BOOLEAN RELATION THEORY

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BABY BRT

BRT is always based on a choice of BRT setting. A BRT setting is a pair (V,K), where

V is an interesting family of multivariate functions.
K is an interesting family of sets.

In this talk, we will only consider V,K, where

V is an interesting family of multivariate functions from N into N.
K is an interesting family of subsets of N.

Here N is the set of all nonnegative integers.

BRT is always based on the following dimension suppressing forward imaging operator.

Let f be a k-ary function. I.e., all elements of dom(f) are k-tuples. Let A be a set.

$$fA = f[A^k] = \{f(x_1,\ldots,x_k): x_1,\ldots,x_k \in A\}.$$

BABY BRT

There are two flavors of Baby BRT.

Equational BRT.
Inequational BRT.

In Equational BRT, we focus on all statements of the following form:

FOR ALL $f \in V$, THERE EXISTS $A \in K$, SUCH THAT
A GIVEN BOOLEAN EQUATION HOLDS BETWEEN A,fA.

In Inequational BRT, we focus on all statements of the following form:

FOR ALL $f \in V$, THERE EXISTS $A \in K$, SUCH THAT
A GIVEN BOOLEAN INEQUATION HOLDS BETWEEN A,fA.

Here we use N as the Universal Set for Boolean algebra purposes.

BABY BRT

We now give the two seminal examples of Equational and Inequational Baby BRT.

For the example of Inequational Baby BRT, we use V = MF, K = INF, where

MF is the family of all functions f whose domain is some N^k and whose range is a subset of N.
INF is the family of all infinite subsets of N.

THIN SET THEOREM. $(\forall f \in MF)(\exists A \in INF)(fA \neq N)$.

For the example of Equational Baby BRT, we use V = SD, K = INF, where

SD is the family of strictly dominating f ∈ MF, in the sense that for all $x_1,\ldots,x_k \in N$, $f(x_1,\ldots,x_k) > \max(x_1,\ldots,x_k)$.

COMPLEMENTATION THEOREM. $(\forall f \in SD)(\exists A \in K)(fA = N\backslash A)$.

THIN SET THEOREM

THIN SET THEOREM. $(\forall f \in MF)(\exists A \in INF)(fA \neq N)$.

Proof: Let $f:N^k \to N$. Let p be the number of order types of k-tuples from N. By the infinite Ramsey theorem, we can find infinite A such that f assumes at most one value in {0,...,p} when using arguments from a single order type. Hence f omits at least one value from {0,...,p}. QED

We know that TST is provable in ACA′ but not in ACA_0. Also TST for k = 2 is not provable in WKL_0. These results of ours are proved in

Peter Cholak, Mariagnese Giusto, Jeffry Hirst, and Carl Jockusch, Free sets and reverse mathematics, in: Reverse Mathematics, ed. S. Simpson, Lecture Notes in Logic, Association for Symbolic Logic, 1905. http://www.nd.edu/~cholak/papers/preincollection.html

H. Friedman and S. Simpson, Issues and problems in reverse mathematics, 127-144, in: Computability Theory and Its Applications, ed. Cholak, Lempp, Lerman, Shore, American Mathematical Society, 2000.

It is open whether TST is equivalent to ACA′ over RCA_0, or whether TST for k = 3 is equivalent to ACA′.

COMPLEMENTATION THEOREM

COMPLEMENTATION THEOREM. $(\forall f \in SD)(\exists A \in INF)(fA = N\backslash A)$. In fact, $(\forall f \in SD)(\exists! A \subseteq N)(fA = N\backslash A)$.

Many ways to write fundamental equation $fA = N\backslash A$. E.g.,

$$fA = N\backslash A.$$

$$A = N\backslash fA.$$

$$A \,\dot\cup\, fA = N.$$

Proof: Beautiful way to teach clutter free recursion. Suppose membership in A has been determined for 0,...,n-1. Put $n \in A$ if and only if $n \notin fA$ so far. Since f is strictly dominating, $n \notin fA$ so far is the same as $n \notin fA$ after we are finished. For uniqueness, let A,B obey the equation. Let n be least such that $n \in A \Leftrightarrow n \notin B$. Then $n \in fA \Leftrightarrow n \in fB$, and so $n \in A \Leftrightarrow n \in B$. QED

Closely related to dominators and kernels in graph theory.

