4.4. Proof using 1-consistency.

In this section we show that Propositions A,B can be proved in ACA′ + 1-Con(SMAH). Here 1-Con(T) is the 1-consistency of T, which asserts that "every Σ^0_1 sentence provable in T is true". 1-Con(T) is also equivalent to "every Π^0_2 sentence provable in T is true".

By Lemma 4.2.1, Proposition B implies Proposition A in RCA_0. Hence it suffices to show that Proposition B can be proved in ACA′ + 1-Con(SMAH).

DEFINITION 4.4.1. We write ELG(p,b) for the set of all f ∈ ELG of arity p satisfying the following conditions. For all $x \in N^p$,

i. if $|x| > b$ then $(1 + 1/b)|x| \leq f(x) \leq b|x|$.
ii. if $|x| \leq b$ then $f(x) \leq b^2$.

Note that from Definition 2.1, f ∈ ELG if and only if there exist positive integers p,b such that f ∈ ELG(p,b). Also note that each ELG(p,b) forms a compact subspace of the Baire space of functions from N^k into N.

DEFINITION 4.4.2. Let $p,q,b \geq 1$. A p,q,b-structure is a system of the form

$$M^* = (N^*,0^*,1^*,<^*,+^*,f^*,g^*,c_0^*,\ldots)$$

such that

1. N* is countable. For specificity, we can assume that N* is N.
2. (N*,0*,1*,<*,+*) is a discretely ordered commutative semigroup (see definition below).
3. $+^*:N^{*2} \to N^*$, $f^*:N^{*p} \to N^*$, $g^*:N^{*q} \to N^*$.
4. f* obeys the above two inequalities for membership in ELG(p,b), internally in M*.
5. g* obeys the above two inequalities for membership in ELG(q,b), internally in M*.
6. Let $i \geq 0$. The sum of any finite number of copies of c_i^* is $< c_{i+1}^*$.
7. The c*'s form a strictly increasing set of indiscernibles for the atomic sentences of M*.

Note that the conditions under clauses 4-7 are all universal sentences.

Note that we do not require every element of N* to be the value of a closed term.

DEFINITION 4.4.3. A discretely ordered commutative semigroup is a system (G,0,1,<,+) such that

i. < is a linear ordering of G.
ii. 0,1 are the first two elements of G.
iii. x+0 = x.
iv. x+y = y+x.
v. (x+y)+z = x+(y+z).
vi. x < y → x+z < y+z.
vii. x+1 is the immediate successor of x.

Note that the cancellation law

$$x+z = y+z \rightarrow x = y$$

holds in any discretely ordered commutative semigroup (in this sense), since assuming x+z = y+z, the cases x < y and y < x are impossible.

In any p,q,b-structure, the c_n* have an important inaccessibility condition: any closed term whose value is c_n* is a sum consisting of c_n* and zero or more 0*'s. To see this, write c_n* = t, and write t as a sum, t = s_1 + ... + s_k, k ≥ 1, where each s_i is either a constant or starts with f or g. By 7, c_n* is infinite, and so all s_i that begin with f or g must have immediate subterms < c_n* (using 4,5). Hence all s_i that begin with f or g must be < c_n* (using 4,5,6). Hence all s_i are either < c_n* or are a constant. If no s_i is c_i* then all s_i are < c_n*, violating 6. Hence some s_i is c_n*. By 2, the remaining s_i must be 0.

We can follow the development of section 4.2 starting right after the proof of Lemma 4.2.7. In this rerun, we do not fix f ∈ ELG(p,b), and g ∈ ELG(q,b).

Instead we fix p,q,b,n ≥ 1, a strongly p^{n-1}-Mahlo cardinal κ, and a p,q,b-structure M*, where every element of N* is the value of a closed term in M*. Note that we must have b ≥ 2.

As in the development of section 4.2 after the proof of Lemma 4.2.7, we extend M* to the structure

$$M^{**} = (N^{**},<^{**},0^{**},1^{**},+^{**},f^{**},g^{**},c_0^{**},...,c_\alpha^{**},...),$$

$\alpha < \kappa$.

