WORKING WITH NONSTANDARD MODELS

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http://www.mathpreprints.com/
July 31, 2003

Most of the research in foundations of mathematics that I do in some way or another involves the use of nonstandard models. I will give a few examples, and indicate what is involved.

1. General algebra and measurable cardinals. An unexpectedly direct connection.

2. Borel selection and higher set theory. A descriptive set theoretic context extensively pursued by some functional analysts.

3. Equational Boolean relation theory and Mahlo cardinals. A discrete mathematical context.

4. We conclude with an adaptation of 1 to weak second order logic.

1. GENERAL ALGEBRA AND MEASURABLE CARDINALS.

Some innocent looking statements in general algebra turn out to be equivalent to the existence of a measurable cardinal. In fact, the first measurable cardinal turns out to be clearly identifiable in a basic context in general algebra.

Here an algebra is just a relational structure based on finitely many constant and function symbols and no relation symbols.

PROPOSITION 1.1. Every algebra with a sufficiently large domain has a proper extension with the same countable subalgebras up to isomorphism.

PROPOSITION 1.2. Every algebra with a sufficiently large domain has a proper extension with the same finitely generated subalgebras up to isomorphism.

THEOREM 1.3. Propositions 1.1 and 1.2 are provably equivalent to the existence of a measurable cardinal, over ZFC.

THEOREM 1.4. The least cardinal □, if any, such that every algebra of cardinality ≥ □ has a proper extension with the same countable (alternatively finitely generated) subalgebras up to isomorphism, is the least measurable cardinal.

Note that these Propositions and Theorems use the notion of cardinality, which can be argued to be not strictly algebraic. We give obvious reformulations which do not use the notion of cardinality.

PROPOSITION 1.1′. Every algebra with a sufficiently inclusive domain has a proper extension with the same countable subalgebras up to isomorphism.

PROPOSITION 1.2′. Every algebra with a sufficiently inclusive domain has a proper extension with the same finitely generated subalgebras up to isomorphism.

THEOREM 1.3′. Propositions 1.1′ and 1.2′ are provably equivalent to the existence of a measurable cardinal, over ZFC.

THEOREM 1.4′. Let D be a nonempty set. The following are equivalent.
i) every algebra whose domain includes D has a proper extension with the same countable subalgebras up to isomorphism;
ii) every algebra whose domain includes D has a proper extension with the same finitely generated subalgebras up to isomorphism;
iii) there is a countably additive 0,1 valued measure on all subsets of D, in which singletons have measure 0 and D has measure 1.

We now give a proof of Theorems 1.3 and 1.4.

LEMMA 1.5. Let □ be a measurable cardinal. Every algebra of cardinality ≥ □ has a proper extension with the same countable subalgebras up to isomorphism.

Proof: Let □ be measurable and A be an algebra of cardinality ≥ □. We will assume that the domain of A is a cardinal □ ≥ □. Let j:V □ M be an elementary embedding with critical point □, where M is a transitive class containing all ordinals. j is obtained by taking the ultrapower of V via a □-complete nonprincipal ultrafilter on □.

j(A) is an algebra with domain j(□), and j|dom(A) is an isomorphism from A into j(A).

Since □ is the critical point of j, it is clear that □ is not in the range of j. Hence j|dom(A) is a proper isomorphism from A into j(A).

We will use the important fact that M is closed under □ sequences; i.e., every □ sequence drawn from M is an element of M. In fact, every □ sequence drawn from M is an element of M (see [Ka94], p. 50, Proposition 5.7 d).

Let B be a countable subalgebra of j(A). We wish to show that B is isomorphic to a subalgebra of A. Since the domain of j(A) is an ordinal, B □ M. Let B* be any algebra with domain □ □ which is isomorphic to B. Then B* □ M and there is an isomorphism from B* onto B that lies in M.

In particular, we have that

B* is embeddable in j(A)

holds in M. Since j is the identity on V(□), we have j(B*) = B*, and so

B* is embeddable in A

holds in V.

We have thus shown that A and j(A) have the same countable subalgebras up to isomorphism. Now j(A) is not really an extension of A. But obviously j(A) is isomorphic to an extension A’ of A since A is embeddable into j(A). Clearly A and A’ have the same countable subalgebras up to isomorphism. QED

The reversal involves the interaction of a well founded model and another model which may not be well founded - i.e., nonstandard. Of course, the use of nonstandard models here is not intense, but is still significant.

We fix a cardinal □ such that every algebra of cardinality ≥ □ has a proper extension with the same finitely generated subalgebras up to isomorphism. Our aim is to show that there is a measurable cardinal □ □.

LEMMA 1.6. □ is uncountable.

Proof: Obviously □ is not finite. Suppose □ = □. We claim that (□,0,S) does not have any proper extension with the same finitely generated subalgebras up to isomorphism. This is because in any proper extension, there will be a finitely generated substructure with an element that is not generated from 0, whereas no such substructure exists in (□,0,S). In fact, all substructures of (□,0,S) are (□,0,S). QED

We begin with the structure (V(□+1),□). Replace the binary relation □ with the binary function symbol □* which is the characteristic function of □. Recall that □* is 0,1 valued. Also add the constants 0,1 representing the ordinals 0,1. Thus we have formed (V(□+1),□*,0,1).

