INVARIANT MAXIMALITY AND INCOMPLETENESS

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Abstract. We present new examples of discrete mathematical statements that can be proved from large cardinal hypotheses but not within the usual ZFC axioms for mathematics (assuming ZFC is consistent). These new statements are provably equivalent to Π^0_1 sentences (purely universal statements, logically analogous to Fermat's Last Theorem) - in particular provably equivalent to the consistency of strong set theories, including one that is in explicitly Π^0_1 form. The examples live in the rational numbers, with only order, where the nonnegative integers are distinguished elements. The statements take the general form: every order invariant $W \subseteq Q^{2k}$ has a maximal subset S^2, with an invariance condition. Certain statements of this form are shown to be provably equivalent to the widely believed Con(SRP), and hence unprovable in ZFC (assuming ZFC is consistent). Modifications are made, involving a simple cross section condition, which propels the statement beyond the huge cardinal hierarchy, to attain equivalence with Con(HUGE). We also present some nondeterministic constructions of infinite and finite length with some of the same metamathematical properties. These lead to practical computer investigations designed to provide arguable confirmation of Con(ZFC) and more.

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1. INTRODUCTION.

Mathematicians view mathematics as a special subject with singularly attractive features. Most intuitively feel that the great power and stability of some "rule book for mathematics" is an important component of their relationship with mathematics. The general feeling is that there is nothing substantial to be gained by revisiting the commonly accepted rule book, and they can continue to do all truly significant mathematics without any foundational concerns.

We approach 100 years of ZFC as the accepted rule book for mathematics. Nearly all mathematicians are aware of its existence - if not its details - and accept that they are implicitly working within its scope. Incompleteness is an attack on the power and stability of the rule book. As power and stability are integral parts of the mathematician's relationship with mathematics, Incompleteness is arguably the unique theme in mathematics today that has the real potential of profoundly altering the mathematician's relationship with mathematics at the most fundamental level. This puts Incompleteness in a category all by itself among all mathematical research.

Incompleteness, in the modern sense, was initiated by Kurt

Gödel through his first and second incompleteness theorems, and his relative consistency of the axiom of choice and the continuum hypothesis - with the follow up development of Paul Cohen on AxC and CH. The mathematician's basic instinct is to defend against Incompleteness by requiring that any statement that is known to be beyond the scope of the rule book be "real mathematics". The exact nature of the "real mathematics" requirement for Incompleteness has evolved over time according to the evolving nature of Incompleteness. It was already clear soon after Cohen that mathematicians were at least implicitly putting up a general defense against Incompleteness that was going to hold until there was a radical change in the examples of Incompleteness. An informal notion of "pathological mathematical object" had emerged during the 20th century that has shaped, and continues to shape, the general course of mathematics. Informal discussion has even made it to Wikipedia with the pages "Pathological (mathematics)" and "Well Behaved".

In the late 1960s we formulated what we now call Concrete Mathematical Incompleteness (CMI), aimed at developing examples of Incompleteness not involving, directly or indirectly, pathological objects. Here we are 45 years later. The CMI examples have the important feature that they become provable if we expand the rule book in certain well studied ways (large cardinal hypotheses, or least their consistency). Furthermore, there appears to be no alternative way to expand the rule book to remove the CMI (for many examples we have outright equivalence with their consistency or variants of consistency). With traditional set theoretic incompleteness, V = L is an alternative way to remove the mathematical incompleteness.

We regularly discuss the continually evolving examples of CMI with a range of mathematicians, including top luminaries such as A. Connes, C. Fefferman, H. Furstenburg, T. Gowers, M. Gromov, D. Kazhdan, Y. Manin, B. Mazur, D. Mumford, and looking forward to continuing discussions. All of these top luminaries are fully aware of what is at stake, and had interesting reactions. I thank them all for their valuable feedback.

It is obvious that these mathematicians do not identify, in any way, "fundamental or important mathematics" with "mathematics people have done or are doing now". Our examples are judged by such top luminaries strictly on fundamental mathematical standards of simplicity, naturalness, intrinsic interest, concreteness, and depth. This is fortunate since the integration of CMI with existing concrete mathematical developments will likely occur only at later stages of CMI. On the other hand, very strong mathematicians, even some at the next level below such top luminaries, do tend to identify "fundamental or important mathematics" with "mathematics people have done or are doing now", and ironically with the mathematics that such top luminaries have done or are doing now.

The outer limits of CMI live in the Borel measurable sets and functions between Polish spaces. Some historical highlights of CMI are: long finite sequences, continuous comparability of countable sets of reals, Kruskal's and Higman's theorem, Graph Minor Theorem, Borel Diagonalization, Borel determinacy, Borel selection, and Boolean Relation Theory. A detailed overview of CMI is given in the Introduction of [Fr14].

In [Fr14], we present an earlier development called Boolean Relation Theory (BRT), which represents an earlier state of the art in CMI. The present development has many advantages over BRT, and BRT has some advantages over CMI. We do not view CMI as replacing BRT, as they both have their strengths.

The simplest statements that we prove here from large cardinals, but do not know how to prove in ZFC, are IMR(Ntsf) and IMC(Ntsf) of section 3.1 (proved from large cardinals in section 9.2).

The simplest statements that we prove here from large cardinals, and know cannot be proved in ZFC (assuming ZFC is consistent), are IMR(∀Q≤n,Ntsf) and IMC(∀Q≤n,Ntsf) of section 3.2 (proved from large cardinals in section 9.2).

In section 7, we present a program of clear mathematical interest of a combinatorial nature. This is the Order Theoretic Invariance Program. The goal of the program is to determine all order theoretic invariance properties that can be required of a maximal root (clique) for an arbitrarily given order invariant relation (graph) on $Q[0,1]^k$. At this time, we have only partial results. In particular, IMR(∀Q≤n,Ntsf) and IMC(∀Q≤n,Ntsf) represent partial results, as they can be readily transferred to the $Q[0,1] = Q \cap [0,1]$ setting.

These partial results establish that any carrying out of the Order Theoretic Invariance Program must involve considerably more than ZFC. This is because particular instances have been established using large cardinal hypotheses, that cannot be established using just ZFC. We conjecture that the Order Theoretic Invariance Program will be carried out using the same large cardinal hypotheses that we use here (see section 9). The solution will take the form of an (intelligible) algorithm that, provably using large cardinal hypotheses, determines whether a given order theoretic relation $R \subseteq Q[0,1]^{2k}$ has the property in question: that all order invariant relations (graphs) on $Q[0,1]^k$ have an R invariant maximal root (clique).

IMR(∀Q≤n,Ntsf) and IMC(∀Q≤n,Ntsf) are implicitly arithmetic statements - even implicitly Π^0_1 - because they can rather straightforwardly be shown to be equivalent to the satisfiability of a readily constructed effectively given sequence of sentences in the first order predicate calculus with equality, so that Gödel’s completeness theorem applies.

Nevertheless, it is desirable to have explicitly arithmetic - even Π^0_1 - forms. In section 5, we present an utterly straightforward infinite sequential form of INMC(nsh), an equivalent variant of IMR(∀Q≤n,Ntsf) and IMC(∀Q≤n,Ntsf) that has some advantages and disadvantages. The main disadvantage of INMC(nsh) is gone in the finite form which is presented in section 5. First the infinite sequential form is presented which asserts that a certain infinite construction can be successfully carried out. The most obvious finite form is obtained by truncation, and therefore is an immediate consequence, and is explicitly Π^0_2. However, we have not established that this obvious truncated form is unprovable in ZFC. We then strengthen the statement by putting an estimate on the magnitudes of the numerators and denominators that are used. It then becomes explicitly Π^0_1, and we can show that the resulting statement is unprovable in ZFC, and in fact provably equivalent to Con(SRP) over EFA.

Section 8 on Computer Investigations is of special note. The discussion suggests, in detail, how exhaustive search algorithms based on truncations of the independent Π^0_1 sentences presented here, to finite initial segments, can either demonstrate inconsistencies in certain large cardinal hypotheses, or arguably, to some extent, confirm the consistency of certain large cardinal hypotheses.

All claims made in this paper that are not proved in section 9 are to appear in [Fr?].

2. PRELIMINARIES.

DEFINITION 2.1. Q is the set of rationals, N is the set of nonnegative integers, Z^+ is the set of positive integers. We use i,j,k,n,m,r,s,t for positive integers, and when indicated, for nonnegative integers. For $x,y \in Q^k$, $x < y \leftrightarrow \max(x) < \max(y)$, $x \leq y \leftrightarrow x < y \vee x = y$. For $S \subseteq Q^k$ and $p \in Q$, $S_{<p} = \{x \in S: \max(x) < p\}$, $S_{\leq p} = \{x \in S: \max(x) \leq p\}$. For $S \subseteq Q^k$, S# is the least $A^k \supseteq S \cup \{0\}^k$.

DEFINITION 2.2. A rational interval is a $J \subseteq Q$ such that
i. $(\forall a < b < c$ from $Q)(a,c \in J \rightarrow b \in J)$.
ii. $\inf(J), \sup(J) \in Q \cup \{-\infty,\infty\}$.
We use J exclusively for rational intervals. J can have cardinality 0 or 1.

DEFINITION 2.3. $Q \leq n$ is the rational interval $Q \cap (-\infty,n]$.

2.1. INVARIANCE.

Invariance plays a central role in the development presented here.

DEFINITION 2.1.1. Let R be a binary relation (set of ordered pairs) and W be a set. W is R invariant if and only if for all $(x,y) \in R$, we have $x \in W \leftrightarrow y \in W$.

We will be using invariance of $W \subseteq X$, where X is viewed as the ambient space for W.

DEFINITION 2.1.2. Let R be a binary relation. $W \subseteq X$ is R invariant if and only if for all $(x,y) \in R \cap X^2$, we have $x \in W \leftrightarrow y \in W$.

