5.2. From length 3 towers to length n towers.

In this section, we obtain a variant of Lemma 5.1.7 (Lemma 5.2.12) involving length n towers rather than length 3 towers of infinite sets. However, we only assert that the sets in the length n tower have at least r elements, for any $r \geq 1$. Thus we pay a real cost for lengthening the towers.

Because the sets in the tower are finite and not infinite, certain indiscernibility properties of the first set in the tower must now be stated explicitly as additional conditions. See Lemma 5.2.12, iii), viii). These indiscernibility properties can of course be obtained from the usual infinite Ramsey theorem by taking a subset of the infinite $A \subseteq N$ from Lemma 5.1.7 - but then we would only have a tower of length 3.

We will apply Lemma 5.1.7 with f arising from term assignments. Thus Lemma 5.2.12 uses g and not f.

Recall the definition of the language L (Definition 5.1.8). In order to avoid having to write too many parentheses in terms and formulas of L, we use the following two standard precedence tables.

$$\uparrow$$
$$\bullet$$
$$+,-$$

$$\neg$$
$$\wedge,\vee$$
$$\rightarrow,\leftrightarrow$$

DEFINITION 5.2.1. Let t be a term of L. We write #(t) for the maximum of: the subscripts of variables in t, and the number of occurrences of the symbols

$$0 1 + - \bullet \uparrow ( ) v_1 v_2, \ldots \log$$

We count log as a single symbol. Note that for all $n \geq 0$, $\{t: \#(t) \leq n\}$ is finite.

DEFINITION 5.2.2. Let φ be a quantifier free formula in L. We write $\#(\varphi)$ for the maximum of: the subscripts of variables in φ, and the number of occurrences of the symbols

$$01+-\bullet\uparrow()=<\neg\wedge\vee\rightarrow\leftrightarrow v_1 v_2,\ldots,v_r \text{ log}$$

in φ. Note that for all $n \geq 0$, $\{\varphi: \#(\varphi) \leq n\}$ is finite.

DEFINITION 5.2.3. For all $r \geq 1$, let $\beta(r)$ be the number of terms t in L with $\#(t) \leq r$. We fix a doubly indexed sequence $t[i,r]$ of terms in L, which is defined if and only if $r \geq 1$ and $1 \leq i \leq \beta(r)$. For each $r \geq 1$, the sequence $t[i,r]$, $1 \leq i \leq \beta(r)$, enumerates the terms t with $\#(t) \leq r$, without repetition.

DEFINITION 5.2.4. For all $r \geq 1$, let $\gamma(r)$ be the number of quantifier free formulas φ in L with $\#(\varphi) \leq r$. We fix a doubly indexed sequence $\varphi[i,r]$ of quantifier free formulas in L, which is defined if and only if $r \geq 1$ and $1 \leq i \leq \gamma(r)$. For each $r \geq 1$, the sequence $\varphi[i,r]$, $1 \leq i \leq \gamma(r)$, enumerates the quantifier free formulas φ with $\#(\varphi) \leq r$, without repetition.

We adhere to the convention of displaying all free variables (and possibly additional variables). Thus $t(v_1,\ldots,v_n)$ and $\varphi(v_1,\ldots,v_m)$ respectively indicate that all variables in the term t are among the first n variables $v_1,\ldots,v_n$, and all variables in the quantifier free formula φ are among the first m variables $v_1,\ldots,v_m$.

Note that all terms $t[i,r]$ have variables among $v_1,\ldots,v_r$, and all formulas $\varphi[i,r]$ have variables among $v_1,\ldots,v_r$.

We want to be more specific about the enumerations of terms and formulas in Definitions 5.2.3, 5.2.4.

DEFINITION 5.2.5. Let $r \geq 1$. The enumeration $t[1,r],\ldots,t[\beta(r),r]$ in Definition 5.2.3 is the enumeration of all terms t of L with $\#(t) \leq r$, ordered first by $\#(t)$, and second by the lexicographic ordering of strings of symbols, where, for specificity, the symbols are ordered by

$$01+-\bullet\uparrow()\, v_1 v_2\ldots v_r \text{ log}$$

DEFINITION 5.2.6. Let $r \geq 1$. The enumeration $\varphi[1,r],\ldots,\varphi[\gamma(r),r]$ in Definition 5.2.4 is the enumeration of all quantifier free formulas φ of L with $\#(\varphi) \leq r$, ordered first by $\#(\varphi)$, and second by the lexicographic ordering of strings of symbols, where the symbols are ordered by

$$01+-\bullet\uparrow()=<\neg\wedge\vee\rightarrow\leftrightarrow v_1 v_2 \ldots v_r \log$$

An important consequence of the way we have enumerated terms and formulas is the following.

$$1 \leq i \leq \beta(r) \wedge 1 \leq r \leq r' \rightarrow t[i,r] = t[i,r'].$$
$$1 \leq i \leq \gamma(r) \wedge 1 \leq r \leq r' \rightarrow \varphi[i,r] = \varphi[i,r'].$$

DEFINITION 5.2.7. For $E \subseteq N$ and $r \geq 1$, we write $\alpha(r,E)$ for the set of values of all terms $t[i,r]$, at assignments f to the variables in t, with $rng(f) \subseteq E$, including $t[i,r]$ that are closed.

DEFINITION 5.2.8. For $E \subseteq N$ and integers $p,q \geq 0$, we write $\alpha(r,E;p,q)$ for the set of all nonnegative integers x such that the following holds. There is a term $t[i,r]$ that is not closed, and an assignment f to its variables, with $rng(f) \subseteq E$, such that x is the value of $t[i,r]$ under f, and $x \in [p\max(rng(f)),q\max(rng(f))]$. We refer to p,q as the lower and upper coefficients, respectively.

Note that for $E \subseteq N$, $r \geq 1$, $p,q \geq 0$, $\alpha(r,E;p,q) \subseteq [p\min(E),\infty)$.

Here is a version of Lemma 5.1.7, where the role of f is taken up by α. Recall Definition 5.1.12.

LEMMA 5.2.1. Let $r \geq 1$ and $g \in ELG \cap SD \cap BAF$, $rng(g) \subseteq 6N$. There exist infinite $A \subseteq B \subseteq C \subseteq N\setminus\{0\}$ such that
i) $6\alpha(r,A^*;1,r) \subseteq B \cup gB$;
ii) $6\alpha(r,B^*;1,r) \subseteq C \cup gC$;
iii) $2\alpha(r,A^*;1,r)+1 \subseteq B$;
iv) $3\alpha(r,A^*;1,r)+1 \subseteq B$;
v) $2\alpha(r,B^*;1,r)+1 \subseteq C$;
vi) $3\alpha(r,B^*;1,r)+1 \subseteq C$;
vii) $C \cap gC = \emptyset$;
viii) $A \cap \alpha(r,B^*;2,r) = \emptyset$.