We follow this prior development through the first line of the proof of Theorem 4.2.26.

Thus we have $r \geq 1$, $E \subseteq S \subseteq \kappa$ of order type ω, and sets $E[1] \subseteq ... \subseteq E[n] \subseteq M^{**}[S,r]$ such that

i. $E[1] = \{c_\alpha^{**}: \alpha \in E\}$.
ii. For all $1 \leq i < n$, $f^{**}E[i] \subseteq E[i+1] \cup. g^{**}E[i+1]$.

This construction of $E \subseteq S \subseteq \kappa$ of order type ω uses that κ is strongly p^{n-1}-Mahlo.

In the proof of Theorem 4.2.26, we continued by transferring this situation back into N via an S,r(p+q)-embedding T from M** into M, thus establishing Proposition B with the sets $TE[1] \subseteq ... \subseteq TE[n]$.

Here we want to merely transfer this situation back into M* via an S,r(p+q)-embedding from M** into M*, and then establish uniformities. By Lemma 4.2.12, we use the unique isomorphism from M**<S> onto M* which maps $\{c_\alpha^{**}: \alpha \in S\}$ onto $\{c_j^*: j \geq 0\}$.

As in section 4.2, for $r \geq 1$, we write M*[r] for the set of all values of closed terms of length $\leq r$ in M*.

Thus we obtain $r \geq 1$ and infinite sets $D[1] \subseteq ... \subseteq D[n] \subseteq M^*[r]$ such that

iii. $D[1] \subseteq \{c_j^*: j \geq 0\}$.
iv. For all $1 \leq i < n$, $f^*D[i] \subseteq D[i+1] \cup. g^*D[i+1]$.

We summarize this modified development as follows.

LEMMA 4.4.1. Let $p,q,b,n \geq 1$. The following is provable in SMAH. Let $M^* = (N^*,0^*,1^*,<^*,+^*,f^*,g^*,c_0^*,...)$ be a p,q,b-structure. There exist $r \geq 1$ and infinite sets $D[1] \subseteq ... \subseteq D[n] \subseteq M^*[r]$ such that $D[1] \subseteq \{c_j^*: j \geq 0\}$, and for all $1 \leq i < n$, $f^*D[i] \subseteq D[i+1] \cup. g^*D[i+1]$. Furthermore, this entire Lemma, starting with "Let p...", is provable in RCA_0.

Proof: Let p,q,b,n,M* be as given. Proceed as discussed above. One of the important points is that we only need $M^* = (N^*,0^*,1^*,<^*,+^*)$ to obey the axioms for a discretely ordered commutative group. QED

By using Lemma 4.4.1, we will no longer need to refer back to section 4.2.

We can obviously view clauses 3-7 in the definition of p,q,b-structure as universal axioms. Recall that b is a standard integer.

We now introduce the notion of p,q,b;r-structure, which is a level r approximation to a p,q,b-structure.

DEFINITION 4.4.4. Let p,q,b,r ≥ 1. A p,q,b;r-structure is a system of the form

$$M^* = (N^*,0^*,1^*,<^*,+^*,f^*,g^*,c_0^*,...)$$

such that the following holds.

a. Clauses 1,2,3 in the definition of p,q,b-structure, without change.
b. All instantiations of the universal sentences under clauses 4-7, by closed terms of length ≤ r. Here length counts the total number of occurrences of constant and function symbols that appear.

In particular, we are using the following specialization of clause 7 in the definition of p,q,b-structure:

7′. The c*′s form a strictly increasing set of indiscernibles for the atomic sentences of M* whose terms are of length ≤ r.

Again, we do not require that every element of N* be the value of a closed term.