Some authors use "complete invariance" for our notion of invariance.

THEOREM 2.1.1. Let R' be the least equivalence relation containing R. The following are equivalent.
i. $W \subseteq X$ is R invariant.
ii. $W \subseteq X$ is R' invariant.
iii. W is $R \cap X^2$ invariant.
iv. W is $R' \cap X^2$ invariant.
v. W is the intersection of an R invariant set with X.

vi. W is the union of equivalence classes of the equivalence relation $R' \cap X^2$.

DEFINITION 2.1.3. Let $S \subseteq Q^k$ and $R \subseteq Q^{2k}$. $R[S] = \{y \in Q^k: (\exists x \in S)(R(x,y))\}$.

We treat unary functions as sets of ordered pairs. Thus if $R \subseteq Q^{2k}$ is a function from Q^k into Q^k, then R[S] is the same as the forward image of R on S as a function.

Almost all of our statements call for a root or a clique. The root or clique is a subset of an implied ambient space that is clear from the context. This convention will be reiterated when we introduce roots and cliques.

2.2. ORDER INVARIANCE.

DEFINITION 2.2.1. $x,y \in Q^k$ are order equivalent if and only if for all $1 \leq i,j \leq k$, $x_i < x_j \leftrightarrow y_i < y_j$.

We also use the following more general definition in section 9.2.

DEFINITION 2.2.2. Let (A,<) and (B,<') be linear orderings. $x \in A^k$ and $y \in B^k$ are order equivalent if and only for all $1 \leq i,j \leq k$, $x_i < x_j \leftrightarrow y_i < y_j$.

THEOREM 2.2.1. For all k, the number of equivalence classes of order equivalence on Q^k is finite.

DEFINITION 2.2.3. ot(k) is the number of equivalence classes of order equivalence on Q^k.

THEOREM 2.2.2. $ot(k) = \Sigma_{i \leq k} f(k,i)$, where f(k,i) is the number of surjective maps from {1,...,k} onto {1,...,i}. ot(k) is asymptotic to $k!/(2\ln^{k+1}2)$. The number of order invariant subsets of Q^k is $2^{ot(k)}$.

The above asymptotic result is from [Gr62], which also presents the following table of exact values:

ot(1) = 1
ot(2) = 3
ot(3) = 13
ot(4) = 75
ot(5) = 541
ot(6) = 4,683
ot(7) = 47,293

ot(8) = 545,835
ot(9) = 7,087,261
ot(10) = 102,247,563
ot(11) = 1,622,632,573
ot(12) = 28,091,567,595
ot(13) = 526,858,348,381
ot(14) = 10,641,342,970,443

Order invariant sets play a crucial role here.

DEFINITION 2.2.4. $W \subseteq Q^k$ is order invariant if and only if $W \subseteq Q^k$ is R invariant, where R is order equivalence on Q^k.

THEOREM 2.2.3. Let $W \subseteq Q^k$. The following are equivalent.
i. W is the union of equivalence classes of order equivalence on Q^k.
ii. W can be defined as a Boolean combination of inequalities $v_i < v_j$, $1 \leq i,j \leq k$.
iii. W is 0-definable over $(Q,<)$.

Order invariance extends to ambient spaces via Definition 2.1.2.

DEFINITION 2.2.5. Let $X \subseteq Q^k$. $W \subseteq X$ is order invariant if and only if $W \subseteq X$ is R invariant, where R is order equivalence on Q^k.

Order invariant $W \subseteq J^{2k}$ will play an important role in the theory. By writing $J^{2k} = J^k \times J^k$, we view W as a binary relation on J^k.

THEOREM 2.2.4. Let $|J| > 1$. The number of order invariant $W \subseteq J^{2k}$ is $2^{ot(2k)}$. The number of order invariant graphs on J^k is $2^{(ot(k)+ot(2k))/2}$.

Graphs and order invariant graphs are introduced in section 2.6.

2.3. N/ORDER INVARIANCE.

We will go beyond order invariance in the following natural way.

DEFINITION 2.3.1. $x,y \in Q^k$ are N/order equivalent if and only if for all $1 \leq i,j \leq k$, $(x_i < x_j \leftrightarrow y_i < y_j) \wedge (x_i \in N \leftrightarrow y_i \in N)$. $S \subseteq Q^k$ is N/order invariant if and only if S is R invariant, where R is N/order equivalence on Q^k.

THEOREM 2.3.1. $W \subseteq Q^k$ is N/order invariant if and only if W can be defined as a Boolean combination of statements $v_i < v_j$, $v_i \in N$, $1 \le i,j \le k$. There are finitely many N/order invariant subsets of Q^k.

N/order invariance of $R \subseteq Q^{2k}$ is used in section 4.

2.4. RESTRICTED SHIFTS.

The shift function on Q^k is very familiar: just add 1 to all coordinates.

DEFINITION 2.4.1. A restricted shift on Q^k is an $f:Q^k \to Q^k$ such that each f(x) is obtained from x by adding 1 to zero or more coordinates of x. I.e., $f(x)-x:Q^k \to \{0,1\}^k$.

Here are some simple examples of restricted shift functions.

DEFINITION 2.4.2. The shift on Q^k adds 1 to all coordinates. The N shift adds 1 to all coordinates in N. The nonnegative shift adds 1 to all nonnegative coordinates.

We shall see that the last of these restricted shifts play a significant role in the development, beginning with section 3.4.

An additional restricted shift called the N tail shift is crucial for the theory, and is introduced in section 3.1.

2.5. (Q,N,+,<).

(Q,N,+,<) is a well known tame structure, with quantifier elimination in an appropriately extended language. We use its definable sets to formulate strong forms of some conjectures (see section 7). We follow the usual convention in model theory that definability allows parameters, and 0-definability does not.

An ultimate goal is to determine exactly which relations on Q^k, definable over (Q,N,+,<) as subsets of Q^{2k}, can be used for the invariance of the roots and cliques throughout the paper. At this point, we do not even know exactly which relations on Q^k, definable over (Q,<) as subsets of Q^{2k}, can be used for many of the results.

2.6. GRAPHS.

All mathematicians are familiar with binary relations, and many are familiar with graphs. Therefore it is advantageous to use binary relations. However, there are some advantages to using graphs rather than relations.

DEFINITION 2.6.1. A graph is a pair $G = (V,E)$, where V is a set, $E \subseteq V^2$, and E is irreflexive and symmetric on V. We say that G is a graph on V. The elements of V are called vertices, and the elements of E are called edges. Two vertices are adjacent if and only if their order pair is an edge.

DEFINITION 2.6.2. Let $G = (J^k,E)$ be a graph and $p \in Q$. $G_{\leq p}$ is the graph $(J_{\leq p}{}^k,E_{\leq p})$.

3. INVARIANT MAXIMAL ROOTS AND CLIQUES.

DEFINITION 3.1. S is a root of $W \subseteq J^{2k}$ if and only if $S \subseteq J^k$ and $S^2 \subseteq W$. S is a maximal root of $W \subseteq J^{2k}$ if and only if S is a root of W and every root $S' \supseteq S$ of $W \subseteq J^{2k}$ is S.

DEFINITION 3.2. S is a clique in the graph G on J^k if and only if $S \subseteq J^k$ and every two distinct elements of S are adjacent in G. S is a maximal clique in G if and only if S is a clique in G and every clique $S' \supseteq S$ in G is S.

The roots of $W \subseteq J^{2k}$ always have J^k as the implied ambient space. The cliques of a graph on J^k always have J^k as the implied ambient space. Invariance conditions take into account the ambient space as discussed in section 2.1.

In Definitions 3.1 and 3.2, it clearly suffices to use only single point extensions S' of S. Maximal roots and cliques can also be defined by a particularly simple set equation.

THEOREM 3.1. (RCA_0). Let $W \subseteq J^{2k}$. S is a maximal root of W if and only if $S = \{x \in J^k: \text{for all } y \in S, (x,y),(y,x),(x,x) \in W\}$. Let G be a graph on J^k. S is a maximal clique in G if and only if $S = \{x \in J^k: x \text{ is adjacent in G to all } y \in S\backslash\{x\}\}$.

We use such set equations in sections 3.4 and 6.

THEOREM 3.2. (RCA_0). Every $W \subseteq J^{2k}$ has a maximal root. Every graph on J^k has a maximal clique. (ACA_0). Every root of every $W \subseteq J^{2k}$ can be extended to a maximal root of W. Every clique in every graph G on J^k can be extended to a maximal clique in G.

THEOREM 3.3. The second pair of claims of Theorem 3.2 are provably equivalent to ACA_0 over RCA_0, even for J = Q and order invariant W,G.

3.1. MAXIMAL ROOTS AND CLIQUES IN Q^k.

We focus on the statements

every order invariant $W \subseteq Q^{2k}$ has an R invariant maximal root
every order invariant graph on Q^k has an R invariant maximal clique

where $R \subseteq Q^{2k}$ is tame. We first try the basic restricted shifts from Definition 2.4.2.

YHEOREM 3.1.1. (RCA_0). Let $k \geq 2$. "Every order invariant $W \subseteq Q^{2k}$ has a shift (N shift) invariant maximal root" fails. "Every order invariant graph on Q^k has a shift (N shift) invariant maximal clique" fails.

We do not know if Theorem 3.1.1 holds for the nonnegative shift. However, see section 3.4.

This sets the stage for another restricted shift - the N tail shift.

DEFINITION 3.1.1. The N tail of $x \in Q^k$ consists of the x_i such that every $x_j \geq x_i$ lies in N. The N tail shift of $x \in Q^k$ results from x by adding 1 to the N tail of x.