Proof: Let r,g be as given. We define $f \in ELG \cap SD \cap BAF$ of arity $\beta(r)+12+r$ as follows. Let $x^* =$

$$(y_1,\ldots,y_{\beta(r)},z_1,\ldots,z_6,w_1,\ldots,w_6,x_1,\ldots,x_r) \in N^{\beta(r)+12+r}.$$

Let i,j,k be greatest such that

$$y_1 = ... = y_i$$
$$z_1 = ... = z_j$$
$$w_1 = ... = w_k$$

respectively.

Define f(x*) =

$$jt[i,r](x_1,...,x_r)+k-1 \text{ if}$$
$$|x^*|+1,2|x^*| \leq jt[i,r](x_1,...,x_r)+k-1 \leq r|x^*|;$$
$$\max(|x^*|+1,2|x^*|) \text{ otherwise.}$$

Clearly f ∈ ELG ∩ SD ∩ BAF. We claim that for any D ⊆ N, $2 \leq p \leq 6$, and $0 \leq q \leq 5$,

$$\alpha(r,D^*;2,r) \cup p\alpha(r,D^*;1,r)+q \subseteq fD.$$

To see this, let $u \in \alpha(r,D^*;2,r)$, $v \in p\alpha(r,D^*;1,r)+q$, and write

$$u = t[i,r](x_1,...,x_r)$$
$$v = pt[i',r](x_1,...,x_r)+q$$

where $x_1,...,x_r \in D^*$, $1 \leq i,i' \leq \beta(r)$, $2|x_1,...,x_r| \leq u \leq r|x_1,...,x_r|$, $|x_1,...,x_r| \leq v \leq r|x_1,...,x_r|$, and t[i,r],t[i',r] are not closed.

First let $y_1 = ... = y_i = \min(D)$, $y_{i+1} = ... = y_{\beta(r)} = |x_1,...,x_r|$, $z_1 = w_1 = \min(D)$, $z_2 = ... = z_6 = w_2 = ... = w_6 = |x_1,...,x_r|$. Then $f(y_1,...,y_{\beta(r)},z_1,...,z_6,w_1,...,w_6,x_1,...,x_r) = u \in fD$.

Now let $y_1 = ... = y_i = \min(D)$, $y_{i+1} = ... = y_{\beta(r)} = |x_1,...,x_r|$, $z_1 = ... = z_p = \min(D)$, $z_{p+1} = ... = z_6 = |x_1,...,x_r|$, $w_1 = ... = w_{q+1} = \min(D)$, $w_{q+2} = ... = w_6 = |x_1,...,x_r|$.

It is obvious that $f(y_1,...,y_{\beta(r)},z_1,...,z_6,w_1,...,w_6,x_1,...,x_r) = v \in fD$.

Now apply Lemma 5.1.7 to f,g to obtain A,B,C ⊆ N\{0} with the properties i)-viii) cited there.

From the demonstrated claim, we have

$$6\alpha(r,A^*;1,r) \subseteq fA.$$
$$6\alpha(r,B^*;1,r) \subseteq fB.$$
$$2\alpha(r,A^*;1,r)+1 \subseteq fA.$$
$$3\alpha(r,A^*;1,r)+1 \subseteq fA.$$
$$2\alpha(r,B^*;1,r)+1 \subseteq fB.$$
$$3\alpha(r,B^*;1,r)+1 \subseteq fB.$$
$$\alpha(r,B^*;2,r) \subseteq fB.$$

We now obtain i)- viii) here immediately from the i)-viii) of Lemma 5.1.7. QED

We are now going to define three properties of finite length towers of sets, of increasing strength: r,g-good for aN, r,g-great for aN, and r,g-terrific for aN. The notion of r-good generalizes some properties from Lemma 5.2.1.

DEFINITION 5.2.9. Let $n \geq 3$, $r,a \geq 1$, and $g \in ELG \cap SD \cap BAF$. We say that $(D_1,...,D_n)$ is r,g-good for aN if and only if

i) $D_1 \subseteq ... \subseteq D_n \subseteq N\backslash\{0\}$;
ii) for all $x < y$ from D_1, $x\uparrow < y$;
iii) for all $1 \leq i \leq n-1$, $a\alpha(r,D_i^*;1,r) \subseteq D_{i+1} \cup gD_{i+1}$;
iv) for all $1 \leq i \leq n-1$, $2\alpha(r,D_i^*;1,r)+1 \subseteq D_{i+1}$;
v) for all $1 \leq i \leq n-1$, $3\alpha(r,D_i^*;1,r)+1 \subseteq D_{i+1}$;
vi) $D_n \cap gD_n = \emptyset$;
vii) $D_1 \cap \alpha(r,D_2^*;2,r) = \emptyset$.

The following proves the existence of length 3 towers that are r,g-good for 6N.

LEMMA 5.2.2. Let $r \geq 1$ and $g \in ELG \cap SD \cap BAF$, $rng(g) \subseteq 6N$. There exists (A,B,C) which is r,g-good for 6N, where A is infinite.

Proof: Let r,g be as given, and let $A,B,C \subseteq N\backslash\{0\}$ be as given by Lemma 5.2.1. Set $D_1 = A$, $D_2 = B$, $D_3 = C$. Obviously i),iii)- vii) hold in the definition of r,g-good for 6N. However ii) may fail. We can obviously shrink A so that ii) holds, keeping A infinite, and retaining i),iii)-vii). QED

We now want to define certain $g \in ELG \cap SD \cap BAF$ so that any r,g-good sequence for N codes up the truth values of existential closures of quantifier free formulas $\varphi[i,r]$, $1 \leq i \leq \gamma(r)$, in a convenient uniform way. This introduces a kind of quantifier elimination.

DEFINITION 5.2.10. Let $r \geq 1$ and $g \in ELG \cap SD \cap BAF$, where $rng(g) \subseteq 24N$. We define $\tau(g,r) \in ELG \cap SD \cap BAF$ as follows. $\tau(g,r)$ has arity $\gamma(r)+k+r+1$, where k is the arity of g. Let $x^* = (y_1,\ldots,y_{\gamma(r)},z_1,\ldots,z_k,x_1,\ldots,x_r,w) \in N^{\gamma(r)+k+r+1}$. Let $i \in [1,\gamma(r)]$ be greatest such that $1 \leq i \leq \gamma(r)$ and $y_1 = \ldots = y_i$.

case 1. $|x^*| = w \wedge x_1,\ldots,x_r < w \wedge \varphi[i,r](x_1,\ldots,x_r)$. Define $\tau(g,r)(x^*) = 24\gamma(r)w+24i+6$.

case 2. $|x^*| = |z_1,\ldots,z_k| \wedge x_1 = \ldots = x_r = w$. Define $\tau(g,r)(x^*) = g(z_1,\ldots,z_k)$.

case 3. Otherwise. Define $\tau(g,r)(x^*) = 24|x^*|+12$.