We fix K to be a generous finite set of prenex formulas in □*,0,1,=, which are closed under subformulas. We add Skolem functions for the formulas in K in a standard way. Specifically, take each of the formulas in K and add a Skolem function for the outermost quantifier alternation, whose arity is equaled to the number of outermost universal quantifiers. In particular, if the formula begins with an existential quantifier, then we use a Skolem constant. As usual, these Skolem functions are only required to provide witnesses if a witness exists.

We write L* for the language based on □*,0,1,=, and the finitely many Skolem functions introduced in the previous paragraph.

Note that the conjunction of the assertions that each of these Skolem functions are actually Skolem functions for the intended formula is naturally expressed as a universal sentence in L*.

It is convenient to also add the unary function symbol S for the successor function on □, which is 0 off of □. We write the resulting algebra as (V(□+1),□*,0,1,S,...). Note that □ □ V(□+1) since □ is an uncountable cardinal. In fact, □ □ V(□+1).

We fix an algebra B = (E,□*′,0,1,S′,...) which properly extends (V(□+1),□*,0,1,S,...), where the two algebras have the same finitely generated substructures up to isomorphism.

LEMMA 1.7. (V(□+1),□*,0,1,S,...) and (E,□*′,0,1,S′,...) satisfy the same purely universal sentences in L*. The former is an elementary substructure of the latter with respect to the formulas in L*.

Proof: Suppose the latter satisfies an existential sentence □. Then □ holds in some finitely generated subalgebra of the latter, by simply taking the subalgebra generated by witnesses for □. Hence □ holds in some finitely generated subalgebra of the former. Hence □ holds in the former.

For the second claim, note that since the two algebras satisfy the same purely universal sentences in L*, the Skolem functions used for the former are also Skolem functions for the latter. The claim then follows by induction on the complexity of the formulas involved. QED

LEMMA 1.8. For all x □ E\□, S′(x) = 0.

Proof: Let x □ E\□, S′(x) ≠ 0. Look at the subalgebra of (E,□*′,S′,0,1,...) generated by x. This subalgebra has an element, x, with S′(x) ≠ 0, that is not generated from 0 and S′. However, in any subalgebra of (V(□+1),□*,S,0,1,...), every element, y, with S(y) ≠ 0, must be generated from 0 and S. So (E,□*′,S′,0,1,...) and (V(□+1),□*,S,0,1,...) do not have the same finitely generated subalgebras up to isomorphism. QED

At this point, we revert to a single binary relation. Thus we use (V(□+1),□) instead of (V(□+1),□*,S,0,1,...), and (E,□′) instead of (E,□*′,S′,0,1,...). We have x □′ y □ □*(x,y) = 1, and x □′ y □ E*′(x,y) = 1.

Since (V(□+1),□) is a standard model of the cumulative hierarchy on a successor ordinal, we see that (E,□′) is a possibly nonstandard model of the cumulative hierarchy on a successor "ordinal". The "ordinals" of (E,□′) may be nonstandard. We will be using the "generosity" of K, and Lemma 1.7.

In (V(□+1),□), □ is the first limit ordinal, and its □ predecessors are exactly the x such that S(x) ≠ 0. Hence in (E,□*′,S′,0,1,...), □ is the first limit ordinal, and its □′ predecessors are exactly the x such that S′(x) ≠ 0.

LEMMA 1.9. For all x, x □′ □ if and only if x □ □.

Proof: Let x □′ □. Since "(□y □ □)(S(y) = 1)" holds in (V(□+1),□), it holds in (E,□′). Hence S′(x) = 1. By Lemma 1.8, x □ E\□. Therefore x □ □. QED

We summarize where we are right now. (V(□+1),□) is a substructure of (E,□′). In fact, it is an elementary substructure of (E,□′) for the generous finite set of formulas K closed under subformulas. Furthermore, □ is an uncountable cardinal, and the two structures have the same □, and the same predecessors of □. Thus in the appropriate sense, the first structure is standard (well founded), but the second structure is an □ model that may be nonstandard.

LEMMA 1.10. Suppose □ < □+1 and (□x □′ □)(x □ □). Then (□x □′ V(□))(x □ V(□)). I.e., if the ordinals in (E,□′) are standard below the ordinal □, then the cumulative hierarchy in (E,□′) on □ is also standard.

Proof: By induction on □. This is trivial for □ = 0 since V(0) = ∅.

Let □ = □+1 and (□x □′ □)(x □ □). Then (□x □′ □)(x □ □ x = □). Hence (□x □′ □)(x □ □). By the induction hypothesis, (□x □′V(□))(x □ V(□)). Let y □′ V(□). Then in (E,□′), we have □ = □+1 and y □ V(□). Hence (□x □′ y)(x □ V(□)). Therefore y has the same □′ predecessors as an element of V(□). Since (E,□′) satisfies extensionality, y □ V(□).