EXAMPLE: The N tail of (-1,0,3,7/2,5,5,8) consists of both copies of 5 and the single copy of 8. The N tail shift of (-1,0,3,7/2,5,5,8) is (-1,0,3,7/2,6,6,9). The coordinates are in numerical order for the reader's convenience.

INVARIANT MAXIMAL ROOTS (Ntsf). IMR(Ntsf). For all k, every order invariant $V \subseteq Q^{2k}$ has an N tail shift invariant maximal root.

INVARIANT MAXIMAL CLIQUES (Ntsf). IMC(Ntsf). For all k, every order invariant graph on Q^k has an N tail shift invariant maximal clique.

Here Ntsf is read "N tail shift function".

IMR((Ntsf) and IMC(Ntsf) are the simplest statements presented here that we prove from large cardinals, but do not know how to prove in ZFC. However, see IMR(($\forall$Q≤n,Ntsf), IMC($\forall$Q≤n,Ntsf) in section 3.2.

We now sharpen N tail shift invariance, using the N tail shift relation.

DEFINITION 3.1.2. The N tail shift relation on Q^k is given by $R(x,y) \leftrightarrow (\exists n \geq 0)$(y results from adding 1 to the part of the N tail of x that is ≥ n).

INVARIANT MAXIMAL ROOTS (Ntsr). IMR(Ntsr). For all k, every order invariant $W \subseteq Q^{2k}$ has an N tail shift relation invariant maximal root.

INVARIANT MAXIMAL CLIQUES (Ntsr). IMC(Ntsr). For all k, every order invariant graph on Q^k has an N tail shift relation invariant maximal clique.

Here Ntsr is read "N tail shift relation".

Following Theorem 2.1.1, we use the least equivalence relation on Q^k containing the N tail shift relation. We call this equivalence relation, N tail equivalence, and the corresponding invariance notion, N tail invariance.

INVARIANT MAXIMAL ROOTS (Nteq). IMR(Nteq). For all k, every order invariant $W \subseteq Q^{2k}$ has an N tail invariant maximal root.

INVARIANT MAXIMAL CLIQUES (Nteq). IMC(Nteq). For all k, every order invariant graph on Q^k has an N tail invariant maximal clique.

Here Nteq is read "N tail equivalence".

N tail equivalence has the following simple direct definition.

THEOREM 3.1.2. (RCA_0). $x,y \in Q^k$ are N tail equivalent if and only if x,y are order equivalent and identical off of their respective N tails.

THEOREM 3.1.3. (RCA_0). IMR(Q,Ntsf), IMR(Q,Ntsr), IMR(Q,Nteq), IMC(Q,Ntsf), IMC(Q,Ntsr), IMC(Q,Nteq) are equivalent. They are provable in SRP^+. Furthermore, they are provable in WKL_0 + Con(SRP).

We do not know if these six statements are provable in ZFC or even in RCA_0. The second and third claims are proved here as part of Corollary 9.2.12.

In section 3.2, we will extend these three statements from Q^k to rational intervals, where we claim their equivalence with Con(SRP) over WKL_0.

3.2. MAXIMAL ROOTS AND CLIQUES IN J^k.

Note that we are not claiming any unprovability results in section 3.1 when we use J = Q. Recall from Definition 2.2 that J always denotes a rational interval.

INVARIANT MAXIMAL ROOTS/CLIQUES (intervals).
IMR/C(intervals). Let k ≥ 2 and J be given. The following seven statements are equivalent.
i. For all k, every order invariant $W \subseteq J^{2k}$ has an N tail shift invariant (N tail shift related invariant, N tail invariant) maximal root.
ii. For all k, every order invariant graph on J^k has an N tail shift invariant (N tail shift related invariant, N tail invariant) maximal clique.
iii. It is not the case that: J contains its nonnegative integer left endpoint and the length of J is at least 2.

THEOREM 3.2.1. In IMR/C(intervals), RCA_0 proves the equivalence of all six forms of i,ii, and the implication from any of these six forms to iii. The implication from iii to any of the six forms of i,ii is provably equivalent to Con(SRP) over WKL_0. IMR/C(intervals) is provable in SRP^+ but not in any consistent fragment of SRP proving RCA_0. Furthermore, it is provably equivalent to Con(SRP) over WKL_0.

Here are six statements corresponding to Con(SRP). They use rational intervals J = Q≤n.

INVARIANT MAXIMAL ROOTS
(∀Q≤n,Ntsf),(∀Q≤n,Ntsr),(∀Q≤n,Nteq)).
IMR(∀Q≤n,Ntsf),IMR(∀Q≤n,Ntsr),IMR(∀Q≤n,Nteq). For all k,n, every order invariant $W \subseteq Q{\leq}n^{2k}$ has an N tail shift invariant (N tail shift related invariant, N tail invariant) maximal root.

INVARIANT MAXIMAL CLIQUES
(∀Q≤n,Ntsf),(∀Q≤n,Ntsr),(∀Q≤n,Nteq)).

IMC(∀Q≤n,Ntsf),IMC(∀Q≤n,Ntsr),IMC(∀Q≤n,Nteq). For all k,n, every order invariant graph on Q≤n has an N tail shift invariant (N tail shift related invariant, N tail invariant) maximal clique.

THEOREM 3.2.2. IMR(∀Q≤n,Ntsf), IMR(∀Q≤n,Ntsr), IMR(∀Q≤n,Nteq), IMC(∀Q≤n,Ntsf), IMC(∀Q≤n,Ntsr), IMC(∀Q≤n, Nteq) are provable in SRP^+ but not provable in any consistent fragment of SRP that proves RCA_0. Furthermore, they are provably equivalent to Con(SRP) over WKL_0. These claims remain unchanged if we require that the maximal roots (cliques) be recursive in 0', resulting in explicitly arithmetic sentences. IMR(∀Q≤n,Ntsf), IMR(∀Q≤n,Ntsr), IMR(∀Q≤n,Nteq), IMC(∀Q≤n,Ntsf), IMC(∀Q≤n,Ntsr), IMC(∀Q≤n,Nteq) can be put in Π^0_1 form via the Gödel completeness theorem.

The provability claims in Theorem 3.2.2 are proved here as part of Corollary 9.2.12.

IMR(∀Q≤n,Ntsf), IMC(∀Q≤n,Ntsf) are the simplest statements that we prove here from large cardinals, and know cannot be proved in ZFC (assuming ZFC is consistent). Competitors are ISMR(Ntsf) and ISMC(Ntsf) in section 3.3, which do have the advantage of using J = Q.

3.3. STEP MAXIMAL ROOTS AND CLIQUES IN Q^k.

DEFINITION 3.3.1. A step maximal root of $W \subseteq J^{2k}$ is an $S \subseteq J^k$ such that for all $n \geq 0$, $S_{\leq n}$ is a maximal root of $W_{\leq n}$.

DEFINITION 3.3.2. A step maximal clique of the graph G on J^k is an $S \subseteq J^k$ such that for all $n \geq 0$, $S_{\leq n}$ is a maximal clique in $G_{\leq n}$.

THEOREM 3.3.1. (ACA_0). For all k, every $W \subseteq Q^{2k}$ has a step maximal root. For all k, every order invariant graph on Q^k has a step maximal clique.

YHEOREM 3.3.2. (RCA_0). Let $k \geq 2$ and $p \in Q$. "Every order invariant $W \subseteq Q^{2k}$ has a shift (N shift) invariant step maximal root" fails. "Every order invariant graph G on Q^k has a shift (N shift) invariant step maximal clique" fails.

We do not know if Theorem 3.3.2 holds for the nonnegative shift. However, see section 3.4.

INVARIANT STEP MAXIMAL ROOTS ($\forall Q \leq n$,Ntsf),($\forall Q \leq n$,Ntsr),($\forall Q \leq n$,Nteq)). ISMR($\forall Q \leq n$,Ntsf),ISMR($\forall Q \leq n$,Ntsr),ISMR($\forall Q \leq n$,Nteq). For all k,n, every order invariant $W \subseteq Q{\leq}n^{2k}$ has an N tail shift invariant (N tail shift related invariant, N tail invariant) step maximal root.

INVARIANT STEP MAXIMAL CLIQUES ($\forall Q \leq n$,Ntsf),($\forall Q \leq n$,Ntsr),($\forall Q \leq n$,Nteq)). ISMC($\forall Q \leq n$,Ntsf),ISMC($\forall Q \leq n$,Ntsr),ISMC($\forall Q \leq n$,Nteq). For all k,n, every order invariant graph on $Q{\leq}n^k$ has an N tail shift invariant (N tail shift related invariant, N tail invariant) step maximal clique.

THEOREM 3.3.3. ISMR(Ntsf), ISMR(Ntsr), ISMR(Nteq), ISMC(Ntsf), ISMC(Ntsr), ISMC(Nteq) are provable in SRP^+ but not provable in any consistent fragment of SRP that proves RCA_0. Furthermore, they are provably equivalent to Con(SRP) over WKL_0. These claims remain unchanged if we require that the step maximal roots and cliques be recursive in 0', resulting in explicitly arithmetic sentences. ISMR(Ntsf), ISMR(Ntsr), ISMR(Nteq), ISMC(Ntsf), ISMC(Ntsr), ISMC(Nteq) can be put in Π^0_1 form via the Gödel completeness theorem.

The following implication is immediate.

THEOREM 3.3.4. (RCA_0). ISMR(Nteq) implies all eighteen titled statements in sections 3.1 - 3.3, excluding IMR/C(intervals).

Theorem 3.3.4 is also true for IMR/C(intervals). This is proved in [Fr?].

We prove ISMR(Nteq) in WKl_0 + Con(SRP) in section 9.2. This establishes the same for all of the eighteen statements, via Theorem 3.3.4.

3.4. INDUCTIVELY MAXIMAL CLIQUES IN Q^k.

To simplify the discussion, we now use graphs and cliques rather than relations and roots. The same results hold for relations and roots.