We now establish some useful coding properties of $\tau(g,r)$.

LEMMA 5.2.3. $\tau(g,r) \in ELG \cap SD \cap BAF$. The values arising out of the above three cases are mutually disjoint, and lie in 6N. Let $E \subseteq N$. For all $w \in E$ and $1 \leq i \leq \gamma(r)$, $24\gamma(r)w+24i+6 \in \tau(g,r)E \leftrightarrow (\exists v_1,\ldots,v_r \in E)(v_1,\ldots,v_r < w \wedge \varphi[i,r](v_1,\ldots,v_r))$. $gE = \tau(g,r)E \cap 24N$.

Proof: Note that in case 1, $\gamma(r),w \geq 1$, and $24w \leq 24\gamma(r)w+24i+6 \leq 100\gamma(r)w$. Hence

$$|x^*|+1,24|x^*| \leq \tau(g,r)(x^*) \leq 100\gamma(r)|x^*|.$$

In case 2,

$$|x^*| = |z_1,\ldots,z_k|$$
$$|\tau(g,r)(x^*)| = |g(z_1,\ldots,z_k)|.$$

In case 3, $|x^*| \geq 1$, and

$$24|x^*| < \tau(g,r)(x^*) \leq 36|x^*|.$$

Therefore $\tau(g,r) \in ELG \cap SD \cap BAF$.

Since $rng(g) \subseteq 24N$, the values arising out of the three cases are mutually disjoint. Also note that the w,i used in case 1 can be recovered from any value of $\tau(g,r)$ obtained by case 1. This is because $1 \leq i \leq \gamma(r)$ in case 1.

Let $E \subseteq N$ and $w \in E$. First suppose $24\gamma(r)w+24i+6 \in \tau(g,r)E$. Then $24\gamma(r)w+24i+6$ must arise out of case 1, with, say, $x^* \in$

$E^{\gamma(r)+k+r+1}$. Then the w,i used in case 1 must be this w,i. Hence the $x_1,\ldots,x_r$ used in case 1 must be < w, and $\varphi[i,r](x_1,\ldots,x_r)$.

Conversely, suppose $x_1,\ldots,x_r \in E \cap [0,w)$ and $\varphi[i,r](x_1,\ldots,x_r)$. Then we can choose $y_1 = \ldots = y_i = x_1$ and $y_{i+1} = \ldots = y_{\gamma(r)} = z_1 = \ldots = z_k = w$. Then case 1 applies, $y_1,\ldots,y_{\gamma(r)},z_1,\ldots,z_k,w \in E$, and i is greatest such that $y_1 = \ldots = y_i$. Hence $\tau(g,r)(y_1,\ldots,y_{\gamma(r)},z_1,\ldots,z_k,x_1,\ldots,x_r,w) = 24\gamma(r)w+24i+6$.

For the final claim, note that every element of gE arises out of case 2, since we can set $y_1 = \ldots = y_{\gamma(r)} = x_1 = \ldots = x_r = w = z_1$, taking $z_1,\ldots,z_k$ to be arbitrary elements of E. On the other hand, all elements of $\tau(g,r)E$ lying in 24N must arise out of case 2, in which case they must lie in gE. QED

DEFINITION 5.2.11. Throughout the book, we will use the logical construction

$$\varphi_1 \leftrightarrow \ldots \leftrightarrow \varphi_k$$

for

$$(\varphi_1 \leftrightarrow \varphi_2) \wedge (\varphi_2 \leftrightarrow \varphi_3) \wedge \ldots \wedge (\varphi_{k-1} \leftrightarrow \varphi_k).$$

LEMMA 5.2.4. Let $r \geq 1$, $g \in ELG \cap SD \cap BAF$, $rng(g) \subseteq 24N$, and (A,B,C) be $100\gamma(r),\tau(g,r)$-good for 6N. Then

i) for all $1 \leq i \leq \gamma(r)$ and $x \in B^*$,

$$(\exists v_1,\ldots,v_r \in C)(v_1,\ldots,v_r < x \wedge \varphi[i,r](v_1,\ldots,v_r)) \leftrightarrow 24\gamma(r)x+24i+6 \notin C;$$

ii) for all $1 \leq i \leq \gamma(r)$ and $x \in A^*$,

$$(\exists v_1,\ldots,v_r \in B)(v_1,\ldots,v_r < x \wedge \varphi[i,r](v_1,\ldots,v_r)) \leftrightarrow (\exists v_1,\ldots,v_r \in C)(v_1,\ldots,v_r < x \wedge \varphi[i,r](v_1,\ldots,v_r)) \leftrightarrow 24\gamma(r)x+24i+6 \notin B \leftrightarrow 24\gamma(r)x+24i+6 \notin C.$$

iii) (A,B,C) is r,g-good for 24N.

Proof: Let r,g,A,B,C be as given. For claim i), let $1 \leq i \leq \gamma(r)$, $x \in B^*$. Then $4\gamma(r)x+4i+1 \in \alpha(100\gamma(r),B^*;1,100\gamma(r))$. To see this, note that $\gamma(r),x \geq 1$, $2x \leq 4\gamma(r)x+4i+1 \leq 100\gamma(r)x$. Also $4\gamma(r)x+4i+1$ is a term t(x) with $\#(t) \leq 100\gamma(r)$.

By clauses iii),vi) in the definition of $100\gamma(r),\boldsymbol{\tau}(g,r)$-good for 6N, we have

$$24\gamma(r)x+24i+6 \in C \cup \boldsymbol{\tau}(g,r)C.$$
$$C \cap \boldsymbol{\tau}(g,r)C = \emptyset.$$

By the above and Lemma 5.2.3,

$$(\exists v_1,\ldots,v_r \in C)(v_1,\ldots,v_r < x \wedge \varphi[i,r](v_1,\ldots,v_r)) \leftrightarrow$$
$$24\gamma(r)x+24i+6 \in \boldsymbol{\tau}(g,r)C \leftrightarrow 24\gamma(r)x+24i+6 \notin C.$$

For claim ii), let $1 \le i \le \gamma(r)$ and $x \in A^*$. Then $4\gamma(r)x+4i+1 \in \boldsymbol{\alpha}(100\gamma(r),A^*;1,100\gamma(r))$. By clauses iii),iv),vi) in the definition of $100\gamma(r),\boldsymbol{\tau}(g,r)$-good for 6N, we have

$$24\gamma(r)x+24i+6 \in B \cup \boldsymbol{\tau}(g,r)B$$
$$B \cap \boldsymbol{\tau}(g,r)B = \emptyset.$$