DEFINITION 4.4.5. A p,q,b;r;n-special structure is a p,q,b;r-structure M* where there exist infinite $D_1 \subseteq ... \subseteq D_n \subseteq M^*[r/(p+q)]$ such that
i. For all $1 \leq i < n$, $f^*D_i \subseteq D_{i+1} \cup. g^*D_{i+1}$.
ii. $D_1 \subseteq \{c_j:^* j \geq 0\}$.

We use M*[r/(p+q)] instead of M*[r] since in clause i, we are applying f*,g* to p,q, terms, respectively, and want all relevant terms to have length at most r.

DEFINITION 4.4.6. The r-type of a p,q,b;r-structure M* is the set of all closed atomic sentences, whose terms have length ≤ r, involving only the constants $0,1,c_0,...,c_{2r}$, which hold in M*. Thus r-types are finite sets.

DEFINITION 4.4.7. A p,q,b;r-type is the r-type of a p,q,b;r-structure. A p,q,b;r;n-special type is the r-type of a p,q,b;r;n-special structure.

LEMMA 4.4.2. Let M* be a p,q,b;r-structure. Then M* is a p,q,b;r;n-special structure if and only if the r-type of M* is a p,q,b;r;n-special type.

Proof: Let M* be a p,q,b;r-structure. First suppose that M* is a p,q,b;r;n-special structure. Then by definition, the r-type of M* is a p,q,b;r;n-special type.

Conversely, suppose the r-type τ of M* is a p,q,b;r;n-special type. Let M*′ be a p,q,b;r;n-special structure of r-type τ.

Let $D_1 \subseteq ... \subseteq D_n \subseteq$ M*′[r/(p+q)] be infinite, where
i. For all $1 \leq i < n$, $f^*D_i \subseteq D_{i+1} \cup. g^*D_{i+1}$.
ii. $D_1 \subseteq \{c_j^*: j \geq 0\}$.

We can obviously come up with an infinite list of atomic sentences whose terms are of length ≤ r, whose truth in M*′ witnesses that M*′ is a p,q,b;r;n-special structure. These include the atomic sentences with terms of length ≤ r that justify that M*' is a p,q,b;r-structure, and the atomic sentences with terms of length ≤ r that justify the special clauses i,ii just above. This uses the fact that the lengths of $f(s_1,...,s_p)$, $g(t_1,...,t_q)$ are ≤ r provided the lengths of $s_1,...,s_p,t_1,...,t_q$ are ≤ r/(p+q). But since M* and M*′ have the same r-type, they agree on all such statements. Hence M* is a p,q,b;r;n-special structure. QED

We can view the following as a uniform version of Lemma 4.4.1.

LEMMA 4.4.3. Let p,q,b,n ≥ 1. The following is provable in SMAH. There exist r ≥ 1 such that every p,q,b;r-structure is p,q,b;r;n-special. Furthermore, this entire Lemma, starting with “Let p...” is provable in RCA_0.

Proof: Fix p,q,b,n ≥ 1. We now argue in SMAH. Suppose this is false. Let T be the following theory in the language of p,q,b-structures.

i. Let $r \geq 1$. Assert the axioms for being a p,q,b;r-structure.
ii. Let $r \geq 1$ and τ be a p,q,b;r;n-special type. Assert that τ is not the r-type of the p,q,b;r-structure.

We claim that every finite subset of T is satisfiable. To see this, let r be an upper bound on the r's used in the finite subset. By hypothesis, there exists a p,q,b;r-structure M* that is not a p,q,b;r;n-special structure. Fix r,M*.

We claim that M* satisfies the finite subset of T. Let τ be the r-type of the p,q,b;r-structure M*.

Obviously M* satisfies all instances of i) for $r' \leq r$. Now let $1 \leq r' \leq r$ and τ' be a p,q,b;r';n-special type. Suppose that τ' is the correct r'-type of M*. I.e., M* has r'-type τ'. By Lemma 4.4.2, M* is a p,q,b;r';n-special structure. Since M* is a p,q,b;r-structure, M* is a p,q,b;r;n-special structure. This is a contradiction.