Let □ be a limit ordinal and (□x □′□)(x □ □). Let □ < □. Then (□x □′ □)(x □ □). Hence (□x □′ □)(x □ □).

By induction hypothesis, (□□ < □)(□x □′ V(□))(x □ V(□)). Let x □′ V(□). We want x □ V(□). Since (E,□′) satisfies "□ is a limit ordinal and V(□) is the union of the V(□), □ < □", we let □ □′ □ be such that x □′ V(□) holds in (E,□′). Then □ < □, and x □′ V(□). Hence x □ V(□). QED

LEMMA 1.11. There is an ordinal in the sense of (E,□′) that is not an ordinal < □+1.

Proof: Suppose that every ordinal in the sense of (E,□′) is an ordinal < □+1. Then the hypothesis of Lemma 1.10 holds for every □ < □+1. Hence by Lemma 1.10, for all □ < □+1, (□x □′ V(□))(x □ V(□)). Hence for all ordinals □ in (E,□′), (□x □′ V(□))(x □ V(□)).

Now (□x)(□□)(x □ V(□)) holds in (V(□+1),□), and therefore in (E,□′). Hence (□x □ E)(□□)(x □ V(□)). Therefore V(□+1) = E, which is a contradiction. QED

LEMMA 1.12. There exists x □′ □, x □ V(□+1).

Proof: Since "□ is the largest ordinal" holds in (V(□+1),□), it also holds in (E,□′). So any ordinal x in the sense of (E,□′) has x □′ □. Apply Lemma 1.11. QED

By Lemma 1.12, we now fix □ to be least ordinal such that there exists x □′ □, x □ V(□+1). Clearly □ □ □. By Lemma 1.10, □ is also the least ordinal such that there exists x □′ □ such that x □ □.

LEMMA 1.13. □ is a limit ordinal.

Proof: (E,□′) satisfies "□ is nonempty" and so (V(□+1),□) also satisfies "□ is nonempty". Hence □ > 0. Suppose □ = □+1. Let x □′ □, x □ V(□+1). Clearly (E,□′) satisfies "□ = □+1". So x □′ □ or x = □. Both alternatives violate x □ V(□+1). QED

We now fix x □′ □, x □ V(□+1).

We define a nonprincipal ultrafilter W over □. I.e.,

i) W □ P(□);
ii) for all y □ □, y □ W   □\y □ W;
iii) for all y,z □ W, y □ z □ W;
iv) for all y □ W, |y| ≥ 2.

Take W to be the set of all A □ □ such that x □′ A.

LEMMA 1.14. W is a nonprincipal ultrafilter over □.

Proof: i) is immediate. Let y □ □. Let z = □\y. "y □ z = □" holds in (V(□+1),□), and so in (E,□′). Since x □′ □, we have x □′ y   x □′ □\y. Hence y □ W   □\y □ W.

Let y,z □ W, and u = y □ z. "y □ z = u" holds in (V(□+1),□), and so in (E,□'). Since x □' y,z, we have x □' u. Hence u □ W.

Let y □ W. Then x □' y. So "y is nonempty" holds in (E,□'), and hence in (V(□+1),□). Now suppose y = {z}. Then "every element of y is z" holds in (V(□+1),□), and hence in (E,□'). Hence "x = z" holds in (E,□'). Therefore x = z □ □, which is a contradiction. QED

We now show that W is countably complete; i.e., the intersection of any infinite sequence of elements of W is an element of W.

LEMMA 1.15. W is a countably complete nonprincipal ultrafilter over □.

Proof: Let $D_0,D_1,\ldots$ all lie in W. Then x □' $D_0,D_1,\ldots$. Let D be the intersection of the D's, and f = {<n,D_n>: n □ □}.

Obviously "f is a function with domain □" holds in (V(□+1),□). Hence "f is a function with domain □" holds in (E,□'). Also "D is the intersection of the values of f" holds in (V(□+1),□). Hence "D is the intersection of the values of f" holds in (E,□').

We claim that for all y □ E, if "y is a value of f" holds in (E,□'), then y is a value of f.

To see this, assume "y is a value of f" holds in (E,□'), and let n □ E be such that "y = f(n)" holds in (E,□'). Since "f is a function with domain □" holds in (E,□'), we see that n □' □. By the crucial Lemma 1.9, n □ □. Now "v = f(n)" holds in (V(□+1),□), where v is f(n). Hence "v = f(n)" holds in (E,□'), and hence y = v = f(n).

We now claim that "x lies in every value of f" holds in (E,□'). Let y be such that "y is a value of f" holds in (E,□'). Then y is a value of f, and hence x □' y.

Obviously "D is the intersection of the values of f" holds in (E,□') since it holds in (V(□+1),□). Hence x □' D by the previous claim. Therefore D □ W. QED

LEMMA 1.16. There is a measurable cardinal □ □.