By the above and Lemma 5.2.3,

$$(\exists v_1,\ldots,v_r \in B)(v_1,\ldots,v_r < x \wedge \varphi[i,r](v_1,\ldots,v_r)) \leftrightarrow$$
$$24\gamma(r)x+24i+6 \in \boldsymbol{\tau}(g,r)B \leftrightarrow 24\gamma(r)x+24i+6 \notin B.$$

Hence

$$(\exists v_1,\ldots,v_r \in C)(v_1,\ldots,v_r < x \wedge \varphi[i,r](v_1,\ldots,v_r)) \rightarrow$$
$$24\gamma(r)x+24i+6 \notin C \rightarrow$$
$$24\gamma(r)x+24i+6 \notin B \rightarrow$$
$$(\exists v_1,\ldots,v_r \in B)(v_1,\ldots,v_r < x \wedge \varphi[i,r](v_1,\ldots,v_r)) \rightarrow$$
$$(\exists v_1,\ldots,v_r \in C)(v_1,\ldots,v_r < x \wedge \varphi[i,r](v_1,\ldots,v_r))$$

and so all of the above $\rightarrow$ are also $\leftrightarrow$.

For claim iii), by the definition of $100\gamma(r),\boldsymbol{\tau}(g,r)$-good for 6N, we have

$$6\boldsymbol{\alpha}(100\gamma(r),A^*;1,100\gamma(r)) \subseteq B \cup \boldsymbol{\tau}(g,r)B$$
$$6\boldsymbol{\alpha}(100\gamma(r),B^*;1,100\gamma(r)) \subseteq C \cup \boldsymbol{\tau}(g,r)C$$
$$2\boldsymbol{\alpha}(100\gamma(r),A^*;1,100\gamma(r))+1 \subseteq B$$
$$3\boldsymbol{\alpha}(100\gamma(r),A^*;1,100\gamma(r))+1 \subseteq B$$
$$2\boldsymbol{\alpha}(100\gamma(r),B^*;1,100\gamma(r))+1 \subseteq C$$
$$3\boldsymbol{\alpha}(100\gamma(r),B^*;1,100\gamma(r))+1 \subseteq C$$
$$C \cap \boldsymbol{\tau}(g,r)C = \emptyset$$
$$A \cap \boldsymbol{\alpha}(100\gamma(r),B^*;2,100\gamma(r)) = \emptyset$$
$$\text{for all } x < y \text{ from } A,\ x\!\uparrow\, < y.$$

By Lemma 5.2.3, $gB = \tau(g,r)B \cap 24N$ and $gC = \tau(g,r)C \cap 24N$. Hence the conditions

$$24\alpha(r,A^*;1,r) \subseteq B \cup gB$$
$$24\alpha(r,B^*;1,r) \subseteq C \cup gC$$
$$2\alpha(r,A^*;1,r)+1 \subseteq B$$
$$3\alpha(r,A^*;1,r)+1 \subseteq B$$
$$2\alpha(r,B^*;1,r)+1 \subseteq C$$
$$3\alpha(r,B^*;1,r)+1 \subseteq C$$
$$C \cap gC = \emptyset$$
$$A \cap \alpha(r,B^*;2,r) = \emptyset$$
$$\text{for all } x < y \text{ from } A,\ x\uparrow < y$$

follow immediately. Therefore (A,B,C) is r,g-good for 24N. QED

We now define r,g-great towers, which feature a special form of indiscernibility for terms. We also define r,g-terrific towers, which feature a special form of indiscernibility for quantifier free formulas. We will only use r,g-terrific towers of length 3.

DEFINITION 5.2.12. Let $n \geq 3$, $r,a \geq 1$, and $g \in ELG \cap SD \cap BAF$. We say that $(D_1,\ldots,D_n)$ is r,g-great for aN if and only if

i) $(D_1,\ldots,D_n)$ is r,g-good for aN;
ii) Let $1 \leq i \leq \beta(2r)$, $x_1,\ldots,x_{2r} \in D_1$, $y_1,\ldots,y_r \in \alpha(r,D_2)$, where $(x_1,\ldots,x_r),(x_{r+1},\ldots,x_{2r})$ have the same order type and min, and $y_1,\ldots,y_r \leq \min(x_1,\ldots,x_r)$. Then
$t[i,2r](x_1,\ldots,x_r,y_1,\ldots,y_r) \in D_3^* \leftrightarrow$
$t[i,2r](x_{r+1},\ldots,x_{2r},y_1,\ldots,y_r) \in D_3^*$.

DEFINITION 5.2.13. Let $r,a \geq 1$, and $g \in ELG \cap SD \cap BAF$. We say that (A,B,C) is r,g-terrific for aN if and only if

i) (A,B,C) is r,g-great for aN;
ii) A is infinite;
iii) for all $1 \leq i \leq \gamma(r)$,

$$(\exists v_1,\ldots,v_r \in B)(\varphi[i,r](v_1,\ldots,v_r)) \leftrightarrow$$
$$(\exists v_1,\ldots,v_r \in C)(\varphi[i,r](v_1,\ldots,v_r)).$$

We now derive an essentially well known infinitary combinatorial lemma. E.g., see [Sc74].

LEMMA 5.2.5. Let D be an infinite subset of N and $r \geq 1$. Let $f:N \to N$, and $R_1,\ldots,R_s$ be a finite list of subsets of N^{2r}. There exists an infinite $D' \subseteq D$ such that the following holds. Let $1 \leq i \leq s$, $x_1,\ldots,x_{2r} \in D'$, and $y_1,\ldots,y_r \in N$, where $(x_1,\ldots,x_r)$ and $(x_{r+1},\ldots,x_{2r})$ have the same order type and min, and $y_1,\ldots,y_r \leq f(\min(x_1,\ldots,x_r))$. Then $R_i(x_1,\ldots,x_r,y_1,\ldots,y_r) \leftrightarrow R_i(x_{r+1},\ldots,x_{2r},y_1,\ldots,y_r)$.

Proof: Let $D,r,f,R_1,\ldots,R_s$ be as given. Here we write $R_i(z_1,\ldots,z_{2r})$ for $(z_1,\ldots,z_{2r}) \in R_i$. We will partition the ordered 2r tuples from N into finitely many pieces as follows. Let $x_1,\ldots,x_{2r} \in N$ be given. We partition $(x_1,\ldots,x_{2r})$

a. first according to the order type of $(x_1,\ldots,x_{2r})$.
b. second according to the set of all $i \in [1,s]$ such that for all $y_1,..,y_r \leq f(\min(x_1,\ldots,x_{2r}))$, $R_i(x_1,\ldots,x_r,y_1,\ldots,y_r) \leftrightarrow R_i(x_{r+1},\ldots,x_{2r},y_1,\ldots,y_r)$.