THEOREM (von Neumann 1944). Every finite dag has a unique kernel and unique dominator.

J. Von Neumann and O. Morgenstern, Theory of Games and Economic Behavior, Princeton University Press, Princeton, (1944).

BABY BRT CLASSIFICATIONS

Note that there are $2^{2^2} = 16$ Boolean inequivalent Boolean expressions in two variables. From this, we see that, in two variables, there are 16 Boolean inequivalent Boolean equations, and 16 Boolean inequivalent Boolean inequations.

Hence on each BRT setting (V,K), there are 16 statements in equational Baby BRT, and 16 statements in inequational Baby BRT. This is because we are dealing with Boolean equations/inequations in A,fA.

The book treats them all for V = MF, SD, and with K = INF. There are no surprises. One variant of the Thin Set Theorem arises in this way.

$$(\forall f \in MF)(\exists A \in INF)(fA \cup A \neq N).$$

This can be easily derived from TST in RCA_0.

EXTENDED BABY BRT CLASSIFICATIONS

In Extended Baby BRT, we use A,fA, but also fU, where U is the universal set. In the present BRT settings, U is just N.

The number of Boolean equations/inequations, up to Boolean equivalence, is $2^{2^3} = 256$.

It begins to be important to have a good way to write and to organize Boolean equations. Inequations are replaced by equations, by moving to the dual statements

$$(\exists f \in V)(\forall A \in K)(s = t).$$

The best way to write a Boolean equation in $B_1,\ldots,B_n$ is as a finite set of inclusions of the form

$$B_{i1} \cap \ldots \cap B_{ip} \subseteq B_{j1} \cup \ldots \cup B_{iq}$$

where $i_1,\ldots,i_p,j_1,\ldots,j_q$ is a permutation of $1,\ldots,n$, and $i_1 < \ldots < i_p$, and $j_1 < \ldots < j_q$. The degenerate cases are written

$$B_{i1} \cap \ldots \cap B_{ip} = \emptyset$$

$$B_{j1} \cup \ldots \cup B_{jq} = U$$

EXTENDED BABY BRT CLASSIFICATIONS

In the case at hand, we are using the three Boolean atoms A,fA,fN. The number of such Boolean inclusions is $2^3 = 8$. These inclusions can be simplified using the obvious fA ⊆ fN. This reduces the number from 8 to 6. Thus only $2^6 = 64$ statements need be considered.

In addition, we can organize the subsets of these 6 inclusions according to increasing cardinality.

Then if the statement in equational/inequational BRT is incorrect with a given set of inclusions, then we do not have to consider any superset of this set of inclusions.

This analysis was carried out for V = MF, SD (and more), and for K = INF. Some additional complications:

$$(\forall f \in SD)(\exists A \in INF)(A \cap fA = \emptyset \wedge A \subseteq fN \wedge fN \subseteq A \cup fA).$$

BEYOND BABY BRT

We now consider equational/inequational BRT on settings (MF,INF), (SD,INF), with one function and TWO sets. So we are looking at all statements of the form

$$(\forall f \in V)(\exists A,B \in K)(s = t \text{ in } A,B,fA,fB)$$
$$(\exists f \in V)(\forall A,B \in K)(s = t \text{ in } A,B,fA,fB)$$

where we have again used the dual for inequational BRT.

The number of component inclusions is $2^4 = 16$, and the number of sets of inclusions, which is the same as the number of statements, is $2^{16} = 65{,}536$.

