda \). Let \( B = \{ w : w \in \text{ERT}(f) v \} \). Then \( B \) is an unbounded subset of \( \lambda \).

\( B \) naturally gives rise to a regressive function \( f^*:[B]^{n+2} \to \lambda \) by taking \( f^*(x) = f(x \cup \{ \gamma \}) \), where \( \gamma \in B \), \( \gamma > \max(x) \).

Note that this definition is independent of the choice of \( \gamma \).

By the induction hypothesis, let \( E \subseteq B \) be min homogenous for \( f^* \), \( E \) of order type \( \alpha \). Then \( E \subseteq B \subseteq A \) is min homogenous for \( f \). QED

**DEFINITION 4.1.8.** For all ordinals \( \alpha \), let \( \alpha' \) be the least infinite cardinal \( > \alpha \). Let \( f:[A]^n \to \kappa \). We say that \( f \) is next regressive if and only if every \( f(x_1, \ldots, x_n) < \min(x_1, \ldots, x_n)^+ \).

**LEMMA 4.1.3.** Let \( n \geq 0 \), \( \kappa \) a strongly \( n \)-Mahlo cardinal, and \( A \subseteq \kappa \) be unbounded. For all \( \alpha < \kappa \), there exists \( E \subseteq A \) of order type \( \alpha \) such that for all \( i \in \omega \), \( E \) is min homogenous for \( f_i \).

Proof: This is by a straightforward modification of the proof of Lemma 4.1.2. Modify the definition of the Erdős-Rado tree \( \text{ERT}(f) \) accordingly, and derive a similar upper bound on the number of siblings of a vertex in \( \text{ERT}(f) \). QED

Let \( n \geq 1 \) and \( f:[A]^n \to \kappa \). We wish to define \( n+1 \) kinds of infinite sets \( E \subseteq A \) for \( f \).

**DEFINITION 4.1.9.** We say that \( E \) is of kind 0 for \( f \) if and only if \( f \) is constant on \( [E]^n \), where the constant value is less than the strict sup of \( E \).
DEFINITION 4.1.10. We say that $E$ is of kind $1 \leq j \leq n$ for $f$ if and only if the following holds. For all $\{x_1, \ldots, x_n\} < \{x_1, \ldots, x_j, y_{j+1}, \ldots, y_n\} \subseteq E$, $f(x_1, \ldots, x_n) = f(x_1, \ldots, x_j, y_{j+1}, \ldots, y_n)$ is greater than every element of $E < x_j$ and smaller than every element of $E > x_j$.

For $E \subseteq \text{On}$ and $\delta < \text{ot}(E)$, we write $E[\delta]$ for the $\delta$-th element of $E$.

We fix $H: \text{On}^{\omega} \to \text{On}\setminus\{0\}$, where $H$ is one-one and for all $x \in \text{On}^{\omega}$, $H(x) < \max(x)^+$.

LEMMA 4.1.4. Let $n \geq 1$, $\kappa$ a strongly $n$-Mahlo cardinal, and $A \subseteq \kappa$ unbounded. For all $i \in \omega$, let $f_i: [A]^{n+1} \to \kappa$. For all $\alpha < \kappa$, there exists $E \subseteq A$ of order type $\alpha$ such that the following holds. For all $i \in \omega$, there exists $0 \leq j \leq n+1$ such that $E$ is of kind $j$ for $f_i$.

Proof: Let $n, \kappa, A, f_i, \alpha$ be as given. We can assume that $\alpha > \omega$, $A \subseteq \kappa \setminus \omega$, and there is an infinite cardinal strictly between any two elements of $A$. We can also assume that for all $\alpha_1, \ldots, \alpha_{n+1} < \beta$ from $A$, $f_i(\alpha_1, \ldots, \alpha_{n+1}) < \beta$.

For all $i \in \omega$, define $g_{i,0}(u, x_1, \ldots, x_{n+1}) < 1+f_i(x_1, \ldots, x_{n+1})$ if $f_i(x_1, \ldots, x_{n+1}) \leq u$; 0 otherwise.

For $1 \leq j \leq n+1$, define $g_{i,j}(u, x_{j+1}, \ldots, x_{n+2})$ as follows. Let $z_1 < \ldots < z_j < u$ be such that $f_i(z_1, \ldots, z_j, x_{j+1}, \ldots, x_{n+1}) \neq f_i(z_1, \ldots, z_j, x_{j+2}, \ldots, x_{n+2})$ and $f_i(z_1, \ldots, z_j, x_{j+1}, \ldots, x_{n+1}) \leq u$. Set $g_{i,j}(u, x_{j+1}, \ldots, x_{n+2}) = H(z_1, \ldots, z_j, f_i(z_1, \ldots, z_j, x_{j+1}, \ldots, x_{n+1}))$. If such $z$’s do not exist, then set $g_{i,j}(u, x_{j+1}, \ldots, x_{n+2}) = 0$.