By the compactness theorem, T is satisfiable. Let M* satisfy T. By Lemma 4.4.1, let r be such that M* is p,q,b;r;n-special. Let τ be the r-type of M*. Then τ is a p,q,b;r;n-special type. By axioms ii) above, τ is not the r-type of M*. This is a contradiction. QED

LEMMA 4.4.4. There is a presentation of a primitive recursive function $Q(p,q,b,r,\tau)$ such that the following is provable in RCA_0. $Q(p,q,b,r,\tau) = 1$ if and only if τ is a p,q,b;r-type (as a Gödel number).

Proof: We give the following necessary and sufficient finitary condition for τ to be a p,q,b;r-type.

1. τ is a set of atomic sentences in $0,1,<,+,f,g,c_0,\ldots,c_{2r}$ whose terms have length $\leq r$, involving only the constants $0,1,c_0,\ldots,c_{2r}$.

2. There is a system $V^* = (D,E,0^*,1^*,<^*,+^*,f^*,g^*,c_0^*,\ldots,c_{2r}^*)$ which obeys the following conditions.

i. D,E have cardinality at least 1 and at most some specific iterated exponential in p,q,r.
ii. $0^*,1^* \in D$.
iii. $+^*:D^2 \to E$.

iv. $f^*:D^p \to E$.
v. $g^*:D^q \to E$.
vi. D is the set of values of the closed terms of length ≤ r.
vii. E is D union the values of $+^*,f^*,g^*$.
vii. All axioms in clause b in the definition of p,q,b;r-structure hold in V*.
viii. All sentences in τ hold in V*.
ix. All atomic sentences in $0,1,<,+,f,g,c_0,\ldots,c_{2r}$ outside τ, with terms of length ≤ r, fail in V*.

This condition is necessary because such a structure V* can be obtained from any p,q,b;r-structure M* of r-type τ by taking D to be the set of values of closed terms in M* of length ≤ r, restricting M* in the obvious way. The atomic sentences in $0,1,<,+,f,g,c_0,\ldots,c_{2r}$ that hold in V* are the same as those that hold in M*, which are the elements of τ.

For the other direction, let τ,V* be given as above. Using the indiscernibility in ix, we can canonically stretch V* to

$$W^* = (D',E',0^*,1^*,<^{*\prime},+^{*\prime},f^{*\prime},g^{*\prime},c_0^{*\prime},c_1^{*\prime},\ldots)$$

which obviously obeys clause 1 and clauses 2i-2ix above, modified to incorporate all constant symbols $c_0,c_1,\ldots$. We now have all of the conditions we need for being a p,q,b;r-structure except that we only have $D' \subseteq E'$. However, this is easily remedied without affecting the properties of W* by taking the domain to be E′, and extending $+^{*\prime},f^{*\prime},g^{*\prime}$ arbitrarily to the tuples from E′ that are not tuples from D′, into E′. This resulting modification of W* is a p,q,b,r-structure with r-type τ. QED

Let τ be a p,q,b;r-type. We want to express

1) τ is a p,q,b;r;n-special type

as a sentence $\lambda(k,n,p+q+2,R_1,\ldots,R_{n-1})$ of section 4.3, and then apply Theorem 4.3.8.

Recall that 1) is equivalent to the condition

2) there exists a p,q,b;r-structure M* of r-type τ and infinite sets $D_1 \subseteq \ldots \subseteq D_n \subseteq M^*[r/(p+q)]$ such that
i. For all $1 \leq i < n$, $f^*D_i \subseteq D_{i+1} \cup. g^*D_{i+1}$.
ii. $D_1 \subseteq \{c_j^*: j \geq 0\}$.

We now put this in a more syntactic form.

DEFINITION 4.4.8. A p,q,r-term is a closed term in 0,1,+,f,g and constants $c_0, c_1, \ldots$ of length at most r.

We identify M*[r] with the set of all p,q,r-terms. Of course, a given element of M*[r] may be the value of many p,q,r-terms.