By Ramsey's theorem, let $D' \subseteq D$ be infinite, where any two $(x_1,\ldots,x_{2r}) \in D'^{2r}$ with the same order type lie in the same partition.

Let $1 \leq i \leq s$ and μ be the order type of an element of N^r. We say that $(x_1,\ldots,x_{2r})$ is μ-special if and only if

i) $(x_1,\ldots,x_r)$ and $(x_{r+1},\ldots,x_{2r})$ have order type μ;
ii) $\min(x_1,\ldots,x_r) = \min(x_{r+1},\ldots,x_{2r})$;
iii) if $x_{r+j} > \min(x_1,\ldots,x_r)$, then $|x_1,\ldots,x_r| < x_{r+j}$.

The μ-special tuples are exactly the 2r-tuples of some particular order type depending on μ. Hence for each μ,i, we have

1) for all μ-special $(x_1,\ldots,x_{2r}) \in D'^{2r}$, we have: for all $y_1,\ldots,y_{2r} \leq f(\min(x_1,\ldots,x_{2r}))$, $R_i(x_1,\ldots,x_r,y_1,\ldots,y_r) \leftrightarrow R_i(x_{r+1},\ldots,x_{2r},y_1,\ldots,y_r)$; or

2) for all μ-special $(x_1,\ldots,x_{2r}) \in D'^{2r}$, we have: $\neg$(for all $y_1,\ldots,y_r \leq f(\min(x_1,\ldots,x_{2r}))$, $R_i(x_1,\ldots,x_r,y_1,\ldots,y_r) \leftrightarrow R_i(x_{r+1},\ldots,x_{2r},y_1,\ldots,y_r)$).

Suppose 2) holds for μ. Let $\alpha_1,\alpha_2,\ldots$ be elements of N^r where each 2r-tuple (α_j,α_{j+1}) is μ-special. For each $j < k$ from $[1,\infty)$, let $h(j,k)$ be some counterexample $(y_1,\ldots,y_r)$ given by 2) for $(x_1,\ldots,x_{2r}) = (\alpha_j,\alpha_k)$.

Obviously h is bounded by $f(\min(\alpha_1))$. By Ramsey's theorem, h is constant on the $j < k$ drawn from some infinite subset of N. But $h(j,k) = h(j,p) = h(k,p)$ is obviously impossible for $j < k < p$. We conclude that 2) fails. Hence 1) holds for μ.

We have thus shown that for all μ,i, 1) holds. To complete the argument, let $1 \leq i \leq s$, $x_1,\ldots,x_{2r} \in D'$, and $y_1,\ldots,y_r \in N$, where $(x_1,\ldots,x_r)$ and $(x_{r+1},\ldots,x_{2r})$ have the same order type and min, and $y_1,\ldots,y_r \leq f(\min(x_1,\ldots,x_r))$. Let the order type of $(x_1,\ldots,x_r)$ be μ. Choose $x_1',\ldots,x_r' \in D'$ such that $(x_1,\ldots,x_r,x_1',\ldots,x_r')$ and $(x_{r+1},\ldots,x_{2r},x_1',\ldots,x_r')$ are μ-special. By 1),

$$R_i(x_1,\ldots,x_r,y_1,\ldots,y_r) \leftrightarrow R_i(x_1',\ldots,x_r',y_1,\ldots,y_r).$$
$$R_i(x_{r+1},\ldots,x_{2r},y_1,\ldots,y_r) \leftrightarrow R_i(x_1',\ldots,x_r',y_1,\ldots,y_r).$$

Hence

$$R_i(x_1,\ldots,x_r,y_1,\ldots,y_r) \leftrightarrow R_i(x_{r+1},\ldots,x_{2r},y_1,\ldots,y_r)$$

as required. QED

We now prove the existence of r,g-terrific towers.

LEMMA 5.2.6. Let $r \geq 1$ and $g \in ELG \cap SD \cap BAF$, where $rng(g) \subseteq 24N$. There exists (A,B,C) which is r,g-terrific for 24N.

Proof: Let r,g be as given. By Lemma 5.2.2, there exists (A,B,C) which is $100\gamma(r),\tau(g,r)$-good for 6N, where A is infinite. By Lemma 5.2.4, (A,B,C) is r,g-good for 24N, and satisfies clauses i) and ii) in Lemma 5.2.4.

For all $1 \leq i \leq \beta(2r)$, let $R_i \subseteq N^{2r}$ be given by

$$R_i(x_1,\ldots,x_{2r}) \leftrightarrow t[i,2r](x_1,\ldots,x_{2r}) \in C^*.$$

Apply Lemma 5.2.5 to these R_i with D = A to obtain $A' \subseteq A$, A' infinite, such that (A',B,C) is r,g-great for 24N.

To see that (A',B,C) is r,g-terrific for 24N, we need only verify clause iii) in that definition. Since (A,B,C) satisfies clause ii) in Lemma 5.2.4, we have that for all $1 \leq i \leq \gamma(r)$ and $x \in A^*$,

$$(\exists v_1,\ldots,v_r \in B)(v_1,\ldots,v_r < x \wedge \varphi[i,r](v_1,\ldots,v_r)) \leftrightarrow (\exists v_1,\ldots,v_r \in C)(v_1,\ldots,v_r < x \wedge \varphi[i,r](v_1,\ldots,v_r)).$$

Since A* is infinite, we have

$$(\exists v_1,\ldots,v_r \in B)(\varphi[i,r](v_1,\ldots,v_r)) \leftrightarrow (\exists v_1,\ldots,v_r \in C)(\varphi[i,r](v_1,\ldots,v_r)).$$

QED

We remark that, using Lemma 5.2.5, we can obtain ii) in the definition of r,g-great with $\alpha(r,D_2)$ replaced by N. However, if we formulated r,g-greatness in such a strong form, we would not be able to push down from C to B in Lemma 5.2.8.

LEMMA 5.2.7. For all $n \geq 3$ and $k,p,r \geq 1$, there exists $m \geq 1$ such that the following holds. Let $g \in ELG \cap SD \cap BAF$ be k-ary, $a \geq 1$, and $(D_1,\ldots,D_n)$ be r,g-great for aN, $|D_1| = p$. There exists $(D_1',\ldots,D_n')$ which is r,g-great for aN, where $D_1' = D_1$, each $D_i' \subseteq D_i$, and $|D_n'| \leq m$.