Note that each $g_{i,j}$ is next regressive. By Lemma 4.1.3, let $E' \subseteq A \setminus \omega$ be min homogeneous for all $g_{i,j}$, where $E'$ has cardinality $\geq \aleph_\omega(\alpha+\omega)$ = the first strong limit cardinal $> \alpha+\omega$.

We can partition the tuples from $E'$ of length $\leq 2n+2$ in a strategic way, with $2^{\omega}$ pieces, and apply the Erdős-Rado theorem to obtain $E \subseteq E'$ with order type $\alpha$, with the following three properties. Write $E[1], E[2], \ldots$ for the first $\omega$ elements of $E$. Let $i \in \omega$.

1) For all $\{x_1, \ldots, x_{n+1}\} \in [E]^{n+1}$, $f_i(x_1, \ldots, x_{n+1}) \in E \to f_i(x_1, \ldots, x_{n+1}) \in \{x_1, \ldots, x_{n+1}\}$.
2) Suppose \( f_i \{E[2], \ldots, E[n+2]\} = f_i \{E[n+3], \ldots, E[2n+3]\} \). Then \( f_i \) is constant on \([E]^{n+1}\).

3) Suppose \( 1 \leq j \leq n+1 \), and \( f_i \{E[2], E[4], \ldots, E[2j], E[2j+4], E[2j+6], \ldots, E[2n+4]\} \subseteq (E[2j-1], E[2j+1]) \). Then \( E \) is of kind \( j \) for \( f_i \).

For the remainder of the proof, we fix \( i \in \omega \). The first case that applies is the operative case.

**case 1.** \( f_i \{E[2], E[4], \ldots, E[2n+2]\} \leq E[1] \). Then 
\[
g_{i,0} \{E[1], E[2], E[4], \ldots, E[2n+2]\} = 1 + f_i \{E[2], E[4], \ldots, E[2n+2]\} > 0.
\]
Since \( E \) is min homogenous for \( g_{i,0} \) we see that for all \( x, y \in [E]^{n+1} \) such that \( \min(x), \min(y) \geq E[2] \), we have 
\[
g_{i,0} (\{E[1]\} \cup x) = g_{i,0} (\{E[1]\} \cup y) = 1 + f_i (x) = 1 + f_i (y).
\]
In particular, \( f_i \{E[2], \ldots, E[n+2]\} = f_i \{E[n+3], \ldots, E[2n+3]\} \). By 2), \( f_i \) is constant on \([E]^{n+1}\). Hence \( E \) is of kind 0 for \( f_i \).

**case 2.** Let \( j \) be the greatest element of \([1, n+1]\) such that 
\( f_i \{E[2], E[4], \ldots, E[2n+2]\} \in (E[2j-1], E[2j+1]) \). Note that 
\[
g_{i,j} \{E[2j+1], E[2j+2], E[2j+4], \ldots, E[2n+4]\} = g_{i,j} \{E[2j+1], E[2j+4], E[2j+6], \ldots, E[2n+6]\}.
\]
Suppose the main clause in the definition of 
\( g_{i,j} \{E[2j+1], E[2j+2], E[2j+4], \ldots, E[2n+4]\} \) holds, with \( z_1 < \ldots < z_j \leq E[2j+1] \). Since \( H \) is nonzero, the main clause in the definition of 
\( g_{i,j} \{E[2j+1], E[2j+4], E[2j+6], \ldots, E[2n+6]\} \) holds with, say, \( w_1 < \ldots < w_j \leq E[2j+1] \). Hence 
\[
H (z_1, \ldots, z_j, f_i (z_1, \ldots, z_j, E[2j+2], E[2j+4], \ldots, E[2n+2])) = H (w_1, \ldots, w_j, f_i (w_1, \ldots, w_j, E[2j+4], E[2j+6], \ldots, E[2n+4])).
\]
Therefore \( z_1, \ldots, z_j = w_1, \ldots, w_j \), respectively, and 
\[
f_i (z_1, \ldots, z_j, E[2j+2], E[2j+4], \ldots, E[2n+2]) = f_i (w_1, \ldots, w_j, E[2j+4], E[2j+6], \ldots, E[2n+4]).
\]
This contradicts the choice of \( z_1, \ldots, z_j \).

Hence the main clause in the definition of 
\( g_{i,j} \{E[2j+1], E[2j+2], E[2j+4], \ldots, E[2n+4]\} \) fails. In particular, it fails with \( z_1, \ldots, z_j = E[2], E[4], \ldots, E[2j] \), respectively. Then 
\[
f_i \{E[2], E[4], \ldots, E[2n+2]\} = f_i \{E[2], E[4], \ldots, E[2j], E[2j+4], E[2j+6], \ldots, E[2n+4]\}.
\]
By 3), \( E \) is of kind \( j \) for \( f_i \).