Here we modify step maximality, with the effect of requiring that maximal cliques be built up smoothly (rather than stepwise) from below going up.

We call the most obvious strengthening, naïve inductive maximality, as it is too strong.

DEFINITION 3.4.1. A naive inductively maximal clique in the graph G on J^k is an S = {x ∈ J^k: x is adjacent to every y < x from S}.

THEOREM 3.4.1. (RCA_0). Let k ≥ 1 and |J| > 1. There is an order invariant graph on J^{2k} without a naïve inductively maximal clique.

DEFINITION 3.4.2. An inductively maximal clique in the graph G on J^k is an S = {x ∈ J^k ∩ S#: x is adjacent to every y < x from S}. The ambient space of S is J^k.

Note that in an inductively maximal clique, any two elements with different max are adjacent. However, distinct elements with the same max may or may not be adjacent.

THEOREM 3.4.2. (RCA_0). For all k, every order invariant graph on Q^k has an inductively maximal clique.

Theorem 3.4.2 is trivial since we can simply arrange that the inductively maximal clique be ∅ or {(0,...,0)}. However, the imposition of an invariance condition radically changes the situation.

INDUCTIVELY MAXIMAL CLIQUES (Nteq). INMC(Nteq). For all k, every order invariant graph on Q^k has an N tail equivalent inductively maximal clique.

We also can use a different kind of invariance condition. The nonnegative shift was introduced in section 2.4. Here is a more powerful operation.

DEFINITION 3.4.3. The N upper shift of S ⊆ Q^k is the union over n ≥ 0 of the result of adding 1 to all coordinates ≥ n of elements of S.

Note that the N upper shift contains the N tail shift.

INDUCTIVELY MAXIMAL CLIQUES (nsh). INMC(nsh). For all k, every order invariant graph on Q^k has an inductively maximal clique containing its nonnegative shift.

INDUCTIVELY MAXIMAL CLIQUES (Nush). INMC(Nush). For all k, every order invariant graph on Q^k has an inductively maximal clique containing its N upper shift.

INDUCTIVELY MAXIMAL CLIQUES (Nteq,Nush). INMC(Nteq,Nush). For all k, every order invariant graph on Q^k has an N tail invariant inductively maximal clique containing its N upper shift.

THEOREM 3.4.3. INMC(Nteq), INMC(nsh), INMC(Nush), INMC(Nteq,Nush) are provable in SRP^+ but not provable in any consistent fragment of SRP that proves RCA_0. Furthermore, they are provably equivalent to Con(SRP) over WKL_0. This holds if we add the requirement that the inductively maximal clique be recursive in 0', resulting in explicitly arithmetic sentences. INMC(Nteq), INMC(nsh), INMC(Nush), INMC(Nteq,Nush) can be put into Π^0_1 form via the Gödel completeness theorem.

4. N TAIL EQUIVALENCE IS MAXIMUM.

We now discuss the clique statements in section 3 with R invariance, where $R \subseteq Q^{2k}$ is N/order invariant. Note that N tail equivalence on Q^k is itself an N/order invariant subset of Q^{2k}.

We first note that requiring $R \subseteq Q^{2k}$ to be order invariant leads to a triviality.

THEOREM 4.1. (RCA_0). Let k,J be given, $|J| > 1$, and $R \subseteq Q^{2k}$ be order invariant. Suppose every order invariant graph on J^k has an R invariant maximal clique $S \subseteq J^k$. Then $(\forall x,y \in Q^k)(R(x,y) \rightarrow x = y)$.

For step maximality and inductive maximality, we have a complete understanding of the statements that result when we use N/order invariant $R \subseteq Q^{2k}$. For maximality in J^k, we need to impose the following natural condition on R: $R \cap N^{2k}$ is order equivalence on N^k.

THEOREM 4.2. (RCA_0). Let $k,n \geq 1$, $R \subseteq Q^{2k}$ be N/order invariant, and $R \cap N^{2k}$ be order equivalence on N^k. Then i ∨ ii → iii.
i. Every order invariant graph on Q^k has an R invariant maximal clique.
ii. Every order invariant graph on $Q_{\leq n}{}^k$ has an R invariant maximal clique.
iii. R is contained in N tail equivalence on Q^k.

THEOREM 4.3. (RCA_0). Let $k \geq 1$ and $R \subseteq Q^{2k}$ be N/order invariant. Then i ∨ ii → iii.

i. Every order invariant graph on Q^k has an R invariant step maximal clique.
ii. Every order invariant graph on Q^k has an R invariant inductively maximal clique.
iii. R is contained in N tail equivalence on Q^k.

NTEQ MAXIMUM. Let $k,n \geq 1$. N tail equivalence on Q^k is the maximum $R \subseteq Q^{2k}$ such that
i. R is N/order invariant.
ii. $R \cap N^{2k}$ is order equivalence on N^k.
iii. Every order invariant graph on $Q_{\leq n}{}^k$ has an R invariant maximal clique.

NTEQ STEP MAXIMUM. Let $k \geq 1$. N tail equivalence on Q^k is the maximum $R \subseteq Q^{2k}$ such that
i. R is N/order invariant.
ii. Every order invariant graph on Q^k has an R invariant step maximal clique.

NTEQ INDUCTIVE MAXIMUM. Let $k \geq 1$. N tail equivalence on Q^k is the maximum $R \subseteq Q^{2k}$ such that
i. R is N/order invariant.
ii. Every order invariant graph on Q^k has an R invariant inductively maximal clique.

THEOREM 4.4. Nteq Maximum, Nteq Step Maximum, and Nteq Inductive Maximum are provable in SRP^+ but not provable in any consistent fragment of SRP that proves RCA_0. Furthermore, they are provably equivalent to Con(SRP) over WKL_0. These claims remain unchanged if we require that the maximal cliques be recursive in 0', resulting in explicitly arithmetic sentences. Nteq Maximum, Nteq Step Maximum, and Nteq Inductive Maximum can be put in Π^0_1 form via the Gödel completeness theorem.

5. FINITE SEQUENTIAL CLIQUES.

We present an explicitly Π^0_1 form of INMC(nsh). Recall the nonnegative shift, nsh, from Definition 2.4.2. The inductively maximal statements have the advantage of a simpler invariance condition involving nsh - merely containing its nonnegative shift. It also has the disadvantage of requiring use of the # operator from Definition 2.1. Remarkably, the finite form retains this advantage but does not need the # operator.

DEFINITION 5.1. Let $x,y \in Q^k$. $x \leq y$ if and only if $\max(x) < \max(y) \vee x = y$.

DEFINITION 5.2. Let $x_1,...,x_r \in Q^k$. $nsh(x_1,...,x_r\}$ is the set of nonnegative shifts of $x_1,...,x_r$.

PROPOSITION 5.1. Let G be an order invariant graph on Q^k, and let $x_1,x_2,... \in Q^k$. There exists $y_1,y_2,... \in Q^k$ such that for all $i \geq 1$, $y_i \leq x_i$ is not adjacent to x_i and adjacent to each $nsh(y_j)$, $j < i$, other than x_i.

THEOREM 5.2. RCA0 proves that INMC(nsh) and Proposition 5.1 are equivalent.

We now truncate Proposition 5.1 in the obvious way.

PROPOSITION 5.3. Let G be an order invariant graph on Q^k, and $x_1,..,x_r \in Q^k$. There exists $y_1,...,y_r$ such that for all $1 \leq i \leq r$, $y_i \leq x_i$ is not adjacent to x_i, but adjacent to each $nsh(y_j) \neq y_i$, $j < i$.

Note that Proposition 5.3 is explicitly $\Pi^0{}_2$. It can be converted to $\Pi^0{}_1$ form by well known decision procedures, or quantitative estimation. In any case, unfortunately, the only proof that we have of Proposition 5.3 merely quotes Proposition 5.1, and so the proof uses large cardinal hypotheses. We do not know if Proposition 5.3 can be proved in ZFC.

We now sharpen Proposition 5.3 by what we call "control". We simply put an upper bound on the norm of y_i in terms of the norm of $\{x_1,...,x_i\}$.

For this purpose, a convenient norm $\#(x_1,...,x_r)$, $x_1,...,x_r \in Q^k$, is defined by putting the x's in reduced form, and adding the magnitudes of the resulting 2kr integers.

PROPOSITION 5.4. Let G be an order invariant graph on Q^k, and $x_1,...,x_r \in Q^k$. There exists $y_1,...,y_r$ such that for all $1 \leq i \leq r$, $y_i \leq x_i$ is not adjacent to x_i, but adjacent to each $nsh(y_j) \neq y_i$, $j < i$, where $\#(y_i) \leq \#(x_1,...,x_i)^4$.

THEOREM 5.5. Proposition 5.4 is provably equivalent to Con(SRP) over EFA.

Proposition 5.4 is explicitly $\Sigma^0{}_2$. However, c can be specified to be a small universal integer, and Proposition 5.4 becomes explicitly $\Pi^0{}_1$. Probably exponent 2 works in Proposition 5.4, as well as much sharper estimates.

6. INDUCTIVELY MAXIMAL CLIQUES IN $Q^{\leq k}$.

DEFINITION 6.1. $Q^{\leq k}$ is the set of tuples from Q of nonzero length at most k. Let $S,S' \subseteq Q^{\leq k}$. The 1-sections of $S \subseteq Q^{\leq k}$ are the subsets of Q obtained from S by fixing any k-1 coordinates to be specific rational numbers. S 1-contains S' if and only if S contains S', and every 1-section of S' with finite sup is a 1-section of S.

DEFINITION 6.3. Let G be a graph on $Q^{\leq k}$. An inductively maximal Q^k clique of G is an $S \subseteq Q^{\leq k}$ such that $S \cap Q^k = \{x \in S\#: x$ is adjacent in G to every $y < x$ from $S\}$.