DEFINITION 4.4.9. We let τ^* be the set of all atomic sentences obtained from elements of τ by replacing c's by c's in an order preserving way.

3) there exist infinite sets $T_1 \subseteq \ldots \subseteq T_n$ of p,q,r/(p+q)-terms such that
i. For any two distinct elements t,t' of T_n, $t = t' \notin \tau^*$.
ii. Every $t \in T_1$ is some c_k.
iii. Let $1 \leq i < n$ and $t_1,\ldots,t_p \in T_i$. Then there exists $t \in T_{i+1}$ such that $f(t_1,\ldots,t_p) = t \in \tau^*$, or there exist $t_1',\ldots,t_q' \in T_{i+1}$ such that $f(t_1,\ldots,t_p) = g(t_1',\ldots,t_q') \in \tau^*$.
iv. Let $t,t_1,\ldots,t_q \in T_n$. Then $g(t_1,\ldots,t_q) = t \notin \tau^*$.
v. For all $k \geq 0$ and $t_1,\ldots,t_p \in T_n$, $f(t_1,\ldots,t_p) = c_k \notin \tau^*$.

LEMMA 4.4.5. The following is provable in RCA_0. Let $p,q,b,n,r \geq 1$ and τ be a p,q,b;r-type. Then conditions 1)-3) are equivalent.

Proof: Let τ be a p,q,b;r-type. It is obvious that 1),2) are equivalent. So assume 2) holds. We derive 3). Let M* be a p,q,b;r-structure of r-type τ, and $D_1 \subseteq \ldots \subseteq D_n \subseteq$ M*[r/(p+q)] be infinite sets such that

i. For all $1 \leq i < n$, $fD_i \subseteq D_{i+1} \cup. gD_{i+1}$.
ii. $D_1 \subseteq \{c_j^*: j \geq 1\}$.

For each $x \in D_n$, pick a p,q,r/(p+q)-term x# of least possible length whose value in M* is x. If x is some c_i^* then make sure that x# is c_i. Set $T_i = \{x\#: x \in D_i\}$.

Since $D_1 \subseteq \ldots \subseteq D_n$, clearly $T_1 \subseteq \ldots \subseteq T_n$. Since every $x \in D_n$ lies in M*[r/(p+q)], clearly every $x\# \in T_n$ has length $\leq$ r/(p+q).

Let $t,t' \in T_n$ be distinct. Write t = x#, t' = y#. Then $x\# \neq y\#$, and so t = t' is false. Hence $t = t' \notin \tau^*$. Let $t \in T_1$.

Write $t = x\#$, $x \in D_1$. Then x is some $c_k{}^*$. Therefore $x\# = c_k$. This establishes 3i and 3ii.

To verify 3iii, let $1 \le i < n$ and $x_1\#,\ldots,x_p\# \in T_i$. Then $x_1,\ldots,x_p \in D_i$. Hence $f^*(x_1,\ldots,x_p) \in f^*D_i \subseteq D_{i+1} \cup. \; g^*D_{i+1}$.

case 1. $f^*(x_1,\ldots,x_p) \in D_{i+1}$. Let the $p,q,r/(p+q)$-term $t \in T_{i+1}$ have the value $f^*(x_1,\ldots,x_p)$ in M^*. Then $f(x_1\#,\ldots,x_p\#) = t$ holds in M^*, and both terms in this equation have length $\le r$. Hence $f(x_1{}^*,\ldots,x_p{}^*) = t \in \boldsymbol{\tau}^*$.

case 2. $f^*(x_1,\ldots,x_p) \in gD_{i+1}$. Let $f^*(x_1,\ldots,x_p) = g^*(y_1,\ldots,y_q)$, where $y_1,\ldots,y_q \in D_{i+1}$. Then $y_1\#,\ldots,y_q\# \in T_{i+1}$. Also $f(x_1{}^*,\ldots,x_p{}^*) = g(y_1{}^*,\ldots,y_q{}^*)$ holds in M^*, and both terms in this equation have length $\le r$. Hence $f(x_1{}^*,\ldots,x_p{}^*) = g(y_1{}^*,\ldots,y_q{}^*) \in \boldsymbol{\tau}^*$.