Proof: Let n,k,p,r,a be as given. Let $g,D_1,\ldots,D_n$ also be as given. We will construct the required $D_1',\ldots,D_n'$ by induction on $1 \leq j \leq n$, in such a way that there is an obvious bound on the cardinality of each D_{j+1}' that depends only on j,k,p,r and not on $a,n,g,D_1,\ldots,D_n$.

Suppose $D_1 = D_1' \subseteq \ldots \subseteq D_j'$ have been defined, $1 \leq j < n$, such that $(\forall i \in [1,j])(D_i' \subseteq D_i)$. We now construct $D_{j+1}' \subseteq D_{j+1}$.

First throw all elements of D_j' into D_{j+1}', and also $\min(D_{j+1})$ into D_{j+1}'. Then for each $x \in a\alpha(r,D_j'^*;1,r)$, throw x into D_{j+1}' if $x \in D_{j+1}$; otherwise find a k-tuple y from D_{j+1} such that $g(y) = x$ and throw $y_1,\ldots, y_k$ into D_{j+1}'. Next, throw all elements of $2\alpha(r,D_j'^*;1,r)+1$, $3\alpha(r,D_j'^*;1,r)+1$, into D_{j+1}'. Note that these elements are in D_{j+1}, because $(D_1,\ldots,D_n)$ is r,g-good.

Finally, if j = 2 then let $1 \leq i \leq \beta(2r)$, $x_1,\ldots,x_r \in D_1$, and $y_1,\ldots,y_r \in \alpha(r,D_2')$, $y_1,\ldots,y_r \leq \min(x_1,\ldots,x_r)$. If $t[i,2r](x_1,\ldots,x_r,y_1,\ldots,y_r) \in D_3^*$, then throw $t[i,2r](x_1,\ldots,x_r,y_1,\ldots,y_r)$ in D_3'. Otherwise, take no action.

It is clear that $(D_1',\ldots,D_n')$ is r,g-good for aN. We have to verify clause ii) in the definition of r,g-great for aN.

Let $1 \leq i \leq \beta(2r)$, $x_1,\ldots,x_r \in D_1$, $y_1,\ldots,y_r \in \alpha(r,D_2')$, where $y_1,\ldots,y_r \leq \min(x_1,\ldots,x_r)$. We claim that

$$t[i,2r](x_1,\ldots,x_r,y_1,\ldots,y_r) \in D_3'^* \leftrightarrow t[i,2r](x_1,\ldots,x_r,y_1,\ldots,y_r) \in D_3^*.$$

The forward direction is immediate. For the reverse direction, first note that $\min(D_3) = \min(D_3')$ by construction. If the right side holds, then $t[i,2r](x_1,\ldots,x_r,y_1,\ldots,y_r)$ has been thrown into D_3', and since $t[i,2r](x_1,\ldots,x_r,y_1,\ldots,y_r) > \min(D_3) = \min(D_3')$, the left side follows.

Now let $1 \leq i \leq \beta(2r)$, $x_1,\ldots,x_{2r} \in D_1$, $y_1,\ldots,y_r \in \alpha(r,D_2')$, where $(x_1,\ldots,x_r)$ and $(x_{r+1},\ldots,x_{2r})$ have the same order type and min, and $y_1,\ldots,y_r \leq \min(x_1,\ldots,x_r)$. We must verify that

$$t[i,2r](x_1,\ldots,x_r,y_1,\ldots,y_r) \in D_3'^* \leftrightarrow t[i,2r](x_{r+1},\ldots,x_{2r},y_1,\ldots,y_r) \in D_3'^*.$$

By the above, this is equivalent to

$$t[i,2r](x_1,\ldots,x_r,y_1,\ldots,y_r) \in D_3^* \leftrightarrow t[i,2r](x_{r+1},\ldots,x_{2r},y_1,\ldots,y_r) \in D_3^*$$

which follows from the hypothesis on $(D_1,\ldots,D_n)$ - in particular, from ii) in the definition of r,g-great.

It is clear that we can write m as a specific iterated exponential in n,k,p,r. QED

We show that, at the cost of increasing r to much larger s, we can guarantee that for any s,g-terrific tower (A,B,C), any r,g-great tower contained in C can be shrunk to an r,g-great tower contained in B.

LEMMA 5.2.8. Let $n \geq 3$, $k,p,r \geq 1$, and $g \in ELG \cap SD \cap BAF$ be k-ary. There exists $s \geq 1$ such that the following holds. Let (A,B,C) be s,g-terrific for 24N. Let $(D_1,\ldots,D_n)$ be r,g-great for 24N, $|D_1| = p$, and $D_n \subseteq C$. Then some $(D_1',\ldots,D_n')$ is r,g-great for 24N, where $|D_1'| = p$ and $D_n' \subseteq B$ is finite.

Proof: Let n,k,p,r,g be as given. Let $m \geq 1$ be given by Lemma 5.2.7, with a = 24, which depends only on n,k,p,r.

Let $s >> n,k,p,r,m$ and the presentation of g. (Some specific iterated exponential in n,k,p,r,m, and the size of the presentation of g, will suffice). Let (A,B,C) be s,g-terrific for $24N$. Let $(D_1,...,D_n)$ be r,g-great for $24N$, $|D_1| = p$, and $D_n \subseteq C$.

By Lemma 5.2.7, the following statement is true:

*) there exists $(D_1,...,D_n)$ which is r,g-great for $24N$, where $|D_1| = p$ and $D_n = \{x_1,...,x_m\} \subseteq C$.

We claim that *) asserts the existence of $x_1,...,x_m \in C$ such that a quantifier free formula $\varphi(x_1,...,x_m)$ in L holds. This crucially depends on the fact that $g \in BAF$. The actual formula depends on n,k,p,r, and the function g.

To see this, $\varphi(x_1,...,x_m)$ asserts that $x_1,...,x_m$ can be arranged into sets $D_1 \subseteq ... \subseteq D_n = \{x_1,...,x_m\}$, where $(D_1,...,D_n)$ is r,g-great for $24N$. We have to put clauses i),ii) in Definition 5.2.12, with $a = 24$, in quantifier free form.

Each arrangement of $x_1,...,x_m$ into sets $D_1 \subseteq ... \subseteq D_n = \{x_1,...,x_m\}$ is given by a double sequence x_{ij}, $1 \leq i \leq n$, $1 \leq j \leq m$, where the x_{ij} are among the variables $x_1,...,x_m$. So we disjunct over the finitely many such double sequences of variables.