**case 3.** Otherwise. Then 
\[
f_i \{E[2], E[4], \ldots, E[2n+2]\} \in \{E[1], E[3], \ldots, E[2n+1]\}, \text{ or } f_i \{E[2], E[4], \ldots, E[2n+2]\} \geq E[2n+3].
\]
The first disjunct is impossible by 1), and the second disjunct is impossible by the assumption on \( A \).
We have thus shown that for some \( j \in [0, n+1] \), \( E \) is of kind \( j \) for \( f_i \). Since \( i \) is arbitrarily chosen from \( \omega \), we are done.

QED

**Definition 4.1.11.** Let \( f: [A]^n \rightarrow \kappa \) and \( E \subseteq A \). We define \( fE \) to be the range of \( f \) on \( [E]^n \).

**Lemma 4.1.5.** Let \( n, m \geq 1, \kappa \) a strongly \( n \)-Mahlo cardinal, and \( A \subseteq \kappa \) unbounded. For all \( i \in \omega \), let \( f_i : [A]^{n+1} \rightarrow \kappa \), and let \( g_i : [A]^m \rightarrow \omega \). There exists \( E \subseteq \kappa \) of order type \( \omega \) such that

1. for all \( i \in \omega \), \( f_i \) is either constant on \( [E]^{n+1} \), with constant value \( < \text{sup}(E) \), or \( f_i E \) is of order type \( \omega \) with the same sup as \( E \);
2. for all \( i \in \omega \), \( g_i \) is constant on \( [E]^m \).

**Proof:** Let \( n, m, \kappa, A, f_i, g_i \) be as given. Apply Lemma 4.1.4 to obtain \( E' \subseteq \kappa \) of order type \( \aleph_0(\omega) \) such that the following holds. For all \( i \in \omega \) there exists \( 0 \leq j \leq n+1 \) such that \( E \) is of kind \( j \) for \( f_i \). By the Erdös-Rado theorem, let \( E \subseteq E' \) be of order type \( \omega \), where for all \( i \in \omega \), \( g_i \) is constant on \( [E]^m \). Write \( E = \{E[1], E[2], \ldots\} \).

Let \( i \in \omega \) and \( E \) be of kind \( j \) for \( f_i \). If \( j = 0 \) then \( f_i \) is constant on \( [E]^{n+1} \), where the constant value is less than \( \text{sup}(E) \).

Now suppose \( 1 \leq j \leq n+1 \). For all \( \{x_1, \ldots, x_{n+1}\} \),
\[
\{x_1, \ldots, x_j, y_{j+1}, \ldots, y_{n+1}\} \subseteq E, \quad f_i\{x_1, \ldots, x_{n+1}\} = f_i\{x_1, \ldots, x_j, y_{j+1}, \ldots, y_{n+1}\} \text{ is greater than every element of } E \text{ and smaller than every element of } E > x_j.
\]
Since we can set \( x_j \) to vary among \( E[j], E[j+1], \ldots \), we see that \( f_i E \) has the same sup as \( E \). In particular, \( f_i E \) is infinite.

Also, for any particular \( E[p] \), the values \( f_i\{x_1, \ldots, x_{n+1}\} < E[p] \), \( x_1 < \ldots < x_{n+1} \subseteq A \), can arise only if \( x_j \leq E[p+1] \). Since the arguments \( x_{j+1}, \ldots, x_{n+1} \) don't matter (kind \( j \) for \( f_i \)), there are at most finitely many such values.

We have shown that \( f_i E \) has at most finitely many elements not exceeding any given element of \( E \). Therefore \( f_i E \) has order type \( \leq \omega \). Since \( f_i E \) is infinite, the order type of \( f_i E \) is \( \omega \). QED

We now switch over to ordered tuples. Let \( f : A^n \rightarrow \kappa \) and \( E \subseteq A \). Here we also define \( fE \) to be the range of \( f \) on \( E^n \).
LEMMA 4.1.6. Let $n, m \geq 1$, $\kappa$ a strongly $n$-Mahlo cardinal, and $A \subseteq \kappa$ unbounded. For all $i \in \omega$, let $f_i : A^{n+1} \to \kappa$, and let $g_i : A^m \to \omega$. There exists $E \subseteq \kappa$ of order type $\omega$ such that

i) for all $i \geq 1$, $f_i E$ is either a finite subset of $\text{sup}(E)$, or of order type $\omega$ with the same $\text{sup}$ as $E$;

ii) for all $i \in \omega$, $g_i E$ is finite.