To verify 3iv, let $x\#,x_1\#,\ldots,x_q\# \in T_n$. Then $g(x_1\#,\ldots,x_q\#) = x\# \notin \boldsymbol{\tau}^*$ because $g^*(x_1,\ldots,x_q) \ne x$ in M^*.

To verify 3v, let $k \ge 0$ and $x_1\#,\ldots,x_p\# \in T_n$. Then $f(x_1\#,\ldots,x_p\#) = c_k \notin \boldsymbol{\tau}^*$ because $f^*(x_1,\ldots,x_p) \ne c_k{}^*$ in M^*.

Now assume that 3) holds. We establish 2). Let $T_1 \subseteq \ldots \subseteq T_n$ be infinite sets of $p,q,r/(p+q)$-terms such that

i. For any two distinct elements t,t' of T_n, $t = t' \notin \boldsymbol{\tau}^*$.
ii. For all $t \in T_1$ there exists $k \ge 0$ such that t is c_k.
iii. Let $1 \le i < n$ and $t_1,\ldots,t_p \in T_i$. Then there exists $t \in T_{i+1}$ such that $f(t_1,\ldots,t_p) = t \in T_{i+1}$, or there exist $t_1',\ldots,t_q' \in T_{i+1}$ such that $f(t_1,\ldots,t_p) = g(t_1',\ldots,t_q') \in \boldsymbol{\tau}^*$.
iv. Let $t,t_1,\ldots,t_q \in T_n$. Then $g(t_1,\ldots,t_q) = t \notin \boldsymbol{\tau}^*$.
v. For all $k \ge 0$ and $t_1,\ldots,t_p \in T_n$, $f(t_1,\ldots,t_p) = c_k \notin \boldsymbol{\tau}^*$.

Let M^* be any $p,q,b;r$-structure of r-type $\boldsymbol{\tau}$. For each $1 \le i \le n$, let D_i be the set of values of terms in T_i. Then $D_1 \subseteq \ldots \subseteq D_n \subseteq M^*[r/(p+q)]$.

Let $1 \le i < n$ and $x \in f^*D_i$. We claim that $x \in D_{i+1} \cup g^*D_{i+1}$.

To see this, write $x = f^*(x_1,\ldots,x_p)$, $x_1,\ldots,x_p \in D_i$, and let $t_1,\ldots,t_p \in T_i$ have values $x_1,\ldots,x_p$, respectively. By 3iii, let $t \in T_{i+1}$, where $f(t_1,\ldots,t_p) = t \in \boldsymbol{\tau}^*$, or there exists $t_1',\ldots,t_q' \in T_{i+1}$ such that $f(t_1,\ldots,t_p) = g(t_1',\ldots,t_q') \in \boldsymbol{\tau}^*$.

case 1. $f(t_1,...,t_p) = t \in \tau^*$. Then $f^*(x_1,...,x_p) = x \in D_{i+1}$.

case 2. Let $t_1',...,t_q' \in T_{i+1}$, where $f(t_1,...,t_p) = g(t_1',...,t_q') \in \tau^*$. Let the values of $t_1',...,t_q'$ be $y_1,...,y_q \in D_{i+1}$, respectively. Then $f^*(x_1,...,x_p) = g^*(y_1,...,y_q)$.

Now suppose $x \in D_{i+1} \cap gD_{i+1}$. Let x be the value of $t \in T_{i+1}$, and write $x = g(y_1,...,y_q)$, $y_1,...,y_q \in D_{i+1}$. Let $t_1,...,t_q \in T_{i+1}$ have values $y_1,...,y_q$, respectively. By 3iv, $g(t_1,...,t_q) = t \notin \tau^*$. Since both terms in this equation have length $\leq r$, we see that $g(t_1,...,t_q) = t$ is false in M^*. Hence $g^*(y_1,...,y_q) \neq x$. This is a contradiction.