Proof: By Lemma 1.15, there is a countably complete nonprincipal ultrafilter over the ordinal □ □ □. Let □ be the least ordinal such that there is a countably complete nonprincipal ultrafilter over □. Then □ □ □ is also the least cardinal such that there is a countably complete nonprincipal ultrafilter over □. Hence □ is a measurable cardinal □ □ (see [Ka94], p. 23). QED

THEOREM 1.3. Propositions 1.1 and 1.2 are provably equivalent to the existence of a measurable cardinal, over ZFC.

Proof: By Lemmas 1.5 and 1.16. QED

THEOREM 1.4. The least cardinal □, if any, such that every algebra of cardinality ≥ □ has a proper extension with the same countable (alternatively finitely generated) subalgebras up to isomorphism, is the least measurable cardinal.

Proof: Let □ be the least cardinal such that every algebra of cardinality ≥ □ has a proper extension with the same countable subalgebras up to isomorphism. By Lemma 1.16, there is a measurable cardinal □ □. Let □ be a measurable cardinal □ □. By Lemma 1.5, every algebra of cardinality ≥ □ has a proper extension with the same countable subalgebras up to isomorphism. Since □ is least with this property, we have □ = □. Hence there is no measurable cardinal < □. Since there is a measurable cardinal □ □, □ is the least measurable cardinal.

The argument can be repeated using "finitely generated" in place of "countable", since Lemma 1.16 was proved using "finitely generated". QED

For an announcement of many further results along these lines, see [Fr03a].

2. BOREL SELECTION AND HIGHER SET THEORY.

We extract material from [Fr03] to give the flavor of the applications of nonstandard models. Here these applications are much more intense than in the previous section, although the models are still slightly standard - they are □-models.

Let S a set of ordered pairs, and A be a set. We say that f is a selection for S on A iff dom(f) = A and $(\forall x \in A)$, $(x,f(x)) \in S$.

f is a selection for S iff f is a selection for S on dom(S) = $\{x:(\exists y)((x,y) \in S)\}$.

THEOREM 2.1. Let $S \subseteq N^N \times N^N$ be Borel and $E \subseteq N^N$ be Borel. If there exists a continuous selection for S on every compact subset of E, then there exists a continuous selection for S on E.

Theorem 2.1 is due to Debs and Saint Raymond of Paris VII, [DSR01X], using Borel Determinacy. We showed that it cannot be proved using just countably many iterations of the power set operation.

The same results apply to the simpler

THEOREM 2.2. Let $S \subseteq N^N \times N^N$ be Borel. If there is a constant selection for S on every compact set, then there is a Borel selection for S.

Theorem 2.2 is also due to Debs and Saint Raymond, in [DSR99] and [DSR01X], using that the set of all Borel selectors of a Borel relation can be coded by a Π^1_1 set of reals.

PROPOSTION 2.3. Let $S \subseteq N^N \times N^N$ be Borel. If there is a Borel selection for S on every compact set, then there is a Borel selection for S.

Proposition 2.3 turns out to be independent of ZFC despite being only a Borel statement. In fact, we have shown that this is equivalent to “some function strictly dominates all functions constructible in any given function”.

DOM: $(\forall f)(\exists g)(\forall h \in L[f])$(g eventually dominates h).

PROPOSTION 2.4. Let $S \subseteq N^N \times N^N$ be Borel. If there is a Borel selection for S on every compact set, then there is a Borel selection for S.

There is a way of proving Proposition 2.4 using Borel determinacy if one can bound the levels of the Borel selections for S over compact sets. Using DOM and various absoluteness arguments, we obtain such a bound in [Fr03].

We use nonstandard models to show that Theorem 2.2 cannot be proved with just countably many iterations of the power set operation.

More specifically, in [Fr03] we show that ZFC\P + V = L + Theorem 2.2 proves that for all □ < $\square_1$ there is an □-model of ZFC\P + V(□) exists. We sketch the ideas.

We work in ZFC\P + V = L + □ < $\square_1$ + for all countable □, there is a set of rank □ □ lying in L(□+1)\L(□). Our job is to refute Theorem 2.2.

Let KP′ = KP with infinity. Let W = class of all (□,R) such that

1. (□,R) satisfies KP′ + V = L.
2. There is an internal ordinal in (□,R) whose set of predecessors is of type □.
3. (□,R) satisfies "for all □, there is a set of rank □ □ lying in L(□+1)\L(□)."

W must be plentiful since L($\square_1$) satisfies these things, and we are working within ZFC\P. Obviously W is Borel.

Let (□,R) □ W. At each level of the constructible hierarchy in (□,R), we can consider the family of sets of rank ≤ □ "present" there. This attaches a transitive set of rank □ □+1 to each "ordinal" of (□,R). These are strictly increasing under inclusion because of condition 3 on elements of W.

Write (□,R)′ for the union of these sets. I.e., the set of all sets of rank □ □ "present" in (□,R).