We have not been able to handle all of these statements. However, we have been able to handle the easier statements

$$(\forall f \in V)(\exists A,B \in K)(A \subseteq B \wedge s = t \text{ in } A,B,fA,fB)$$
$$(\exists f \in V)(\forall A,B \in K)(A \subseteq B \Rightarrow s = t \text{ in } A,B,fA,fB)$$

We refer to this as equational/inequational BRT in A,B,fA,fB,$\subseteq$. The number of relevant inclusions is cut from 16 to 9, so that there are a total of 512 sets of inclusions, or statements, to be analyzed.

We use a tree like methodology to organize the work.

$A,B,fA,fB,\subseteq$

Some new phenomena come up when we are in $A,B,fA,fB,\subseteq$, on (SD,INF).

$(\forall f \in SD)(\exists A,B \in INF)(A \subseteq B \wedge B \mathbin{\dot\cup} fA = N \wedge A = B \cap fB)$.
$(\forall f \in SD)(\exists A,B \in INF)(A \subseteq B \wedge A \mathbin{\dot\cup} fB = N \wedge fA \subseteq B \wedge B \cap fB \subseteq fA)$.
$\neg(\forall f \in SD)(\exists A,B \in INF)(A \subseteq B \wedge A \cap fB = \emptyset \wedge fB \subseteq B)$.

We expect an explosion of new phenomena in the much harder A,B,fA,fB.

We also worked out equational/inequational BRT in $A,B,fA,fB,\subseteq$ on the five BRT settings

(SD,INF), (ELG ∩ SD,INF).
(ELG,INF), (EVSD,INF).
(MF,INF).

where ELG is “expansive linear growth”, and “EVSD is “eventually strictly dominating”. f in ELG if and only if f in MF and

there exist rational constants $c,d > 1$ such that for all but finitely many $x \in dom(f)$,

$$c|x| \leq f(x) \leq d|x|$$

where $|x|$ is the maximum coordinate of the tuple x.

MAHLO CARDINALS

The strongly 0-Mahlo cardinals are the strongly inaccessible cardinals cardinals (uncountable regular strong limit cardinals). The strongly n+1-Mahlo cardinals are the infinite cardinals all of whose closed unbounded subsets contain a strongly n-Mahlo cardinal.

These cardinals have delicious combinatorial properties going back to James Schmerl's Ph.D. thesis under Jack Silver in the 1970s.

Here is the particular combinatorial principle tailor made for applications to BRT:

Let $n,m \geq 1$, κ a strongly n-Mahlo cardinal, and $A \subseteq \kappa$ unbounded. For all $i \in \omega$, let $f_i:A^{n+1} \to \kappa$, and let $g_i:A^m \to \omega$. There exists E of order type ω such that
i) for all $i \geq 1$, f_iE is either a finite subset of sup(E), or of order type ω with the same sup as E;
ii) for all $i \geq 1$, g_iE is finite.

$SMAH^+ = ZFC + (\forall n < \omega)(\exists\kappa)(\kappa$ is a strongly n-Mahlo cardinal).
$SMAH = ZFC + \{(\exists\kappa)(\kappa$ is a strongly n-Mahlo cardinal)$\}_{n<\omega}$.

EQUATIONAL BRT
TWO FUNCTIONS, THREE SETS
A,B,C,fA,fB,fC,gA,gB,gC

With two functions and three sets, we have Boolean inequations in nine Boolean variables. There are 512 basic inclusions, and 2^{512} sets of basic inclusions, or statements. Without major new ideas, this is ridiculously hopeless.

THEOREM. There is an instance of equational BRT in A,B,C,fA,fB,fC,gA,gB,gC on (ELG,INF) that is provable in $SMAH^+$ but not in SMAH.

CONJECTURE. Every instance of equational BRT in A,B,C,fA,fB,fC,gA,gB,gC on (ELG,INF) is provable or refutable in $SMAH^+$. However, we cannot replace $SMAH^+$ by SMAH.

So we must investigate this Conjecture for fragments. We have explored one particular fragment, but there are others that remain to be investigated.

Firstly, the independent example only uses A,B,C,fA,fB,gB,gC. So the numbers are reduced from 512,2^{512}, down to 128,2^{128}. Still daunting. Also, the independent example stays independent with $A \subseteq B \subseteq C$ added.