According to Definition 5.2.12, we assert

i. $(\{x_{11},...,x_{1m}\},...,\{x_{n1},...,x_{nm}\})$ is r,g-good for $24N$.
ii. Let $1 \leq i \leq \beta[2r]$, $x_1,...,x_{2r} \in \{x_{11},...,x_{1m}\}$, $y_1,...,y_r \in \alpha(r,\{x_{21},...,x_{2m}\})$, where $(x_1,...,x_r)$, $(x_{r+1},...,x_{2r})$ have the same order type and min, and $y_1,...,y_r \leq \min(x_1,...,x_r)$. Then

$$t[i,2r](x_1,...,x_r,y_1,...,y_r) \in \{x_{31},...,x_{3m}\}\backslash\{0\} \leftrightarrow t[i,2r](x_{r+1},...,x_{2r},y_1,...,y_r) \in \{x_{31},...,x_{3m}\}\backslash\{0\}.$$

It is clear that ii) is given by a quantifier free formula in L.

As for i), it asserts

i'. $\{x_{11},...,x_{1m}\} \subseteq ... \subseteq \{x_{n1},...,x_{nm}\} \subseteq N\backslash\{0\}$.
ii'. $x_{1i} < x_{1j} \rightarrow x_{1i}\uparrow < x_{1j}$.
iii'. For all $1 \leq i \leq n-1$, $24\alpha(r,\{x_{i1},...,x_{im}\}\backslash\{0\};1,r) \subseteq \{x_{i+1,1},...,x_{i+1,m}\} \cup g\{x_{i+1,1},...,x_{i+1,m}\}$.

iv'. For all $1 \leq i \leq n-1$, $2\alpha(r,\{x_{i1},\ldots,x_{im}\}\backslash\{0\};1,r)+1 \subseteq \{x_{i+1,1},\ldots,x_{i+1,m}\}$;
v'. Same as iv' with 2 replaced by 3.
vi'. $\{x_{n1},\ldots,x_{nm}\} \cap g\{x_{n1},\ldots,x_{nm}\} = \emptyset$.
vii'. $\{x_{11},\ldots,x_{1m}\} \cap \alpha(r,\{x_{21},\ldots,x_{2m}\};2,r) = \emptyset$.

It is now clear that i) is also given by a quantifier free formula.

By the choice of s, write $\varphi = \varphi[i,s]$, where $1 \leq i \leq \gamma(s)$.

By Lemma 5.2.7, we have

$$(\exists v_1,\ldots,v_m \in C)(\varphi[i,s](v_1,\ldots,v_m)).$$

By clause iii) in the definition of s,g-terrific for 24N,

$$(\exists v_1,\ldots,v_m \in B)(\varphi[i,s](v_1,\ldots,v_m)).$$

Hence

$$(\exists v_1,\ldots,v_m \in B)(\exists D_1,\ldots,D_n)((D_1,\ldots,D_n) \text{ is } r,g\text{-great for } 24N \wedge |D_1| = p \wedge D_n = \{v_1,\ldots,v_m\}).$$

I.e., some $(D_1',\ldots,D_n')$ is r,g-great for 24N, where $|D_1'| = p$ and $D_n' \subseteq B$ has at most m elements. QED

DEFINITION 5.2.14. Let s(n,k,p,r,g) be an s given by Lemma 5.2.8.

LEMMA 5.2.9. Let $n \geq 3$, $k,p,r \geq 1$, and $g \in ELG \cap SD \cap BAF$ be k-ary. There exists $t \geq 1$ such that the following holds. Let (A,B,C) be t,g-terrific for 24N. Then some $(D_1,\ldots,D_n)$ is r,g-great for 24N, where $|D_1| = p$ and $D_n \subseteq B$ is finite.

Proof: Let n,k,p,r,g be as given. Let $t = \max\{s(q,k,p,r,g): 3 \leq q \leq n)\}$. Let (A,B,C) be t,g-terrific for 24N. We prove by induction on $3 \leq q \leq n$ that some $(D_1,\ldots,D_q)$ is r,g-great for 24N, where $|D_1| = p$ and $D_n \subseteq B$ is finite.

For the basis case q = 3, apply Lemma 5.2.8 to (D_1,D_2,D_3), where D_1 is any subset of A of cardinality p, and $D_2 = B$, $D_3 = C$. Note that $t \geq s(3,k,p,r,g)$.

Let $3 \leq q < n$ and $(D_1,\ldots,D_q)$ be r,g-great for 24N, where $|D_1| = p$ and $D_q \subseteq B$ is finite.

We claim that $(D_1,...,D_q,C)$ is r,g-great for 24N.

We first verify that $(D_1,...,D_q,C)$ is r,g-good for 24N. In light of the fact that $(D_1,...,D_q)$ is r,g-good for 24N and $q \geq 3$, it suffices to show that

$$24\alpha(r,D_q*;2,r) \subseteq C \cup gC$$
$$2\alpha(r,D_q*;2,r)+1 \subseteq C$$
$$3\alpha(r,D_q*;2,r)+1 \subseteq C$$
$$C \cap gC = \emptyset.$$

These are immediate since $D_q \subseteq B$ and (A,B,C) is r,g-good for 24N.

Clause ii) in the definition of $(D_1,...,D_q,C)$ is immediate since $q \geq 3$ and $(D_1,...,D_q)$ is r,g-great for 24N.

Now apply Lemma 5.2.8 to $(D_1,...,D_q,C)$ to obtain a sequence $(D_1',...,D_{q+1}')$ that is r,g-great for 24N, where $|D_1| = p$ and $D_{q+1}' \subseteq B$ is finite. Note that $t \geq s(q+1,k,p,r,g)$. QED

LEMMA 5.2.10. Let $n \geq 3$, $p,r \geq 1$, and $g \in$ ELG $\cap$ SD $\cap$ BAF, where $rng(g) \subseteq 24N$. There exists $(D_1,...,D_n)$ which is r,g-great for 24N, where $|D_1| = p$ and D_n is finite.