Finally, let $x \in D_1$. Then x is the value of a term $t \in T_1$. By 3ii, t is some c_k. Hence x is some c_k^*. QED

We can conveniently represent the p,q,r-terms as elements of N^k in the following way. This integer k will be set below.

DEFINITION 4.4.10. Two p,q,r-terms have the same shape if and only if the second can be obtained from the first by replacing c's by c's, where we do not require that equal c's be replaced by equal c's.

Let e be the number of shapes of the p,q,r-terms.

We represent the p,q,r-term σ as follows. Let the shape of σ be $1 \leq i \leq e$. Here the shapes have been arbitrarily indexed without repetition, by $1 \leq i \leq e$.

DEFINITION 4.4.9. The representations of σ are obtained as follows. First write down a sequence of e elements of N, where exactly i of these elements are the same as the first of these elements. Follow this by the sequence of subscripts of the c's that appear from left to right. If this sequence of c's is of length < r then fill it out to length r by repeating the last argument. This results in a representation of σ as an element of N^{e+r}. Obviously, σ will have infinitely many representations.

Set k = e+r. We will use the above representation of p,q,r-terms to write 3) in the form of a sentence $\lambda(k,n,p+q+2,R_1,...,R_{n-1})$, as in section 4.3.

4) There exist infinite sets $B_1 \subseteq \ldots \subseteq B_n \subseteq N^k$ of p,q,r/(p+q)-representations such that
a. Distinct elements of B_n represent distinct p,q,r/(p+q)-terms.
b. For each $1 \leq i \leq n$, let T_i be the p,q,r/(p+q)-terms represented by the elements of B_i. Then $T_1,\ldots,T_n$ obeys 3) above.

Note the use of τ^* in 3). We represent elements of τ^* as a p,q,r-representation followed by two equal elements of N (indicating <), or followed by two unequal elements of N (indicating =), followed by a p,q,r-representation. Keep in mind that the lengths of p,q,r-representations are fixed at k = e+r. Hence representations of elements of τ^* are fixed at length k+2+k = 2k+2. If τ is a p,q,b;r-type, then τ is finite and τ^* is order invariant.

LEMMA 4.4.6. The following is provable in RCA_0. Let $p,q,b,n,r \geq 1$ and τ be a p,q,b;r-type. Conditions 1)-4) are each equivalent to $\lambda(k,n,p+q+2,R_1,\ldots,R_{n-1})$, for some order invariant relations $R_1,\ldots,R_{n-1} \subseteq N^{2k(p+q+2)}$ obtained explicitly from p,q,b,n,r,τ.

Proof: We argue in RCA_0. Let $p,q,b,n,r \geq 1$ and τ be a p,q,b;r-type. It is clear that 3) is equivalent to 4), and hence by Lemma 4.4.5, 1)-4) are equivalent. We now exclusively use clause 4.

$B_1 \subseteq \ldots \subseteq B_n$ asserts, for each $1 \leq i < n$, that $(\forall x \in B_i)(\exists y \in B_{i+1})(x = y)$.

"Distinct elements of B_n represent distinct p,q,r/(p+q)-terms" is of the form $(\forall x,y \in B_n)(S(x,y))$.

"Distinct elements t,t' of the corresponding T_n have $t = t' \notin \tau^*$" is of the form $(\forall x,y \in B_n)(S(x,y))$.

Clause 3ii for the corresponding T_1 is of the form $(\forall x \in B_1)(S(x))$.

Clause 3iii for the corresponding T's is of the form $(\forall i \in [1,n))(\forall x_1,\ldots,x_p \in B_i)(\exists y_1,\ldots,y_q \in B_{i+1})(S(x_1,\ldots,x_p, y_1,\ldots,y_q))$.