EQUATIONAL BRT
TWO FUNCTIONS, THREE SETS
FRAGMENTS OF A,B,C,fA,fB,fC,gA,gB,gC

THEOREM. There is an instance of equational BRT in A,B,C,fA,fB,gB,gC,$\subseteq$ on (ELG,INF) that is provable in $SMAH^+$ but not in SMAH.

CONJECTURE. Every instance of equational BRT in A,B,C,fA,fB,gB,gC,$\subseteq$ on (ELG,INF) is provable or refutable in $SMAH^+$, However, we cannot replace $SMAH^+$ by SMAH.

Obviously, this Conjecture is a lot more amenable than the one with A,B,C,fA,fB,fC,gA,gB,gC, but absent a number of new ideas, it still looks out of reach.

Our BRT book is based on an entirely different fragment. We go back to A,B,C,fA,fB,fC,gA,gB,gC as the starting point, without $\subseteq$. Instead we work with inclusions among disjoint unions.

For background, let us rewrite the Complementation Theorem with '$\cup.$'.

COMPLEMENTATION THEOREM. $(\forall f \in SD)(\exists A \in INF)(A \cup. fA = N)$.

EQUATIONAL BRT
2 FUNCTIONS, 3 SETS
DISJOINT UNION INCLUSIONS

TEMPLATE. For all $f,g \in ELG$, there exist $A,B,C \in INF$ such that

$$X \cup. fY \subseteq V \cup. gW$$
$$P \cup. fR \subseteq S \cup. gT$$

where X,Y,V,W,P,R,S,T are among the letters A,B,C.

This Template has $3^8 = 6561$ instances. There is an obvious symmetry: permute A,B,C, and switch the two disjoint union inclusions. This defines an equivalence relation on Template instances, whose equivalence classes generally have 12 elements.

THEOREM. All but 12 instances of the Template are provable or refutable in RCA_0. The 12 exceptions are symmetric, and are provable in $SMAH^+$.

PRINCIPAL EXOTIC CASE. For all $f,g \in ELG$, there exist $A,B,C \in INF$ such that

$$A \cup. fA \subseteq C \cup. gB$$
$$A \cup. fB \subseteq C \cup. gC$$

There are 12 Exotic Cases, one Principal Exotic Case.

EQUATIONAL BRT
2 FUNCTIONS, 3 SETS
DISJOINT UNION INCLUSIONS

PRINCIPAL EXOTIC CASE. For all f,g ∈ ELG, there exist A,B,C ∈ INF such that

$$A \cup. fA \subseteq C \cup. gB$$
$$A \cup. fB \subseteq C \cup. gC$$

THEOREM. The Exotic Cases are provably equivalent to 1-Con(SMAH) over ACA'.

The remaining 6561-12 = 6549 cases involve numerous tricky combinatorial arguments. There are a total of 574 cases up to symmetry, of which only one cannot be decided in RCA_0 - the 12 Exotic Cases.

TEMPLATE*. For all f,g ∈ ELG, there exist arbitrarily large finite A,B,C ⊆ N such that

$$X \cup. fY \subseteq V \cup. gW$$
$$P \cup. fR \subseteq S \cup. gT$$

where X,Y,V,W,P,R,S,T are among the letters A,B,C.

THEOREM. All instances of Template* are provable or refutable in RCA_0.

EQUATIONAL BRT
2 FUNCTIONS, 3 SETS
DISJOINT UNION INCLUSIONS

TEMPLATE. For all f,g ∈ ELG, there exist A,B,C ∈ INF such that

$$X \cup. fY \subseteq V \cup. gW$$
$$P \cup. fR \subseteq S \cup. gT$$

TEMPLATE*. For all f,g ∈ ELG, there exist arbitrarily large finite A,B,C ⊆ N such that

$$X \cup. fY \subseteq V \cup. gW$$
$$P \cup. fR \subseteq S \cup. gT$$

BRT TRANSFER. Template and Template* are equivalent.