Proof: Let n,p,r,g be as given. Let g be k-ary. Let t be given by Lemma 5.2.9. By Lemma 5.2.6, let (A,B,C) be t,g-terrific for 24N. By Lemma 5.2.9, let $(D_1,...,D_n)$ be r,g-great for 24N, where $|D_1| = p$ and D_n is finite. QED

LEMMA 5.2.11. Let $r \geq 3$ and $g \in$ ELG $\cap$ SD $\cap$ BAF, where $rng(g) \subseteq 24N$. There exists $(D_1,...,D_r)$ such that
i) $D_1 \subseteq ... \subseteq D_r \subseteq N\backslash\{0\}$;
ii) $|D_1| = r$ and D_r is finite;
iii) for all $x < y$ from D_1, $x\uparrow < y$;
iv) for all $1 \leq i \leq r-1$, $24\alpha(r,D_i*;1,r) \subseteq D_{i+1} \cup gD_{i+1}$;
v) for all $1 \leq i \leq r-1$, $2\alpha(r,D_i*;1,r)+1$, $3\alpha(r,D_i*;1,r)+1 \subseteq D_{i+1}$;
vi) $D_r \cap gD_r = \emptyset$;
vii) $D_1 \cap \alpha(r,D_2*;2,r) = \emptyset$;
viii) Let $1 \leq i \leq \beta(2r)$, $x_1,...,x_{2r} \in D_1$, $y_1,...,y_r \in \alpha(r,D_2)$, where $(x_1,...,x_r)$ and $(x_{r+1},...,x_{2r})$ have the same order type and min, and $y_1,...,y_r \leq \min(x_1,...,x_r)$. Then
$t[i,2r](x_1,...,x_r,y_1,...,y_r) \in D_3* \leftrightarrow$
$t[i,2r](x_{r+1},...,x_{2r},y_1,...,y_r) \in D_3*$.

Proof: Immediate from Lemma 5.2.10 and the definition of r,g-great for 24N, setting n,p,r there to be r here. QED

We now eliminate the use of the D_i*.

LEMMA 5.2.12. Let $r \geq 3$ and $g \in ELG \cap SD \cap BAF$, where $rng(g) \subseteq 48N$. There exists $(D_1,...,D_r)$ such that
i) $D_1 \subseteq ... \subseteq D_r \subseteq N\backslash\{0\}$;
ii) $|D_1| = r$ and D_r is finite;
iii) for all $x < y$ from D_1, $x\uparrow < y$;
iv) for all $1 \leq i \leq r-1$, $48\alpha(r,D_i;1,r) \subseteq D_{i+1} \cup gD_{i+1}$;
v) for all $1 \leq i \leq r-1$, $2\alpha(r,D_i;1,r)+1$, $3\alpha(r,D_i;1,r)+1 \subseteq D_{i+1}$;
vi) $D_r \cap gD_r = \emptyset$;
vii) $D_1 \cap \alpha(r,D_2;2,r) = \emptyset$;
viii) Let $1 \leq i \leq \beta(2r)$, $x_1,...,x_{2r} \in D_1$, $y_1,...,y_r \in \alpha(r,D_2)$, where $(x_1,...,x_r)$ and $(x_{r+1},...,x_{2r})$ have the same order type and min, and $y_1,...,y_r \leq \min(x_1,...,x_r)$. Then
$t[i,2r](x_1,...,x_r,y_1,...,y_r) \in D_3 \leftrightarrow$
$t[i,2r](x_{r+1},...,x_{2r},y_1,...,y_r) \in D_3$.

Proof: Let r,g be as given. Let $g:N^k \rightarrow 48N$.

Define $g':N^{k+1} \rightarrow 24N$ by $g'(x_1,...,x_{k+1}) = g(x_1,...,x_k)$ if $x_{k+1} < x_1,...,x_k$; $48|x_1,...,x_{k+1}|+24$ otherwise.

Note that $rng(g') \subseteq 24N$, and $g' \in ELG \cap SD \cap BAF$. Let $D_1,...,D_n \subseteq N$ be given by Lemma 5.2.11 applied to $r+1,g'$. In particular, $|D_1| = r+1$.

We now verify that D_1*,...,D_r* is as required.

For claim i), since $D_1 \subseteq ... \subseteq D_r$, we have $\min(D_1) \geq ... \geq \min(D_r)$. We claim that D_1* $\subseteq ... \subseteq D_r$*. To see this, let $n \in D_i$*. Then $n \in D_{i+1}$, $n > \min(D_i) \geq \min(D_{i+1})$, $n \in D_{i+1}$*.

For claim ii), since $|D_1| = r+1$, we have $|D_1*| = r$. since D_r is finite, D_r* is finite.

Claim iii) is immediate from iii) of Lemma 5.2.11.

For claim iv), let $1 \leq i \leq r-1$, $x \in 48\alpha(r,D_i*;1,r)$. Then $x > \min(D_i) \geq \min(D_{i+1})$. By Lemma 5.2.11 iv), $x \in D_{i+1} \cup g'D_{i+1}$. If $x \in D_{i+1}$ then $x \in D_{i+1}$*. If $x \in g'D_{i+1}$ then $x \in g(D_{i+1}*)$, because x must arise from the first clause in the definition of g'.

For claim v), let $1 \leq i \leq r-1$, $x \in 2\alpha(r,D_i^*;1,r)+1 \cup 3\alpha(r,D_i^*;1,r)+1$. Then $x > \min(D_i) \geq \min(D_{i+1})$. By Lemma 5.2.11 v), $x \in D_{i+1}$. Hence $x \in D_{i+1}^*$.

For vi), we have $D_r \cap g'D_r = \emptyset$. Since $g(D_r^*) \subseteq g'(D_r)$, we have $D_r^* \cap g(D_r^*) = \emptyset$.

Claim vii) is the same as vii) of Lemma 5.2.11.

For claim viii), let $1 \leq i \leq \beta(2r)$. Let $1 \leq i' \leq \beta(2r+2)$ be such that t[i',2r+2] is the result of replacing the variables $v_{r+1},\ldots,v_{2r}$ in t[i,2r] with the variables $v_{r+2},\ldots,v_{2r+1}$.

Let $x_1,\ldots,x_{2r} \in D_1^*$, $y_1,\ldots,y_r \in \alpha(r,D_2^*)$, where $(x_1,\ldots,x_r)$ and $(x_{r+1},\ldots,x_{2r})$ have the same order type and min, and $y_1,\ldots,y_r \leq \min(x_1,\ldots,x_r)$. Clearly

$$t[i,2r](x_1,\ldots,x_r,y_1,\ldots,y_r) = t[i',2r+2](x_1,\ldots,x_r,x_r,y_1,\ldots,y_r,y_r).$$

$$t[i,2r](x_{r+1},\ldots,x_{2r},y_1,\ldots,y_r) = t[i',2r+2](x_{r+1},\ldots,x_{2r},x_{2r},y_1,\ldots,y_r,y_r).$$

By Lemma 5.2.11 viii),

$$t[i',2r+2](x_1,\ldots,x_r,x_r,y_1,\ldots,y_r,y_r) \in D_3^* \leftrightarrow t[i',2r+2](x_{r+1},\ldots,x_{2r},x_{2r},y_1,\ldots,y_r,y_r) \in D_3^*.$$

$$t[i,r](x_1,\ldots,x_r,y_1,\ldots,y_r) \in D_3^* \leftrightarrow t[i,r](x_{r+1},\ldots,x_{2r},y_1,\ldots,y_r) \in D_3^*.$$

QED