Clause 3iv for the corresponding T_n is of the form $(\forall x_1,\ldots,x_{q+1} \in B_n)(S(x_1,\ldots,x_{q+1}))$.

Clause 3v for the corresponding T_n is of the form $(\forall x_1,...,x_{p+1} \in B_n)(S(x_{p+1}) \rightarrow S'(x_1,...,x_{p+1}))$.

Here all the S's are order invariant relations. QED

LEMMA 4.4.7. There is a presentation of a primitive recursive function H such that the following is provable in ACA'. Let $p,q,b,n,r \geq 1$ and τ be a $p,q,b;r$-type. Then $H(p,q,b,r,n,\tau) = 1$ if and only if τ is a $p,q,b;r;n$-special type (as a Gödel number).

Proof: Let p,q,b,r,n,τ be given, where τ is a $p,q,b;r$-type. Apply Lemma 4.4.6 to obtain order invariant $R_1,...,R_{n-1}$. Now apply Theorem 4.3.8. QED

We fix H as given by Lemma 4.4.7.

LEMMA 4.4.8. Let $p,q,b,n \geq 1$. The following is provable in SMAH. $(\exists r)(\forall\tau)(Q(p,q,b,r,\tau) = 1 \rightarrow H(p,q,b,r,n,\tau) = 1)$. Furthermore, this entire Lemma, starting with "Let p...", is provable in RCA_0.

Proof: Let p,q,b,n be as given. By Lemma 4.4.3, SMAH proves the existence of $r \geq 1$ such that every $p,q,b;r$-type is a $p,q,b;r;n$-special type. Now apply Lemmas 4.4.4 and 4.4.7. QED

LEMMA 4.4.9. RCA_0 + 1-Con(SMAH) proves $(\forall p,q,b,n \geq 1)(\exists r)(\forall\tau)(Q(p,q,b,r,\tau) = 1 \rightarrow H(p,q,b,r,n,\tau) = 1)$.

Proof: We argue within RCA_0 + 1-Con(SMAH). Let $p,q,b,n \geq 1$ be given. By Lemma 4.4.8,

1) $(\exists r)(\forall\tau)(Q(p,q,b,r,\tau) = 1 \rightarrow H(p,q,b,r,n,\tau) = 1)$

is provable in SMAH. Note that the quantifier $\forall\tau$ in 1) is bounded. Hence by 1-Con(SMAH), this Σ^0_1 sentence is true. QED

LEMMA 4.4.10. The following is provable in ACA' + 1-Con(SMAH). $(\forall p,q,b,n \geq 1)(\exists r)(\forall\tau)(\tau$ is a $p,q,b;r$-type $\rightarrow \tau$ is a $p,q,b;r;n$-special type).

Proof: By Lemmas 4.4.4, 4.4.7, and 4.4.9. QED

For Propositions C,D, see Appendix A.

THEOREM 4.4.11. Propositions A,B,C,D are provable in ACA′ + 1-Con(SMAH).

Proof: Propositions A,C,D are immediate consequences of Proposition B over RCA_0 (see Lemmas 4.2.1 and 5.1.1). We argue in ACA′ + 1-Con(SMAH). Let p,q,b,n ≥ 1, and f ∈ ELG(p,b), g ∈ ELG(q,b). Let r be given by Lemma 4.4.10. By Ramsey's theorem for 2r-tuples in ACA′, we can find a p,q,b;r-structure M = $(N,0,1,<,+,f,g,c_0,c_1,...)$. Let **τ** be its r-type. By Lemma 4.4.10, **τ** is a p,q,b;n;r-special type. By Lemma 4.4.2, M is a p,q,b;r;n-special structure. Let $D_1 \subseteq ... \subseteq D_n \subseteq N$, where $D_1 \subseteq \{c_0,c_1,...\}$, and each $fD_i \subseteq D_{i+1} \cup. gD_{i+1}$, and $D_1 \cap fD_n = \emptyset$. This is Proposition B, thus concluding the proof. QED