Adjacent Ramsey Theory
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This concerns Special FOM e-Mail Postings number 36 and 37.

Let $k \geq 2$ and $f:N^k \to [1,k]$ and $n \geq 1$ be such that there is no $x_1 < \ldots < x_{k+1} \leq n$ such that $f(x_1,\ldots,x_k) = f(x_1,\ldots,x_{k+1})$. Then we want to find $g:N^{k+1} \to [1,3]$ such that there is no $x_1 < \ldots < x_{k+2} \leq n$ such that $g(x_1,\ldots,x_{k+1}) = g(x_2,\ldots,x_{k+2})$. This reducees adjacent Ramsey in k dimensions with k colors to adjacent Ramsey in k+1 dimensions with 3 colors.

Let $x_1 < \ldots < x_{k+1} \leq n$ be given. Look at $t_1,\ldots,t_{k+1}$, where $t_i = f(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{k+1})$.

case 1. $t_1,\ldots,t_{k+1}$ all = or strictly monotone. Equality violates the hypotheses, whereas strict monotonicity is impossible because f goes into [1,k].

case 2. first i such that t_i,t_{i+1},t_{i+2} not strictly monotone is even. Set $g(x_1,\ldots,x_{k+1}) = 1$.

case 3. first i such that t_i,t_{i+1},t_{i+2} is odd and $t_i < t_{i+1}$. Set $g(x_1,\ldots,x_{k+1}) = 2$.

case 4. first i such that t_i,t_{i+1},t_{i+1} is odd and $t_i \geq t_{i+1}$. Set $g(x_1,\ldots,x_{k+1}) = 3$.

Now let $x_1 < \ldots < x_{k+2} \leq n$ be such that $g(x_1,\ldots,x_{k+1}) = g(x_2,\ldots,x_{k+2})$.

All cases easily lead to contradictions. QED

$s(k,r)$ = least n such that for all $F:N^k \to \{1,\ldots,r\}$ there exists $x_1 < \ldots < x_{k+1} \leq n$ such that F is constant on all subsequences of $x_1,\ldots,x_{k+1}$.

$Adj(k,r)$ = least n such that for all $F:N^k \to \{1,\ldots,r\}$ there exists $x_1 < \ldots < x_{k+1} \leq n$ such that $F(x_1,\ldots,x_k) = F(x_2,\ldots,x_{k+1})$.

THEOREM (Duffus, Lefmann, Rodl 1995). For $k \geq 1$ and $r \geq 4$, $s(k,r) >$ exponential stack of k-1 1.5’s topped off with r.

THEOREM. For $k \geq 2$, $Adj(k+1,3) > s(k,k)$.

THEOREM. Adj(k,2) = 2k+1.

THEOREM. s(k,r) ≤ a stack of k-1 2's topped with a reasonable function of k and r. So Adj(k,3) ≤ a stack of k 2's topped with a reasonable function of k.

Now do Adj(k,2) = 2k+1.

Let $f:N^k \rightarrow \{1,2\}$ be given. f(2,...,k+1} ≠ f(3,...,k+2} ≠ ... ≠ f(k+2,...,2k+1}, alternating with length k+1. But also f(1,...,k} ≠ f(2,...,k,k+2} ≠ ... ≠ f(k,k+2,...,2k) ≠ f(k+2,...,2k+1}, also alternating with length k+1. Hence f(1,...,k) = f(2,...,k+1).

This shows that Adj(k,2) ≤ 2k+1.

Now we need to get an appropriate function on $\{1,...,2k\}^k$.

For k = 1 this is trivial. Suppose f is an appropriate function on $\{2,...,2k+1\}^k$. We now want to construct an appropriate function g on $\{1,...,2k+2\}^{k+1}$.

case 1. x1 ≥ 2. Set $g(x_1,...,x_{k+1}) = f(x_1,...,x_k)$.

case 2. x1 = 1, $x_{k+1} \leq k+1$. Set $g(x_1,...,x_{k+1}) = 1-f(x_2,...,x_{k+1})$.

case 3. $x_1 = 1$, $x_{k+1} = k+2$. Set $g(x_1,...,x_{k+1}) = 1$.

Suppose $g(x_1,...,x_{k+1}) = g(x_2,...,x_{k+2})$.

case 1. $x_1 \geq 2$. $g(x_1,...,x_{k+1}) = f(x_1,...,x_k) = f(x_2,...,x_{k+1})$. Contradiction.

case 2. $x_1 = 1$. Since $x_{k+1} \leq k+1$, we have $g(x_1,...,x_{k+1}) = 1-f(x_2,...,x_{k+1})$. But $g(x_2,...,x_{k+1}) = f(x_2,...,x_{k+1})$. Contradiction. QED