INDUCTIVELY MAXIMAL CLIQUES ($Q^{\leq k}$,1⊇,nsh). INMC($Q^{\leq k}$,1⊇,nsh). For all $k \geq 3$, every order invariant graph on $Q^{\leq k}$ has an inductively maximal Q^k clique that 1-contains its nonnegative shift.

INDUCTIVELY MAXIMAL CLIQUES ($Q^{\leq k}$,1⊇,Nush). INMC($Q^{\leq k}$,1⊇,Nush). For all $k \geq 3$, every order invariant graph on $Q^{\leq k}$ has an inductively maximal Q^k clique that 1-contains its N upper shift.

THEOREM 6.1. INMC($Q^{\leq k}$,1⊇,nsh), INMC($Q^{\leq k}$,1⊇,Nush) are provable in $HUGE^+$, but not in any consistent fragment of HUGE that proves RCA_0. Furthermore, they are provably equivalent to Con(HUGE) over WKL_0. These claims hold if we add the requirement that the maximal clique be recursive in 0', resulting in an explicitly arithmetic sentence. INMC($Q^{\leq k}$,1⊇,nsh), INMC($Q^{\leq k}$,1⊇,Nush) can be put into Π^0_1 form via the Gödel completeness theorem.

7. ORDER THEORETIC INVARIANCE PROGRAM.

DEFINITION 7.1. $R \subseteq Q^m$ is order theoretic if and only if R is definable without quantifiers over the structure (Q,<), with constants allowed in the quantifier free definition. $Q[0,1] = Q \cap [0,1]$.

TEMPLATE. Let $R \subseteq Q[0,1]^{2k}$ be order theoretic. Every order invariant graph on $Q[0,1]^k$ has an R invariant maximal clique.

The Order Theoretic Invariance Program seeks to determine for which R the above Template holds.

CONJECTURE 7.1. Every instance of the above Template is provable or refutable in SRP.

We know that there are particular instances of the Template that are provable in SRP but not in ZFC. This is the case for IMC (invariant maximal cliques) for a specific choice of small k,r. Instead of using the order theoretic relation Nteq on $Q{\leq}n^k$, we can use a corresponding order theoretic relation on $Q[0,1]^k$ where the parameters 1/n,1/(n-1),…,1 in Q[0,1] take the place of the parameters 1,2,…,n in Q≤n. In fact, in this way we see that

THEOREM 7.2. Conjecture 7.1 does not hold for any consistent SRP[k].

In particular, ZFC is far too weak to carry out the Order Theoretic Invariance Program. The results presented here carry out the program in a limited way.

8. COMPUTER INVESTIGATIONS.

In section 8.1, we present a general infinite length construction associated with a slight modification of the INMC(Nteq,Nush) from section 3.4, from Q^k to $Q_{\leq n}{}^k$.

DEFINITION 8.1. Let $S \subseteq Q_{\leq n}{}^k$. Nteq*[S] = Nteq[S] $\cap$ $Q_{\leq n}{}^k$. Nush*[S] is the least S' = Nush[S'] $\cap$ $Q_{\leq n}{}^k$ containing S. N*[S] is 0,...,n, together with all p+i ≤ n, such that p is a coordinate of an element of S, and i ∈ N.

INDUCTIVELY MAXIMAL CLIQUES (∀Q≤n,Nteq*,Nush*).
INMC(∀Q≤n,Nteq*,Nush*). For all k, every order invariant graph on $Q_{\leq n}{}^k$ has an inductively maximal clique $S \subseteq Q_{\leq n}{}^k$ containing Nteq*[S] ∪ Nush*[S].

THEOREM 8.1. INMC(∀Q≤n,Nteq*,Nush*) is provable in SRP^+ but not provable in any consistent fragment of SRP that proves RCA_0. Furthermore, it is provably equivalent to Con(SRP) over WKL_0.

The general infinite length construction is an obvious nondeterministic construction that builds the inductively maximal clique with the required properties.

In section 8.2, we discuss the exhaustive search for finite initial segments of the infinite construction. This exhaustive search can rapidly require far too much computer resources. These practical considerations are addressed in section 8.2.

8.1. INFINITE AND FINITE LENGTH CONSTRUCTIONS.

We fix positive integer parameters k,n ≥ 1. We also fix an order invariant graph G on $Q{\le}n^k$. We nondeterministically build an infinite sequence of finite sets $S_1 \subseteq S_2 \subseteq \ldots \subseteq Q{\le}n^k$.

Recall the definition of ≤ (Definition 2.1).

DEFINITION 8.1.1. A G resolution of $x \in Q{\le}n^k$ is a y ≤ x such that x,y are not adjacent in G.

We begin by choosing a G resolution of each $x \in \{0,\ldots,n\}^k$, and forming the set S_1 of these G resolutions. Suppose S_i has been constructed, i ≥ 1. Choose a G resolution of each $x \in N^*[S_i]^k$, and form the set S_{i+1} consisting of these G resolutions together with S_i.

This construction is considered successful if and only if for $S = \cup_i S_i \subseteq Q{\le}n^k$, Nteq*[S] ∪ Nush*[S] is a clique in G.

If we are only carrying out the construction for r steps, then success is for Nteq*[S] ∪ Nush*[S] to be a clique in G.

THEOREM 8.1.1. The assertion "for all k,n and order invariant graphs G on $Q_{\le n}{}^k$, this construction can be successfully carried out for infinitely many steps" is provable in SRP^+ but not provable in any consistent fragment of SRP that proves RCA_0. Furthermore, it is provably equivalent to Con(SRP) over WKL_0.

THEOREM 8.1.2. The assertion "for all k,n,r and order invariant graphs G on $Q{\le}n^k$, this construction can be successfully carried out for r steps" is provable in SRP^+ but not provable in any consistent fragment of SRP that proves EFA. Furthermore, it is provably equivalent to Con(SRP) over EFA.

Note that the quoted statement in Theorem 8.1.2 is explicitly $\Pi^0{}_2$, and can easily be put into explicitly $\Pi^0{}_1$ form by the remarks made in section 5.

8.2. COMPUTATIONAL ASPECTS.

We know that the construction in section 8.1 can be successfully carried out for any number of steps - even infinitely many steps - using Con(SRP). However, it is not at all clear how you actually carry out the construction, even for modest k,n,r, in the sense of actually obtaining a computer file - i.e., a certificate residing on your computer.

We will discuss the exhaustive search for such certificates, which will be implementable for certain choices of parameters, based on algorithmic efficiencies and computer resources. There seems to be plenty of opportunity here for clever algorithmic design.

If the search comes up empty, then we know that SRP is inconsistent. In fact, if the search comes up empty, we can obtain an actual inconsistency in SRP that corresponds to the trace of the negative computer search.

Thus, if the computer search turns up a certificate, we have arguably confirmed the consistency of at least ZFC and even SRP[k] for small k.

Of course, the "strength" of the confirmation here corresponds to the extent to which the high powered set theory is actually being "engaged". At this point, we do not know how to confidently judge this, but one test is whether the exhaustive searches show that certificates are extremely rare. And of course, we know that large cardinals are, in fact, fully engaged as the parameters approach infinity, because of Theorem 8.1.2.

There are a number of computational issues in order to bring the exhaustive search for certificates down to practice.

1. The choice of positive integer parameters $k,n,r,s,t_1,...,t_r$.
2. The choice of order invariant graph G on $Q{\leq}n^k$.
3. Managing the rationals (easily handled).
4. Resolutions of tuples.

The parameters k,n determine the ambient space $Q_{\leq n}{}^k$. The parameter r is the number of "big stages" in the nondeterministic construction. These big stages result in the finite sets $S_1 \subseteq ... \subseteq S_r$, where the goal is for Nteq*[S] ∪ Nush*[S] to be a clique in G.

Inside each big stage i, there are t_i resolutions, as discussed below.

The order invariant graph G on $Q_{\leq n}{}^k$ is given by a randomly generated $C \subseteq \{1,...,2k\}^{2k}$ of cardinality s, subject to the constraint that no $(x,x) \in C$, and $(x,y) \in C \leftrightarrow (y,x) \in C$. Here $x,y \in Q_{\leq n}{}^k$ are considered to be adjacent in G if and only if (x,y) is order equivalent to an element of C.

In our nondeterministic construction of $S_1,...,S_r$, we will arrange that each $S_i = Nteq*[S_i] \cup Nush*[S_i]$ is a clique in G. The set of candidates for resolutions are, respectively, $\{0,...,n\}^k$, $N*[S_1]^k$, $N*[S_2]^k$, ..., $N*[S_{r-1}]^k$. However, we do not want to be resolving all of these tuples, as would cause a computational explosion. We will resolve only $t_1,...,t_r$ tuples, respectively, from these r large sets. But precisely which tuples are to be selected for resolution?

We envision community wide agreement on a selection process **Ω** for picking elements from A, where $A \subseteq Q_{\leq n}{}^k$ of reasonable size. **Ω** would give a listing of A without repetition, or at least a reasonable number of elements of A without repetition. The tuples to be resolved would be chosen in order from this list. Thus **Ω** sets priorities for the choice of tuples to be resolved. **Ω** should be invariant under order isomorphisms of Q that fix 0,...,n.

Ω should give some bias toward use of coordinates from {0,...,n}, and some bias toward use of coordinates among the lesser numbers in $Q_{\leq n} \setminus \{0,...,n\}$. This way, the engagement with Nteq* and Nush* will be intensified. We expect considerable randomness to be built into **Ω** unless a theory develops that suggests otherwise.

