APPLICATIONS OF LARGE CARDINALS TO BOREL FUNCTIONS

by
Harvey M. Friedman
Department of Mathematics
Ohio State University
friedman@math.ohio-state.edu
www.math.ohio-state.edu/~friedman
November 25, 1997

NOTE: This is work in progress. No proofs are presented. Results are still being checked.

Let R be the set of all real numbers, and let CS(R) be the space of all nonempty countable subsets of R.

The space CS(R) has a unique "Borel structure" in the following sense. Note that there is a natural mapping from R^∞ onto CS(R}; namely, taking ranges. We can combine this with any Borel bijection from R onto R^∞ in order to get a "preferred" surjection $F:R \to CS(R)$.

In what sense is this preferred? Consider the following property * on $F:R \to CS(R)$:

i) F is onto;
ii) $\{(x,y1,y2,...): F(x) = F(y1) \cup F(y2) \cup ...\}$ is a Borel measurable subset of R^∞.

By way of background, we have the following:

THEOREM 1. Let $F,G:R \to CS(R)$ have property *. Then G is the result of composing F with a Borel permutation of R.

In light of Theorem 1, we fix a perferred $\varphi:R \to CS(R)$.

There are two reasonable ways to define the Borel functions F from CS(R) into CS(R).

1. There exists Borel $G:R \to R$ such that $F(\varphi(x)) = \varphi(G(x))$.
2. $\{(x,y): F(\varphi(x)) = \varphi(y)\}$ is Borel measurable subset of R^2.

THEOREM 2. Both of these definitions of Borel $F:CS(R) \to CS(R)$ are equivalent.

The following basic result indicates the liklihood of a substantial theory of the structure of Borel functions on CS(R).

THEOREM 3. Let $F:CS(R) \to CS(R)$ be Borel. Then there exists A such that $F(A) \subseteq A$.

We proved this around 1977. We actually showed that this can be proved in third order arithmetic but not in second order arithmetic. See [Fr].

We now want to talk about a new theorem of this rough form (Borel diagonalization) which is independent of ZFC.

Let X be an uncountable complete separable metric space. Then we can discuss Borel functions on CS(X) in the same manner.

More generally, let Y be an uncountable Borel measurable subset of X. We can also consider CS(Y). Using any Borel measurable bijection between X and Y, we can define the Borel functions on CS(Y).

We say that $x,y \in R^\infty$ are finitely equivalent if and only if y is obtained from x by a permutation of the indices that leaves all but finitely many indices fixed.

We say that $A \subseteq R^\infty$ is finitely invariant if and only if $x \in A$ and $E(x,y)$ implies $y \in A$. We write $FICS(R^\infty)$ for the space of all nonempty finitely invariant countable subsets of R^∞. This is obviously an uncountable Borel subset of $CS(R^\infty)$, and therefore we can consider Borel functions on $FICS(R^\infty)$ in the usual way.

Let $x,y \in R^\infty$. We say that x is a subsequence of y if and only if there is a strictly increasing function $f:N \to N$ such that each $x_i = y_{f(i)}$.

Here is a warmup exercise.

THEOREM 4. Let $G:FICS(R^\infty) \to FICS(R^\infty)$ be Borel. Then there exists A such that every element of G(A) is a subsequence of an element of A.

Theorem 4 has a proof that is closely related to Theorem 3, and so is provable in third order arithmetic but not in second order arithmetic.

We say that $A \in FICS(R^\infty)$ is a chain if and only if for all $x,y \in A$, x is a subsequence of y or y is a subsequence of x.

THEOREM 5. Let $G:FICS(R^\infty) \to FICS(R^\infty)$ be Borel. Then there exists a chain A such that every element of G(A) is a subsequence of an element of A.

It is necessary and sufficient to use infinitely many uncountable cardinals to prove Theorem 5. Theorem 5 cannot be proved in Zermelo set theory, but can be proved in ZF\P + $V(\omega+\omega)$ exists.

Now for the big stuff.

THEOREM 6. Let $G:FICS(R^\infty) \to FICS(R^\infty)$ be Borel. Then there exists A such that all elements of values of G at subsets of A are subsequences of elements of A.

Theorem 6 can be proved from a measurable cardinal, yet not with “every subset of N has a sharp.” Presumably, ZFC + Ramsey cardinal should also not suffice.

Again, in light of Theorems 4,5,6, there should be a substantial structure theory for the Borel functions on the space $FICS(R^\infty)$.

We are working on getting a clean extension of Theorem 6 that would require many measurable cardinals to prove.

[Fr] On the necessary use of abstract set theory, Advances in Math., Vol. 50, No. 3, September 1981, pp. 209-280.