We say that (□,R) is special iff it lies in W and every element of (□,R) is definable in (□,R).

We also say that L(□) is special iff every (□,R) isomorphic to L(□) is special.

Define L(□)′ = (□,R)′, where (□,R) is isomorphic to L(□).

LEMMA 2.5. Let L(□) satisfy KP′, □ < □ < $\square_1$, where a new real appears in L(□+1). Then L(□) is special. There are arbitrarily large countable □ such that L(□) is special.

Let (□,R) □ W be special. We need a special code (□,R)* for (□,R). We use T = set of true sentences in (□,R), T □ N.

(□,R)* is the first real recursive in the double jump of T such that

1. (□,R)* strictly dominates every real recursive in T.
2. For all n, (□,R)*(n) is even iff n □ T.

Note that given a special code, we can recover an underlying special (□,R) uniquely up to isomorphism, quite effectively.

The set of special codes is a Borel set.

We are now ready to define the Borel set S □ N^N x N^N.

Let x,y □ N^N. (x,y) □ S iff either x is not a special code, or x,y are both special codes and

#) Let x = (□,R)*, y = (□,Y)*, effectively chosen. Let U = maximum common initial segment of (□,R)′ and (□,Y)′. Then U is the set of strict predecessors of some element of (□,Y)′.

In the case that the maximum common initial segment of (□,R)′ and (□,Y)′ demonstrate the ill foundedness of both (□,R)′ and (□,Y)′, we take (x,y) □ S - but it doesn’t make any difference what we do in this case.

LEMMA 2.6. There is no Borel selection for S.

LEMMA 2.7. Let V □ N^N be compact. There is a countable ordinal bound to the lengths of the maximum well ordered initial segments of the internal ordinals of every (□,R) □ W such that (□,R)* lies in V.

LEMMA 2.8. There is a constant Borel selection for S on every compact subset of N^N.

We now have the following.

THEOREM 2.9. ZFC\P + V = L + Theorem 2.2 proves that for all □ < $□_1$ there is an □-model of ZFC\P + V(□) exists.

As discussed in [Fr03], we see that using only countably many iterations of the power set operation is insufficient to prove Theorem 2.2. However, using all countable iterations of the power set operation is sufficient to prove Theorem 2.2, and even Theorem 2.1.

As a second, related application of nonstandard models, in [Fr03] we derive DOM from the following.

PROPOSTION 2.3. Let $S \subseteq N^N \times N^N$ be Borel. If there is a Borel selection for S on every compact set, then there is a Borel selection for S.

Since DOM is derivable from Proposition 2.3, we see that Proposition 2.3 is not provable in ZFC. The derivation of Proposition 2.3 in ZFC + DOM appears in [Fr03].

We work in ZFC\P + ¬DOM. We wish to refute Proposition 2.3.

To simplify the notation in the sketch, we will assume that no real dominates every constructible real.

Let X be the class of all (ω,R) satisfying KP' + V = L. X is Borel.

We say (ω,R) is unusual iff every element is definable.

We say $L(\alpha)$ unusual iff $L(\alpha)$ satisfies KP' and every element is definable.

Let $(\omega,R) \in X$ be unusual. The unusual code $(\omega,R)^*$ is defined as follows.

Let T be the theory of (ω,R). $(\omega,R)^*$ is the first sequence recursive in the double jump of T such that

1. $(\omega,R)^*$ eventually strictly dominates every sequence recursive in T.
2. for each n, $(\omega,R)^*(n)$ is even iff $n \in T$. The set of all unusual codes is Borel.

Now define Borel $S' \subseteq N^N \times N^N$ as follows.

Let $x,y \in N^N$. $(x,y) \in S'$ iff

1. x is not an unusual code, or
2. x is an unusual code and y is an infinite backward R chain, where (ω,R) comes from x; or

3. x,y are unusual codes and x is internal to (□,Y), where Y is obtained from y.

Note that S’ is arithmetic.

LEMMA 2.10. If L(□) satisfies KP’, $\square < \square_1$, and L(□+1)\L(□) meets N^N, then L(□) is unusual. There are arbitrarily large $\square < \square_1$ such that L(□) is unusual.

LEMMA 2.11. There is no Borel selection for S’.

LEMMA 2.12. Let V be compact. There exists a countable ordinal bound to the lengths of the maximum well ordered initial segments of the internal ordinals of the (□,R) □ X such that (□,R)* lies in V.

LEMMA 2.13. There is a Borel selection for S’ on every compact subset of N^N.

So we now have that Proposition 2.3 implies DOM over ZFC\P. In particular, Proposition 2.3 is independent of ZFC.

3. EQUATIONAL BOOLEAN RELATION THEORY AND MAHLO CARDINALS.

Equational Boolean Relation Theory (equational BRT) concerns the Boolean equations between sets and their images under multivariate functions.

In [Fr02], we present equational BRT, and give a complete proof of a crucial instance of equational BRT using the existence of Mahlo cardinals of every finite order.