More precisely, in forming S_1, we apply **Ω** to $\{0,...,n\}^k$, and successively resolve the first t_1 elements of $\mathbf{\Omega}(\{0,...,n\}^k)$, throwing the resolutions into S_1. When we insert a new tuple into S_1, we immediately close under Nteq* ∪ Nush*, checking that the S_1 thus far remains a clique in G. If we find otherwise, then we have made an error, and need to try a new resolution.

If we find that we are scheduled to resolve a tuple from the list obtained by applying **Ω** which already has a resolution in S_1, (because there is already some $y \leq x$ nonadjacent to x in S_1), then we move to the next tuple on the list supplied by **Ω**. We proceed until we have put t_1 new resolutions into S_1, satisfying the requirement that the closure under Nteq* ∪ Nush* is a clique in G.

After this process has been completed, we have formed a clique S_1 = Nteq*[S_1] ∪ Nush*[S_1] in G. Now the set of candidates for resolution is N*[S_1]^k. We proceed as above, to construct a clique S_2 = Nteq*[S_2] ∪ Nush*[S_2] in G, after inserting t_2 new resolutions of tuples generated by **Ω**, as before. We continue in this way constructing $S_1,...,S_r$.

For the exhaustive search, we need to consider all possible resolutions. Let's go back to the formation of S_1 from $\{0,...,n\}^k$. We need to form the set of all possible S_1's arising from the above process. I.e., all possible S_1's up to order isomorphism that fixes 0,...,n.

At any stage in the formation we manage the rationals that have been used by forming the list of them in strictly increasing order, including 0,...,n. When making a resolution of a tuple, we usually will be using one or more new rationals, which are to be inserted into this list in the following way. Whenever a new nonnegative rational is inserted, we must immediately also insert the results of adding nonnegative integers, as long as the sum stays ≤ n.

It is now clear what we mean by constructing all possible S_1's, up to isomorphism fixing 0,...,n. We have arrived at a list S[1,1],...,S[1,p_1] of all possible S_1's fixing 0,...,n.

We repeat this process, starting independently with each S[1,i]. Each S[1,i] spawns all possible S_2's containing S[1,i], up to isomorphism fixing 0,...,n. We arrive at a list S[2,1],...,S[2,p_2] of all possible S_2's up to isomorphism fixing 0,...,n.

We eventually obtain a complete listing S[r,1],...,S[r,p_r] of all possible S_r's up to isomorphism. Any one of them provides the sought after certificate. However, if p_r = 0 then there is no certificate, and this yields an inconsistency in SRP[k], for small k.

It is a matter for research to see how to best use computational resources under various adjustments of the parameters, and to see how many sets exist at the last stage.

What values of the parameters are reasonable for initial experimentation, where at least ZFC is engaged?

We recommend initially experimenting with k = n = r = t_1 = t_2 = t_3 = t_4 = 4, and s = 100. There may have to be some experimentation with s in order to obtain traction. We have no way of foretelling that.

We might find that for even modest choices of parameters, it becomes computationally intractable to keep checking for the closure under Nteq* ∪ Nush* being a clique in G. There is the possibility of weakening the closure under Nteq* ∪ Nush* according to some predetermined rule, although this may cause the number of S[i,j]'s in the exhaustive search to grow uncontrollably.

Experts in large cardinals have developed a large amount of experience and intuitions that make them comfortable with Con(SRP). How much credence to give this experience and intuition is a deep question that is difficult to address today.

A possible reservation is that these scholars have a good grasp only of humanly intelligible proofs. It could be that the only inconsistencies in large cardinal hypotheses are not humanly intelligible. Certainly any inconsistency found by one of our proposed exhaustive searches coming up negative, will be completely humanly unintelligible. There is the remarkable possibility, not altogether absurd, that the proposed exhaustive searches actually do generate wholly humanly unintelligible inconsistencies. Note that we have arranged for these exhaustive searches to at least plausibly be intensely engaging the large cardinals through their underlying finite combinatorial structure.

There is a real precedent for surprising findings by exhaustive computer searches that turn up completely humanly unintelligible proofs. There is now an approximately 100 terabyte database evaluating all seven piece chess positions via the Lomonosov supercomputer in Moscow, [Lo13]. It yields a treasure trove of theorems that seem completely beyond human intelligibility, and have already forced chess grandmasters to revise many previously held evaluations.

9. PROOFS OF INVARIANT MAXIMALITY.

In this section we prove ISMC(Nteq) in WKL_0 + Con(SRP). We also show that for all k, SRP proves ISMC(Nteq) for dimension k.

In section 9.1 we discuss the large cardinal hypotheses that are relevant to this paper (except those relevant to section 6). We will be using a considerable portion of the notation from section 9.1 in section 9.2, but not many of the results. Section 9.1 does provide a lot of useful orienting information about the relevant region of large cardinal hypotheses.

In section 9.2, we prove ISMC(Nteq) in SRP^+ and WKL_0 + Con(SRP). We also show that for all k, SRP proves ISMC(Nteq) for dimension k. We focus on ISMC(Nteq) in light of Theorem 3.3.4.

9.1. THE STATIONARY RAMSEY PROPERTY.

All results in this section are taken from [Fr01]. All of these results, with the exception of Theorem 9.1.1, iv ↔ v → vi, are credited in [Fr01] to James Baumgartner. Below, λ always denotes a limit ordinal.

DEFINITION 9.1.1. We say that $C \subseteq \lambda$ is unbounded if and only if for all $\alpha < \lambda$ there exists $\beta \in C$ such that $\beta \geq \alpha$.

DEFINITION 9.1.2. We say that $C \subseteq \lambda$ is closed if and only if for all limit ordinals $x < \lambda$, if the sup of the elements of C below x is x, then $x \in C$.

DEFINITION 9.1.3. We say that $A \subseteq \lambda$ is stationary if and only if it intersects every closed unbounded subset of λ.

DEFINITION 9.1.4. For sets A, let S(A) be the set of all subsets of A. For integers $k \geq 1$, let $S_k(A)$ be the set of all k element subsets of A.

DEFINITION 9.1.5. Let $k \geq 1$. We say that λ has the k-SRP if and only if for every $f:S_k(\lambda) \rightarrow \{0,1\}$, there exists a stationary $E \subseteq \lambda$ such that f is constant on $S_k(E)$. Here SRP stands for "stationary Ramsey property."

The k-SRP is a particularly simple large cardinal property. To put it in perspective, the existence of an ordinal with the 2-SRP is stronger than the existence of higher order indescribable cardinals, which is stronger than the existence of weakly compact cardinals, which is stronger than the existence of cardinals which are, for all k, strongly k-Mahlo (see Theorem 9.1.1 below, and [Fr01], Lemmas 1.11).

Our main results are stated in terms of the stationary Ramsey property. In particular, we use the following extensions of

ZFC based on the SRP.

DEFINITION 9.1.6. SRP^+ = ZFC + "for all k there exists an ordinal with the k-SRP". SRP = ZFC + $\{$there exists an ordinal with the k-SRP$\}_k$. We also use k-SRP for the formal system ZFC + $(\exists\lambda)(\lambda$ has the k-SRP$)$.

For technical reasons, we will need to consider some large cardinal properties that rely on regressive functions.

DEFINITION 9.1.7. We say that $f:S_k(\lambda) \to \lambda$ is regressive if and only if for all $A \in S_k(\lambda)$, if $\min(A) > 0$ then $f(A) < \min(A)$. We say that E is f-homogenous if and only if $E \subseteq \lambda$ and for all $B,C \in S_k(E)$, $f(B) = f(C)$.

DEFINITION 9.1.8. We say that $f:S_k(\lambda) \to S(\lambda)$ is regressive if and only if for all $A \in S_k(\lambda)$, $f(A) \subseteq \min(A)$. (We take $\min(\emptyset) = 0$, and so $f(\emptyset) = \emptyset$). We say that E is f-homogenous if and only if $E \subseteq \lambda$ and for all $B,C \in S_k(E)$, we have $f(B) \cap \min(B \cup C) = f(C) \cap \min(B \cup C)$.

DEFINITION 9.1.9. Let $k \geq 1$. We say that α is purely k-subtle if and only if

i) α is an ordinal;
ii) For all regressive $f:S_k(\alpha) \to \alpha$, there exists $A \in S_{k+1}(\alpha\backslash\{0,1\})$ such that f is constant on $S_k(A)$.

DEFINITION 9.1.10. We say that λ is k-subtle if and only if for all closed unbounded $C \subseteq \lambda$ and regressive $f:S_k(\lambda) \to S(\lambda)$, there exists an f-homogenous $A \in S_{k+1}(C)$.

DEFINITION 9.1.11. We say that λ is k-almost ineffable if and only if for all regressive $f:S_k(\lambda) \to S(\lambda)$, there exists an f-homogenous $A \subseteq \lambda$ of cardinality λ.

DEFINITION 9.1.12. We say that λ is k-ineffable if and only if for all regressive $f:S_k(\lambda) \to S(\lambda)$, there exists an f-homogenous stationary $A \subseteq \lambda$.

THEOREM 9.1.1. Let $k \geq 2$. Each of the following implies the next, over ZFC.
i. there exists an ordinal with the k-SRP.
ii. there exists a (k-1)-ineffable ordinal.
iii. there exists a (k-1)-almost ineffable ordinal.
iv. there exists a (k-1)-subtle ordinal.
v. there exists a purely k-subtle ordinal.

vi. there exists an ordinal with the (k-1)-SRP.
Furthermore, i,ii are equivalent, and iv,v are equivalent. There are no other equivalences. ZFC proves that the least ordinal with properties i - vi (whichever exist) form a decreasing (≥) sequence of uncountable cardinals, with equality between i,ii, equality between iv,v, and strict inequality for the remaining consecutive pairs.