THEOREM. BRT Transfer is provably equivalent to 1-Con(SMAH) over ACA'.

TEMPLATE_2. For all f,g ∈ ELG, there exist A,B,C ∈ INF such that

$$X \cup. fY \subseteq V \cup. gW$$
$$P \cup. fR \subseteq S \cup. gT$$
$$D \cup. fE \subseteq J \cup. gK$$

CONJECTURE. Results extend to Template_2.

EQUATIONAL BRT
2 FUNCTIONS, 3 SETS
DISJOINT UNION INCLUSIONS

TEMPLATE$_3$. For all f,g ∈ ELG, there exist A,B,C ∈ INF such that

$$X \,\dot{\cup}\, Y \subseteq V \,\dot{\cup}\, W$$
$$P \,\dot{\cup}\, R \subseteq S \,\dot{\cup}\, T$$

where X,Y,V,W,P,R,S,T are among A,B,C,fA,fB,fC,gA,gB,gC.

We think that the analogous results hold for Template$_3$. However, the difficulty substantially increases as we move on to triples and more. We can of course add $A \subseteq B \subseteq C$ to the conclusion, lessening the difficulties substantially.

DISJOINT UNION INCLUSION THEORY is a branch of BOOLEAN RELATION THEORY.

PROOF OF PRINCIPAL EXOTIC CASE

PRINCIPAL EXOTIC CASE. For all $f,g \in ELG$, there exist $A,B,C \in INF$ such that

$$A \,\dot{\cup}\, fA \subseteq C \,\dot{\cup}\, gB$$
$$A \,\dot{\cup}\, fB \subseteq C \,\dot{\cup}\, gC$$

We have refuted the Principal Exotic Case for SD, and some other classes of functions. We actually prove the sharper

PROPOSITION B. Let $f,g \in ELG$ and $n \geq 1$. There exist infinite sets $A_1 \subseteq \ldots \subseteq A_n \subseteq N$ such that

i) for all $1 \leq i < n$, $fA_i \subseteq A_{i+1} \,\dot{\cup}\, gA_{i+1}$;

ii) $A_1 \cap fA_n = \emptyset$.

Fix n,f,g. Also fix a strongly p^{n-1}-Mahlo cardinal k, where p is the arity of f. We start with the structure $M = (N,<,0,1,+,f,g)$. By using the infinite Ramsey theorem infinitely many times, we expand M to the structure

$$M^* = (N^*,<^*,0^*,1^*,+^*,f^*,g^*,c_0^*,\ldots)$$

where the constants generate M^*, and the c^*'s are indiscernible with respect to all atomic formulas.

PROOF OF PROPOSITION B

PROPOSITION B. Let $f,g \in ELG$ and $n \geq 1$. There exist infinite sets $A_1 \subseteq \ldots \subseteq A_n \subseteq N$ such that

i) for all $1 \leq i < n$, $fA_i \subseteq A_{i+1} \cup. gA_{i+1}$;

ii) $A_1 \cap fA_n = \varnothing$.

$$M = (N,<,0,1,+,f,g)$$
$$M^* = (N^*,<^*,0^*,1^*,+^*,f^*,g^*,c_0^*,\ldots)$$

where the constants generate M*, and the c*'s are indiscernible with respect to all atomic formulas. The way this is done, any partial subsystem of M* generated r times over the c*'s, can be isomorphically embedded back into M.

Using the indiscernibility, we can transfinitely extend canonically to

$$M^{**} = (N^{**},<^{**},0^{**},1^{**},+^{**},f^{**},g^{**},\ldots,c_\alpha^{**},\ldots)_{\alpha<\kappa}.$$

Unfortunately, M*,M** are not well founded. However, the relevant ordering is $tx < y$, where t is some rational > 1. In M*,M**, this ordering is well founded, exploiting $f,g \in ELG$.