The proof that ZFC plus the existence of a Mahlo cardinal of any specific finite order is not sufficient uses nonstandard models in an extremely intensive way, where there are nonstandard integers. That proof is not presently publicly available. However, we will give an oversimplified sketch of some key ideas.

Rather than repeat the rather full discussion of equational BRT in [Fr02], we move directly to the special corner of equational BRT where we have obtained rather complete information.

We use the space MF(N) of all multivariate functions form N into N, which are functions whose domain is a Cartesian power of N (finite exponent) and whose range is a subset of N.

Let $f \in MF(N)$ and $A \subseteq N$. Define fA to be the set of all values of f at arguments from A.

We say that $f \in MF(N)$ is of expansive linear growth if and only if there exist $c,d > 1$ such that for all but finitely many $x \in dom(f)$,

$$c|x| \leq f(x) \leq d|x|$$

where $|x|$ is the maximum coordinate of the tuple x.

Let ELG(N) be the set of all $f \in MF(N)$ of expansive linear growth. Let INF(N) be the set of all infinite subsets of N.

We use $X \cup. Y$ for $X \cup Y$ together with the commitment that X,Y are disjoint. For example,

$$X \cup. Y \subseteq Z \cup. W$$

means

$$X \cup Y \subseteq Z \cup W \wedge X \cap Y = \emptyset \wedge Z \cap W = \emptyset.$$

Another way of explaining this convention is to think of $\cup.$ as a partially defined binary operation on sets, which is defined if and only if the two sets are disjoint. The additional rather standard convention is that if a logically atomic statement is true then all expressions within it must be defined.

PROPOSITION 3.1. For all $f,g \in ELG(N)$ there exist $A,B,C \in INF(N)$ such that

$$A \cup. fA \subseteq C \cup. gB$$
$$A \cup. fB \subseteq C \cup. gC.$$

A cardinal κ is 0-Mahlo if and only if κ is strongly inaccessible.

A cardinal κ is (n+1)-Mahlo if and only if it is n-Mahlo and every closed and unbounded subset of κ has an element that is n-Mahlo.

Let MAH = ZFC + $\{$there exists an n-Mahlo cardinal$\}_n$. Let MAH+ = ZFC + "for all n there exists an n-Mahlo cardinal".

THEOREM 3.2. MAH+ proves Proposition 3.1.

The proof of Theorem 3.2 is presented in detail in [Fr02].

Nonstandard models are used in an essential way in order to prove the following.

THEOREM 3.3. MAH does not prove Proposition 3.1. ACA + Proposition 3.1 proves Con(MAH).

The overarching idea is to work in ACA + Proposition 3.1, and construct, in a series of many stages, a non ω-model of MAH. Slowly but surely one builds countable structures that more and more resemble a model of set theory with large cardinals.

The two inclusions in Proposition 3.1 are of course rather special, and so this immediately leads to consideration of an arbitrary pattern of letters A,B,C in the two conclusions. There are 8 positions for A,B,C, for a total of 3^8 = 6561 statements.

I.e., we are looking at all 6561 statements of the following form.

PROPOSITION. For all f,g ∈ ELG(N) there exist A,B,C ∈ INF(N) such that

X ∪. fY ⊆ Z ∪. gW
S ∪. fT ⊆ U ∪. gV.

Here X,Y,Z,W,S,T,U,V are among the three letters A,B,C.

We have shown the following.

THEOREM 3.4. Proposition 3.1 and its other 11 symmetric equivalent forms (total of 12) are the only ones among the 6561 that cannot be settled in RCA_0. They are provably equivalent, over ACA, to the 1-consistency of MAH.

We now give some idea of how we construct a non ω-model of MAH using Proposition 3.1.

We need only use Proposition 3.1 for multivariate functions f from N into N which can be defined by finitely many cases, where

i) each case given by a finite set of polynomial inequalities with integer coefficients;
ii) f is defined within each case by a polynomial with integer coefficients;
iii) for all $x \in dom(f)$, $f(x) \geq 2|x|$.

Let V_0 be the set of all such f. As a first step, we derive the following in ACA + Proposition 3.1.

LEMMA 3.5. For all functions $f,g \in V_0$, there exist infinite $A \subseteq B \subseteq C \subseteq N$ such that
i) $fA \subseteq B \cup gB$;
ii) $fB \subseteq C \cup gC$;
iii) $C \cap gC = \emptyset$;
iv) $A \cap fB = \emptyset$.

Here is the next step.

LEMMA 3.6. For all $k \geq 1$ and functions $f,g \in V_0$, there exist infinite $A_1 \subseteq ... \subseteq A_k \subseteq N$ such that for all $1 \leq i \leq k-1$,
i) $fA_i \subseteq A_{i+1} \cup gA_{i+1}$;
ii) $A_k \cap gA_k = \emptyset$;
iii) $A_1 \cap fA_k = \emptyset$.

The next step is to apply the compactness theorem so that we have an infinite tower within a single structure.