Proof: i ↔ ii is from [Fr01], Theorem 1.28, iv ↔ v is from [Fr01], Corollary 2.17. The strict implications ii → iii → iv → vi are from [Fr01], Theorem 1.28. Same references apply for comparing the least ordinals. QED

DEFINITION 9.1.13. We follow the convention that for integers p ≤ 0, a p-subtle, p-almost ineffable, p-ineffable ordinal is a limit ordinal, and that the ordinals that are 0-subtle, 0-almost ineffable, 0-ineffaable, or have the 0-SRP, are exactly the limit ordinals. An ordinal is called subtle, almost ineffable, ineffable, if and only if it is 1-subtle, 1-almost ineffable, 1-ineffable.

DEFINITION 9.1.14. SRP^+ is the formal system ZFC + (∀k)(there exists an ordinal with the k-SRP). SRP is the formal system ZFC + $\{$there exists an ordinal with the k-SRP$\}_k$. For each k, we write SRP[k] for the formal system ZFC + "there exists an ordinal with the k-SRP".

9.2. INVARIANT STEP MAXIMAL ROOTS (Nteq).

In this section, we prove ISMR(Nteq) in SRP^+ and WKL_0 + Con(SRP).

DEFINITION 9.2.1. ISMR(Nteq,k) is ISMR(Nteq) for order invariant $W \subseteq Q^{2k}$ only.

Obviously ISMR(Nteq) and ∀k(ISMR(Nteq,k)) are provably equivalent over RCA_0. We show that for all k, ISMR(Nteq,k) is provable in SRP.

LEMMA 9.2.1. RCA_0 proves ISMR(Nteq,1).

Proof: Let $W \subseteq Q^2$ be order invariant. There are three principal cases.

case 1. W has no (p,p). Use S = ∅.

case 2. W has all (p,p) but not all elements of Q^2. Use S = {-1}.

case 3. W = Q^2. Use S = Q.

QED

We now fix n ≥ k ≥ 2 and λ to be the least (k-1)-subtle ordinal. We derive ISMR(Nteq,k), using λ, within ZFC. We make no claims for the optimality of λ here.

Note that by Theorem 9.1.1, λ is strictly between the least ordinal with the (k-1)-SRP and the least ordinal with the k-SRP (assuming the latter exists). In any case, λ is available to us in SRP.

We first make some definitions in general linear orderings.

For any set U, we think of U^r as the set of functions from {1,...,r} into U. We extend this as follows.

DEFINITION 9.2.2. U^{r-} is the set of all nonempty partial functions from {1,...,r} into U.

DEFINITION 9.2.3. Let (A,<*) be a linear ordering, w ∈ A^{r-}, X ⊆ A^{r-}, S* ⊆ A^k, W* ⊆ A^{2k}, E ⊆ A, x ∈ A. max(w) is the largest value of w under <*. X[<*x] = {y ∈ X: max(y) <* x}. X[≤*x] = {y ∈ X: max(y) ≤* x}. S* is a root of W* if and only if S^{*2} ⊆ W. S* is a maximal root of W* ⊆ A^{2k} if and only if S* is a root of W* and every root S*' ⊇ S* of W* is S*. The E tail of z ∈ A^k consists of the z_i such that every z_j ≥* z_i lies in E. z,w ∈ A^k are E tail equivalent if and only if z,w order equivalent (under <*) and identical off of their E tails. S* is E tail invariant if and only if for all E tail equivalent z,w ∈ S*, z ∈ X ↔ w ∈ X.

DEFINITION 9.2.4. Q[0,1) is the rational interval [0,1). (λ × Q[0,1),$<_\lambda$) is the linear ordering where (α,p) $<_\lambda$ (β,q) ↔ α < β ∨ (α = β ∧ p < q). x $\leq_\lambda$ y ↔ x $<_\lambda$ y ∨ x = y.

It is obvious that (λ × Q[0,1),$<_\lambda$) is a dense linear ordering with the left endpoint (0,0). Since we are going to transfer from the transfinite to Q, and Q has no left endpoint, it is convenient to use a slightly different linear ordering.

DEFINITION 9.2.5. (λ ×' Q[0,1),$<_\lambda$') is the same linear ordering as (λ × Q[0,1),$<_\lambda$), but with the left endpoint

(0,0) removed. $x \leq_\lambda' y \leftrightarrow x <_\lambda' y \vee x = y$. For $x \in (\lambda \times' Q[0,1))^{r-}$, $|x|$ is the largest of the first coordinates of values of x. We also use $\alpha \times' Q[0,1)$ for $(\alpha \times Q[0,1))\backslash\{(0,0)\}$.

DEFINITION 9.2.6. Let $f:S_{k-1}(\lambda) \rightarrow S((\lambda \times' Q[0,1))^{k-})$. f is regressive if and only if for all $x \in S_{k-1}(\lambda)$, $y \in f(x)$, we have $|y| < \min(x)$. $E \subseteq \lambda$ is f-homogenous if for all $x,y \in S_{k-1}(E)$ and $z \in S((\lambda \times' Q[0,1))^{k-})$ with $|z| < \min(x \cup y)$, we have $z \in f(x) \leftrightarrow z \in f(y)$.

LEMMA 9.2.2. λ is a strongly inaccessible cardinal. Let $C \subseteq \lambda$ be closed and unbounded, and let $f_1,\ldots,f_m:S_{k-1}(\lambda) \rightarrow S((\lambda \times' Q[0,1))^{k-})$ be regressive. There exists $E \subseteq C$ of order type ω, where for all i, E is f_i-homogenous.

Proof: λ is a strongly inaccessible cardinal by [Fr01], Lemma 1.10. Let $C,f_1,\ldots,f_m$ be as given. Set C to be the set of uncountable cardinals $< \lambda$, which is closed and unbounded in λ by the first claim. Define $f:S_{k-1}(\lambda) \rightarrow S(\lambda)$ by $f(x) = \langle f_1(x),\ldots,f_m(x)\rangle$. Here we can use any convenient one-one map from $S((\lambda \times' Q[0,1))^{k-})^m$ into $S(\lambda)$ which, for each uncountable cardinal $\kappa < \lambda$, maps $S((\kappa \times' Q[0,1))^{k-})^m$ into $S(\kappa)$. Now f may not be regressive, but clearly f is regressive on $S_{k-1}(C)$. So we take the values of f to be the empty set, off of $S_{k-1}(C)$. According to [Fr01], Lemma 1.6, there exists f-homogenous $E \subseteq C$ of order type ω. Then for all i, E is f_i-homogenous. QED

LEMMA 9.2.3. Let $X \subseteq (\lambda \times' Q[0,1))^k$. There exists $E \subseteq \lambda$ of order type ω such that X is $E \times \{0\}$ tail invariant.

Proof: Let X be as given. We apply Lemma 9.2.2. We define finitely many functions $f:S_{k-1}(\lambda) \rightarrow S((\lambda \times' Q[0,1))^{k-})$ as follows. The $w \in (\lambda \times' \{0\})^{k-}$ with domain not all of $\{1,\ldots,k\}$ (see Definition 9.2.3), fall naturally into finitely many kinds. The kind of w is determined first by its domain (a nonempty subset of $\{1,\ldots,k\}$), and second by the order type of the ordinal components (first terms) of its values, listed from left to right. Write these kinds as $\sigma_1,\ldots,\sigma_m$, without repetition.

We define $f_1,\ldots,f_m:S_{k-1}(\lambda) \rightarrow S((\lambda \times' Q[0,1))^{k-})$ as follows. Let $x \in S_{k-1}(\lambda)$. To evaluate $f_i(x)$, let $y \in \lambda^{k-}$ be unique with kind σ_i, where the values of y form an initial segment of the elements of x. Set $f_i(x) = \{z \in (\min(x) \times' Q[0,1))^{k-}: \mathrm{dom}(z) = \{1,\ldots,k\}\backslash\mathrm{dom}(y) \wedge y \times \{0\} \cup z \in X\}$.

By Lemma 9.2.2, let $E \subseteq \lambda$ be of order type ω and f_i-homogeneous for all $1 \leq i \leq m$. We can assume that $0 \notin E$. Let $u,v \in (\lambda \times' Q[0,1))^k$ be $E \times \{0\}$ tail equivalent, where $u,v \notin (E \times \{0\})^k$. We claim that $u \in X \leftrightarrow v \in X$.

Let $u' \in \lambda^{k-}$ be the restriction of u to its coordinates in $E \times \{0\}$, where all coordinates higher in $<_\lambda'$, are in $E \times \{0\}$. Let $v' \in \lambda^{k-}$ be the restriction of v to its coordinates in $E \times \{0\}$, where all coordinates higher in $<_\lambda'$, are in $E \times \{0\}$. Then u',v' result in the evaluation of some $f_i(x),f_i(y)$, respectively, where $x,y \in S_{k-1}(E)$. Hence $u \in X \leftrightarrow v \in X$.

It remains to show that for order equivalent $u,v \in (E \times \{0\})^k$, $u \in X \leftrightarrow v \in X$. However, this may not be the case. But we can use the ordinary infinite Ramsey theorem to replace E by a suitable subset of E of order type ω for which this is the case. QED

LEMMA 9.2.4. Let $W^* \subseteq (\lambda \times' Q[0,1))^{2k}$. There exists $S^* \subseteq (\lambda \times' Q[0,1))^k$ such that for all $\alpha < \lambda$, $S^*[\leq_\lambda'(\alpha,0)]$ is a maximal root of $W^*[\leq_\lambda'(\alpha,0)]$.