Proof: Let $n, m, \kappa, A, f_i, g_i$ be as given. Each $f_i$ gives rise to finitely many corresponding $f_{i, \sigma}$, where $\sigma$ ranges over the order types of $n+1$ tuples. Also each $g_i$ gives rise to finitely many corresponding $g_{i, \sigma}$, where $\sigma$ ranges over the order types of $m$ tuples. Any $f_i E$ is the union of the $f_{i, \sigma} E$, and any $g_i E$ is the union of the $g_{i, \sigma} E$. Choose $E$ according to Lemma 4.1.5. Then $E$ will be as required. QED

DEFINITION 4.1.12. Let SMAH+ be $\text{ZFC} + (\forall n < \omega) (\exists \kappa) (\kappa$ is a strongly $n$-Mahlo cardinal). Let SMAH be $\text{ZFC} + \{(\exists \kappa) (\kappa$ is a strongly $n$-Mahlo cardinal)$\}_{n < \omega}$.

DEFINITION 4.1.13. Let MAH+ be $\text{ZFC} + (\forall n < \omega) (\exists \kappa) (\kappa$ is an $n$-Mahlo cardinal). Let MAH be $\text{ZFC} + \{(\exists \kappa) (\kappa$ is an $n$-Mahlo cardinal)$\}_{n < \omega}$.

We will use the following (known) relationship between SMAH+, MAH+, SMAH, and MAH.

DEFINITION 4.1.14. The system EFA = exponential function arithmetic is defined to be the system $I \Sigma_0 (\exp)$; see [HP93].

THEOREM 4.1.7. SMAH+ and MAH+ prove the same $\Pi_2^1$ sentences. SMAH and MAH prove the same $\Pi_2^1$ sentences. SMAH is 1-consistent if and only if MAH is 1-consistent. SMAH is consistent if and only if MAH is consistent. These results are provable in EFA.

Proof: We first prove the following well known theorem in ZFC.

1) Let $n \geq 0$. Every $n$-Mahlo cardinal is an $n$-Mahlo cardinal in the sense of $L$.

The basis case asserts that every weakly inaccessible cardinal is a weakly inaccessible cardinal in $L$. This is particularly well known and easy to check.
Fix \( n \geq 0 \) and assume that every \( n \)-Mahlo cardinal is an \( n \)-Mahlo cardinal in \( L \). Let \( \kappa \) be an \( n+1 \)-Mahlo cardinal. Let \( A \subseteq \kappa \), \( A \in L \), where \( A \) is closed and unbounded in \( \kappa \) (in the sense of \( L \)). Let \( \lambda \in A \) be an \( n \)-Mahlo cardinal. Then \( \lambda \in A \) is an \( n \)-Mahlo cardinal in \( L \). Hence \( \kappa \) is an \( n+1 \)-Mahlo cardinal in \( L \).

If \( T \) is a sentence or set of sentences in the language of set theory, then we write \( T^{(L)} \) for the relativization of \( T \) to Gödel's constructible universe \( L \).

For the first claim, let \( \text{SMAH}^+ \) prove \( \phi \), where \( \phi \) is \( \Pi^1_2 \). By Lemma 4.1.1, \( \text{MAH}^+ + \text{GCH} \) proves \( \phi \). Hence \( \text{ZFC} + \text{MAH}^{+(L)} + \text{GCH}^{(L)} \) proves \( \phi^{(L)} \) by, e.g., [Je78], section 12. Therefore \( \text{ZFC} + \text{MAH}^{+(L)} \) proves \( \phi^{(L)} \) by, e.g., [Je78], section 13. By the Shoenfield absoluteness theorem (see, e.g., [Je78], p. 530), \( \text{ZFC} + \text{MAH}^{+(L)} \) proves \( \phi \). By 1), \( \text{MAH}^+ \) proves \( \phi \).

For the second claim, we repeat the proof of the first claim for any specific level of strong Mahloness.

For the third claim, assume \( 1 \)-\text{Con(MAH)}). Let \( \phi \) be a \( \Sigma^0_1 \) sentence provable in \( \text{SMAH} \). By the second claim, \( \phi \) is provable in \( \text{MAH} \). Hence \( \phi \) is true.

For the final claim, assume \( \text{Con(MAH)} \). Then \( \text{MAH} \) does not prove \( 1 = 0 \). By the second claim, \( \text{SMAH} \) does not prove \( 1 = 0 \). Hence \( \text{Con(SMAH)} \). QED

Theorem 4.1.7 tells us that for the purposes of this book, \( \text{SMAH}^+ \) and \( \text{SMAH} \) are equivalent to \( \text{MAH}^+ \) and \( \text{MAH} \). We will always use \( \text{SMAH}^+ \) and \( \text{SMAH} \).