LEMMA 3.7. There exists a countable model M of $I\Sigma_0$ and $A_1 \subseteq A_2 \subseteq ... \subseteq dom(M)$ such that
i) A_1 is unbounded in dom(M) and forms a set of indiscernibles for bounded existential statements whose existential quantifiers range over A_2, with parameters from the $A_i\#$ below the indiscernibles;
ii) for any bounded existential $\varphi(x)$, there exists a linear function f(x) with standard coefficients, such that for all $x \in A_i\#$, $\varphi(x)$ holds over A_{i+1} if and only if $f(x) \in A_{i+1}$;
iii) for all $x \in A_1$, nothing in the A's lies in $(x/2,x)$.

Here E# is the closure of $E \cup \{0,1\}$ under $+,\bullet$.

The next step is to take the submodel of M generated by $0,1,+,\bullet$, and the union of the A's. We inherit the < of M. The elements of A_1 become limit points and appropriate indiscernibles. We also get an appropriate enumeration theorem.

LEMMA 3.8. There exists a countable ordered commutative semiring with unit, M = (D,<,0,1,+,•), and $c_1 < c_2 < \ldots$ unbounded in D, such that
i) the c's are indiscernibles for bounded formulas using parameters below indiscernibles;
ii) for any c_i, every subset of $[0,c_i]$ defined by a formula bounded by c_i with parameters from $[0,c_i]$ is defined by some formula bounded by c_{i+1} with a fixed number of symbols, with a single parameter allowed from $[0,c_{i+1}]$;
iii) the c's are limit points.

With such a structure at hand, we can internally build the constructible universe, and show that there exists, for each k, an internally k-Mahlo cardinal below c_1. The first limit ordinal serves as the ▯, and will of course be nonstandard.

We again warn the reader that this is a simplified sketch which should not be taken too literally.

4. WEAK SECOND ORDER LOGIC AND MEASURABLE CARDINALS.

Here we adapt the proof in section 1 to the case of elementary equivalence and elementary extension in weak second order logic.

WSOL (weak second order logic) is the same as first order predicate calculus with equality, except that we adjoin the quantifier "there are at most finitely many". This has the same effect as adding "there is at least one but at most finitely many", or adding the quantifier "there exist infinitely many".

Since we are moving to a language formulation of the results of section 1, we will use relational structures instead of algebras, but still require that they be in a finite relational type.

However, the results would be the same if we just used algebras as in section 1. In fact, when we reverse Proposition 4.1 below, we will only use algebras, as in section 1.

We also remark that all results in sections 1 and 4 would remain unchanged if we, for example, used countable relational types instead of just finite relational types.

We have the following analogous Propositions and Theorems.

PROPOSITION 4.1. Every structure with a sufficiently large domain has a proper WSOL elementary extension.

PROPOSITION 4.2. Every structure with a sufficiently large domain has a proper WSOL equivalent extension.

THEOREM 4.3. Propositions 4.1 and 4.2 are provably equivalent to the existence of a measurable cardinal, over ZFC.

THEOREM 4.4. The least cardinal κ, if any, such that every structure of cardinality $\geq \kappa$ has a proper WSOL elementary (alternatively equivalent) extension, is the least measurable cardinal.

Here are the obvious reformulations which do not use the notion of cardinality.

PROPOSITION 4.1’. Every structure with a sufficiently inclusive domain has a proper WSOL elementary extension.

PROPOSITION 4.2’. Every structure with a sufficiently inclusive domain has a proper WSOL equivalent extension.

THEOREM 4.3’. Propositions 1.1’ and 1.2’ are provably equivalent to the existence of a measurable cardinal, over ZFC.

THEOREM 4.4’. Let D be a nonempty set. The following are equivalent.
i) every structure whose domain includes D has a proper WSOL elementary extension;
ii) every structure whose domain includes D has a proper WSOL equivalent extension;
ii) there is a countably additive 0,1 valued measure on all subsets of D, in which singletons have measure 0 and D has measure 1.

LEMMA 4.5. Let κ be a measurable cardinal. Every algebra of cardinality $\geq \kappa$ has a proper WSOL elementary extension.

Proof: Let κ be measurable and A be a structure of cardinality $\geq \kappa$. We will assume that the domain of A is a cardinal $\lambda \geq \kappa$. Let $j:V \to M$ be an elementary embedding with critical point κ, where M is a transitive class containing all ordinals. j is obtained by taking the ultrapower of V via a □-complete nonprincipal ultrafilter on □.

j(A) is an algebra with domain j(□), and j|dom(A) is an isomorphism from A into j(A).

Since □ is the critical point of j, it is clear that □ is not in the range of j. Hence j|dom(A) is a proper isomorphism from A into j(A).

Let □(x1,...,xk) be a formula of WSOL with at most the free variables shown. Let $\square_1,\ldots,\square_k$ □ dom(A) = □. By the elementarity of j, we have

V satisfies "A satisfies □($\square_1,\ldots,\square_k$)"

if and only if

M satisfies "j(A) satisfies □(j($\square_1$,)..,j($\square_k$))".