Proof: Let W^* be as given. We build S^* by transfinite recursion along λ. Let S_0^* be the maximal root of $W^*[\leq_\lambda'(0,0)]$, which is $\{(0,0),\ldots,(0,0)\}$ or $\emptyset$. Suppose S_α^* is a maximal root of $W^*[\leq_\lambda'(\alpha,0)]$. Take $S_{\alpha+1}^*$ to be a maximal root of $W^*[\leq_\lambda'(\alpha+1,0)]$ extending S_α^*. Now suppose that for all $\alpha < \gamma < \lambda$, S_α^* is a maximal root of $W^*[\leq_\lambda'(\alpha,0)]$, where γ is a limit ordinal, and we have for all $\beta < \delta < \gamma$, $S_\alpha^* \subseteq S_\beta^*$. Let $S_\gamma^{**} = \cup_{\alpha<\lambda} S_\alpha^*$. Then S_γ^{**} is a maximal root of $W^*[<_\lambda'(\gamma,0)]$. Let S_γ^* be a maximal root of $W^*[\leq_\lambda'(\gamma,0)]$ extending S_γ^{**}. Finally, define $S = \cup_\alpha S_\alpha$. QED

LEMMA 9.2.5. Let $W^* \subseteq (\lambda \times' Q[0,1))^{2k}$. There exists $E \subseteq \lambda$ of order type ω and $S^* \subseteq (\sup(E) \times' Q[0,1))^k$, where for all $\alpha \in E$, $S^*[\leq_\lambda'(\alpha,0)]$ is a maximal root of $W^*[\leq_\lambda'(\alpha,0)]$, and where S^* is $E \times \{0\}$ tail invariant.

Proof: Let W^* be as given. Let S^* be as given by Lemma 9.2.4. By Lemma 9.2.3, let $E \subseteq \lambda$ be of order type ω, where S^* is $E \times \{0\}$ tail invariant. QED

LEMMA 9.2.6. Let $(A,<^*)$ be a dense linear ordering with no endpoints, and $x_1,x_2,\ldots$ be an infinite strictly increasing sequence from A with no upper bound. Let $W' \subseteq Q^{2k}$ and $S' \subseteq A^k$, where for all n, $S'[\leq^* x_n]$ is a maximal root of $W'[\leq^* x_n]$, and S' is $\{x_1,x_2,\ldots\}$ tail invariant. There exists $B \subseteq A$, where $(B,<^* \cap B^2)$ is a countable dense linear ordering with no endpoints, $x_1,x_2,... \in B$, for all n, $(S' \cap B^k)[(\leq^* \cap B^2)x_n]$ is a maximal root of $(W' \cap B^k)[(\leq^* \cap B_2)x_n]$, and where $S' \cap B^2$ is $\{x_1,x_2,...\}$ tail invariant.

Proof: This can be proved either from the basic model theoretic theorem that every structure has a countable elementary substructure (in a finite language), or directly by an easy sequential construction. The former is preferable because the model theoretic setup facilitates the proof of Lemma 9.2.7. We work with the structure $(A,<^*,W',S',x_1,x_2,...)$. The hypothesized properties are all first order. Now simply take any countable elementary substructure $(B,<^* \cap B^2,W' \cap B^{2k},S' \cap B_k,x_1,x_2,...)$. The first order properties still hold, and they have the proper meaning for the desired conclusion. QED

LEMMA 9.2.7. ISMR(Nteq,k).

Proof: Let $W \subseteq Q^{2k}$ be order invariant. Let $W^* = \{x \in (\lambda \times' Q[0,1))^{2k}: (\exists y \in W)(x,y \text{ are order equivalent})\}$. Let $S^* \subseteq (\lambda \times' Q[0,1))^k$ and $E = \{x_1 < x_2 < ...\}$ be given by Lemma 9.2.5. Let $\sup(E) = (\gamma,0)$, where γ is a limit ordinal. Let $A,<^*,W',S'$ be the restrictions of $\lambda \times' Q\{0,1),<_\lambda',W^*,S^*$ to $\gamma \times' Q[0,1)$. Then the hypotheses of Lemma 9.2.6 hold. Note that $W' = \{x \in (\gamma \times' Q[0,1))^{2k}: (\exists y \in W)(x,y \text{ are order equivalent})\}$.

Now apply Lemma 9.2.6. We obtain $(B,<^{**},W'',S'',x_1,x_2,...)$, where $(B,<^{**})$ is a countable dense linear ordering without endpoints, $x_1 <^{**} x_2 <^{**} ...$ has no upper bound in $<^{**}$, each $S''[<^{**}x_i]$ is a maximal root of $W''[<^{**}x_i]$, and S'' is $\{x_1,x_2,...\}$ tail invariant. Once again, $W'' = \{x \in B^{2k}: (\exists y \in W)(x,y \text{ are order equivalent})\}$.

Let h be any isomorphism from $(B,<^{**})$ onto $(Q,<)$ mapping $x_1,x_2,...$ onto $0,1,...$. Then h is an isomorphism from $(B,<^{**},W'',S'',x_1,x_2,...)$ onto $(Q,<,h[S''],h[W''],0,1,...)$. It is clear that $h[W''] = W$, because the definition of W'' as the set of all tuples with certain order types must be preserved under h. It is immediate that $S = h[S'']$ is an N tail invariant step maximal root of W. QED

THEOREM 9.2.8. (ZFC). For all $k \geq 1$, if there exists a (k-1)-subtle ordinal then ISMR(Nteq,k).

Proof: Note that the argument from Lemmas 9.2.2 - 9.2.7 used only that $k \geq 2$, and λ is the least (k-1)-subtle ordinal. Lemma 9.2.1 takes care of the trivial case k = 1. QED

THEOREM 9.2.9. RCA_0 proves that for all k, SRP proves ISMR(Nteq,k). SRP^+ proves ISMR(Nteq).

Proof: The first claim is immediate from the proof of Theorem 9.2.7. It is also clear that the entire argument can be conducted using k as a variable. Therefore SRP^+ proves ISMR(Nteq). QED

It remains to prove ISMR(Nteq) in WKL_0 + Con(SRP). We work in WKL_0 + Con(SRP).

LEMMA 9.2.10. There is a countable M with satisfaction relation which satisfies SRP.

Proof: By the formalized Gödel completeness theorem, which is provable in WKL_0. QED

We now fix M as given by Lemma 9.2.10.

LEMMA 9.2.11. Let k and order invariant $W \subseteq Q^{2k}$ be given. Let K be the finite set of order types of elements of W. There exists (Q*,<*,W*,N*,S*) internal to M, where, according to M,
i. (Q*,<*) is the rationals with its usual linear ordering.
ii. $W^* \subseteq Q^{*k}$ consists of all $x \in Q^{*k}$ with order type in K.
iii. N* is the set of all nonnegative integers.
iv. S* is a step maximal root of W*.
v. S* is N* tail invariant.

Proof: Here k,W,K are standard. There may be nonstandard rationals in Q* and nonstandard integers in N*. The steps in iv are given by elements of N*. The tail invariance uses N* and not N. We simply carry out the proof of Theorem 9.2.7 in the model M. QED

THEOREM 9.2.12. WKL_0 + Con(SRP) proves ISMR(Nteq).

Proof: We work in WKL_0 + Con(SRP), with M provided by Lemma 9.2.9. Let $W \subseteq Q^{2k}$ be order invariant. Let B,W*,<*,W*,N*,S* be as given by Lemma 9.2.11. Obviously (Q*,<*) is a countable dense linear ordering without endpoints, let $x_1 <^* x_2 <^* \ldots$ be elements of N* that have no upper bound in (Q*,<*). Let h be any isomorphism from $(Q^*,<^*,W^*,S^*,x_1,x_2,\ldots)$ onto $(Q,<,W,S,0,1,\ldots)$, $S \subseteq Q^{2k}$. Note that h[W*] = W, and we have set h[S*] = S. Then S is an N tail equivalent step maximal root of W. QED

COROLLARY 9.2.13. SRP^+ and WKL_0 + Con(SRP) prove all eighteen titled statements in sections 3.1 - 3.3, excluding IMR/C(intervals).

Proof: By Theorems 9.2.9, 9.2.12, and 3.3.4. QED

In [Fr?], we derive Con(SRP) from RCA_0 + ISMR(Nteq). This together with Theorem 9.2.12 establishes that ISMR(Nteq) is provably equivalent to Con(SRP) over WKL_0. In [Fr?}, we also show that IMR/C(intervals) is provably equivalent to Con(SRP) over WKL_0.

10. APPENDIX - FORMAL SYSTEMS USED

EFA Exponential function arithemtic. Based on exponentiation and bounded induction. Same as $I\Sigma_0$(exp), [HP93], p. 37, 405.
RCA_0 Recursive comprehension axiom naught. Our base theory for Reverse Mathematics. [Si99,09].
WKL_0 Weak Konig's Lemma naught. Our second level theory for Reverse Mathematics. [Si99,09].
ACA_0 Arithmetic comprehension axiom naught. Our third level theory for Reverse Mathematics. [Si99,09].
ZF(C) Zermelo set theory (with the axiom of choice). ZFC is the official theoretical gold standard for mathematical proofs. [Je14].
SRP[k] ZFC + $(\exists\lambda)(\lambda$ has the k-SRP), for fixed k. Section 9.1.
SRP ZFC + $(\exists\lambda)(\lambda$ has the k-SRP), as a scheme in k. Section 9.1.
SRP^+ ZFC + $(\forall k)(\exists\lambda)(\lambda$ has the k-SRP). Section 9.1.
HUGE[k] ZFC + $(\exists\lambda)(\lambda$ is k-HUGE), for fixed k.
HUGE ZFC + $(\exists\lambda)(\lambda$ is k-huge), as a scheme in k.
$HUGE^+$ ZFC + $(\forall k)(\exists\lambda)(\lambda$ is k-huge).

λ is k-huge if and only if there exists an elementary embedding $j:V(\alpha) \to V(\beta)$ with critical point λ such that $\alpha = j^{(k)}(\lambda)$. (This hierarchy differs in inessential ways from the more standard hierarchies in terms of global elementary embeddings). For more about huge cardinals, see [Ka94], p. 331.

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