PROOF OF PROPOSITION B

PROPOSITION B. Let $f,g \in ELG$ and $n \geq 1$. There exist infinite sets $A_1 \subseteq \ldots \subseteq A_n \subseteq N$ such that

i) for all $1 \leq i < n$, $fA_i \subseteq A_{i+1} \cup. gA_{i+1}$;

ii) $A_1 \cap fA_n = \emptyset$.

$$M = (N,<,0,1,+,f,g)$$
$$M^* = (N^*,<^*,0^*,1^*,+^*,f^*,g^*,c_0^*,\ldots)$$
$$M^{**} = (N^{**},<^{**},0^{**},1^{**},+^{**},f^{**},g^{**},\ldots,c_\alpha^{**},\ldots)_{\alpha<\kappa}.$$

$tx < y$, $t > 1$, is well founded in M,M^*,M^{**}.

The Complementation Theorem has an obvious generalization to well founded structures. So we obtain a unique $A \subseteq N^{**}$ such that

$$A \cup. g^{**}A = N^{**}.$$
$$\{\ldots,c_\alpha^{**},\ldots\} \subseteq A.$$
$$f^{**}A \cap \{\ldots,c_\alpha^{**},\ldots\} = \emptyset.$$

This is much stronger than Proposition B (no straddling!) EXCEPT that it lives in M^{**} and not in M.

PROOF OF PROPOSITION B

PROPOSITION B. Let $f,g \in ELG$ and $n \geq 1$. There exist infinite sets $A_1 \subseteq \ldots \subseteq A_n \subseteq N$ such that

i) for all $1 \leq i < n$, $fA_i \subseteq A_{i+1} \cup. gA_{i+1}$;

ii) $A_1 \cap fA_n = \emptyset$.

$$M = (N,<,0,1,+,f,g)$$

$$M^* = (N^*,<^*,0^*,1^*,+^*,f^*,g^*,c_0^*,\ldots)$$

$$M^{**} = (N^{**},<^{**},0^{**},1^{**},+^{**},f^{**},g^{**},\ldots,c_\alpha^{**},\ldots)_{\alpha<\kappa}$$

$$A \cup. g^{**}A = N^{**}.$$

$$\{\ldots,c_\alpha^{**},\ldots\} \subseteq A.$$

$$f^{**}A \cap \{\ldots,c_\alpha^{**},\ldots\} = \emptyset.$$

We can build a tower of subsets of A, of length n, starting with the c_α's, which is like a Skolem hull construction. We can define Skolem functions whose arguments are the c_α's, that generate all of the elements in this tower, and also generates all of the c_α's that are used in terms representing the elements of A.

We now apply the indiscernibility property of kappa. This enables us to cut down the Skolem hull construction, starting with a set of indiscernible transfinite constants of order type omega.

From the indiscernibility, the c_α's that arise have order type ω. Since the elements in the tower are generated by a bounded number of steps from these c_α's, we see that the tower is isomorphically embeddable in the original structure M.

EFFECTIVITY OF PRINCIPAL EXOTIC CASE

PRINCIPAL EXOTIC CASE. For all f,g ∈ ELG, there exist A,B,C ∈ INF such that

$$A \cup. fA \subseteq C \cup. gB$$
$$A \cup. fB \subseteq C \cup. gC$$

THEOREM. The Principal Exotic Case holds in the arithmetic sets. This fact is provably equivalent to 1-Con(SMAH) over ACA.

OPEN QUESTION. Does the Principal Exotic Case hold in the recursive sets?

We know that the Principal Exotic Case is just as strong even for rather concrete f,g.

We let BAF (basic functions) be the least family of multivariate functions from N into N which are closed under composition and which contain the functions +,-,×,↑,log. Here +,×,↑ are the usual addition, multiplication, and base 2 exponentiation on N. x-y is raised to 0 if negative. log(x) is the base 2 logarithm, where we take the floor. By convention, log(0) = 0.

THEOREM. The Principal Exotic Case holds in the recursive sets - or even the sets with primitive recursive enumeration functions. This fact is provably equivalent to 1-Con(SMAH) over RCA_0.