Now satisfaction for WSOL is easily seen to be very absolute. Therefore

A satisfies □($\square_1,\ldots,\square_k$)

if and only if

j(A) satisfies □(j($\square_1$,)..,j($\square_k$)).

Hence j|dom(A) is a proper WSOL elementary embedding from A into j(A). By a standard construction, there is a proper WSOL elementary extension of A. QED

We fix a cardinal □ such that every algebra of cardinality ≥ □ has a proper WSOL equivalent extension. Our aim is to show that there is a measurable cardinal □ □.

LEMMA 4.6. □ is uncountable.

Proof: Obviously □ is not finite. Suppose □ = □. We claim that (□,0,S,<) does not have any proper WSOL equivalent extension. Here S is the successor operation on □.

To see this, let (A,0,S',<') be a proper WSOL equivalent extension.

First note that in (□,0,S,<)

i) < is a linear ordering of □;
ii) 0 is the < least element;
iii) S is the successor function for <.

Hence

i) <' is a linear ordering of □;
ii) 0 is the < least element;
iii) S' is the successor function for <'.

Since S □ S', it is easy to see that (A,<') must be a proper end extension of (□,<).

We now use WSOL. Note that (□,<) satisfies

(□x)(there are at most (standardly) finitely many y < x)

Hence (A,<) satisfies

(□x)(there are at most (standardly) finitely many y < x).

This contradicts that (A,<') is a proper end extension of (□,<). QED

We now let (V(□+1),□*,0,1,S,...) be the algebra defined after Lemma 1.6 and before 1.7 in section 1. We fix an algebra B = (E,□*',0,1,S',...) which is a proper WSOL equivalent extension of (V(□+1),□*,0,1,S,...).

We now follow the proof that there is a measurable cardinal □ □ given in section 1, which is Lemma 1.16. However, Lemma 1.16 was proved under the assumption that

"every algebra of cardinality ≥ □ has a proper extension with the same finitely generated subalgebras up to isomorphism"

whereas we are operating under the different assumption that

"every algebra of cardinality ≥ □ has a proper WSOL equivalent extension."

LEMMA 4.7. (V(□+1),□*,0,1,S,...) and (E,□*',0,1,S',...) satisfy the same purely universal sentences in L*. The former is an elementary substructure of the latter with respect to the formulas in L*.

Proof: The first claim is by assumption, even for WSOL (for L*). As in the proof of Lemma 1.7, the second claim follows immediately from the first claim. QED

LEMAM 4.8. For all x □ E\□, S′(x) = 0.

Proof: Note that

(□x)(x □ □ □ S(x) ≠ 0)

holds in (V(□+1),□*,0,1,S,...).

By Lemma 4.7,

(□x)(x □ □ □ S′(x) ≠ 0)

holds in (E,□*′,0,1,S′,...).

Also by Lemma 4.7, □ is satisfied to be the first limit ordinal in (E,□*′,0,1,S′,...).

It suffices to show that, in (E,□*′,0,1,S′,...), the elements of □ are exactly the actual elements of □.

We now use WSOL. Clearly

"every element of □ has at most (standardly) finitely many elements"

holds in (V(□+1),□*,0,1,S,...). Hence

"every element of □ has at most (standardly) finitely many elements"

holds in (E,□*′,0,1,S′,...).

Hence every (E,□*′,0,1,S′,...) element of □ is an ordinal of (E,□*′,0,1,S′,...) with (standardly) finitely many (E,□*′,0,1,S′,...) elements. Therefore every (E,□*′,0,1,S′,...), element of □ is an actual element of □. QED

LEMMA 4.9. There is a measurable cardinal □ □.

Proof: The remainder of the proof that there is a measurable cardinal □ □ is identical to the proof of Lemma 1.16 in starting right after the proof of Lemma 1.8. This is because in the proof of Lemma 1.16, the hypothesis in section 1 on □ is not used after the proof of Lemma 1.8.

THEOREM 4.3. Propositions 1.1 and 1.2 are provably equivalent to the existence of a measurable cardinal, over ZFC.

Proof: By Lemmas 4.5 and 4.9. QED

THEOREM 4.4. The least cardinal □, if any, such that every structure of cardinality ≥ □ has a proper WSOL equivalent (alternatively elementary) extension, is the least measurable cardinal.

Proof: Let □ be the least cardinal such that every algebra of cardinality ≥ □ has a proper WSOL elementary extension. By Lemma 4.9, there is a measurable cardinal □ □. Let □ be a measurable cardinal □ □. By Lemma 4.5, every algebra of cardinality ≥ □ has a proper WSOL equivalent extension. Since □ is least with this property, we have □ = □. Hence there is no measurable cardinal < □. Since there is a measurable cardinal □ □, □ is the least measurable cardinal.

The argument can be repeated using WSOL equivalence in place of WSOL elementary, since Lemma 4.9 was proved using WSOL equivalence. QED

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