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EXPLICITLY Π^0_1 STATUS 4/20/18

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ABSTRACT. We begin with the Upper Image Equation $A \cup. R_<[A] = [t]^k$, where $R \subseteq [t]^k \times [t]^k$ is the known and $A \subseteq [t]^k$ is the unknown. This Upper Image Equation (and obvious variants) is easily seen to have a unique solution. Even for very explicit R (order invariant), these unique solutions exhibit computational completeness. Under a weakening of the equation (using "rich over"), we lose uniqueness but gain a structure theory: for all order invariant $R \subseteq [kn]^k \times [kn]^k$, $n >> k$, there exists a nonempty stable $A \subseteq [kn]^k$ such that $A \cup. R_<[A]$ is rich over A. However, this resulting Rich Stability statement is independent of the usual ZFC axioms, and is, in fact, provably equivalent to Con(SRP) over EFA. RS is explicitly Π^0_1 (using $n \geq (8k)!!$).

1. DEFINITIONS

DEFINITION 1.1. N is the set of all nonnegative integers. We use n,m,r,s,t for positive integers unless otherwise indicated. $[t] = \{0,...,t\}$. For $x \in N^k$, max(x) is the largest coordinate of x. Let $R \subseteq [t]^k \times [t]^k$ and $A \subseteq [t]^k$. $R_<[A]$ is the Upper Image of R on A, $\{y: (\exists x \in A)(x R y \wedge \max(x) < \max(y))\}$. We use $\cup.$ for disjoint union. The use of $\cup.$ signals that the left and right sides are disjoint. The Upper Image Equation is the equation $A \cup. R_<[A] = [t]^k$ with known $R \subseteq [t]^k \times [t]^k$ and unknown $A \subseteq [t]^k$.

DEFINITION 1.2. $x,y \in [t]^k$ are order equivalent if and only if for all $1 \leq i,j \leq k$, $x_i < x_j \leftrightarrow y_i < y_j$. $A \subseteq [t]^r$ is order invariant if and only if for all order equivalent $x,y \in [t]^r$, $x \in A \leftrightarrow y \in A$. $R \subseteq [t]^k \times [t]^k$ is order invariant if and only if R is order invariant as a subset of $[t]^{2k}$.

After section 2, we work in $[kn]^k$, $n >> k$, using the distinguished numbers $n,2n,...,kn$. We again work with known order invariant $R \subseteq [kn]^k \times [kn]^k$ and unknown $A \subseteq [kn]^k$.

DEFINITION 1.3. $A \subseteq [kn]^k$ is stable if and only if for all $p \in [n-1]$, $(p,n,2n,...,(k-1)n) \in A \leftrightarrow (p,2n,3n,...,kn) \in A$.

We seek stable solutions to the Upper Image Equation. Unfortunately, this is not possible. Thus instead of requiring that

$A \cup. R_<[A] = [kn]^k$
A stable

we use a weaker condition, and require that

$A \cup. R_<[A]$ is rich over A
A stable

DEFINITION 1.4. EFA is exponential function arithmetic, often written $I\Sigma_0(\exp)$. ZFC is the usual set theoretic axioms for mathematics. SRP = ZFC + $\{(\exists\lambda)(\lambda$ has the k-SRP$)$: $k \geq 1\}$. SRP^+ = ZFC + $(\forall k)(\exists\lambda)(\lambda$ has the k-SRP$)$.

2. UPPER IMAGE EQUATION

UPPER IMAGE EQUATION. UIE. For all $R \subseteq [t]^k \times [t]^k$ there exists $A \subseteq [t]^k$ such that $A \cup. R_<[A] = [t]^k$. A is unique.

UIE is proved in EFA by an obvious inductive argument well worth reconstructing on one's own.

After one has internalized the obvious proof of UIE, one readily senses that UIE is one of those

fundamental trivialities
of finite combinatorial mathematics

We shall see that an in depth investigation of UIE leads us naturally on a path that cannot be kept within the usual ZFC axioms for mathematics.

THEOREM 2.1. A is the solution of some Upper Image Equation with $R \subseteq [t]^k \times [t]^k$ if and only if $(0,...,0) \in A$.

Proof: Assume $(0,...,0) \in A$. Let R be such that

i. For all $x \in A$, there are no R predecessors of x.
ii. For all $x \notin A$, there is exactly one R predecessor of x, namely $(0,...,0)$.

We claim that $A \cup. R_<[A] = [t]^k$. To see this, we have to check that

1) $x \in A$ if and only if x has no R predecessor from A with lower max.

The forward direction is immediate. Suppose $x \notin A$. Then x has an R predecessor from A with lower max, namely $(0,...,0)$. QED

So there cannot be any kind of structure theory for solutions to the Upper Image Equation for general R.

But what about for very explicitly given R? Let's move to this restricted form of UIE.

EXPLICIT UPPER IMAGE EQUATION. EUIE. For all order invariant $R \subseteq [t]^k \times [t]^k$ there exists $A \subseteq [t]^k$ such that $A \cup. R_<[A] = [t]^k$. A is unique.

Now what can we say about the solutions of the Upper Image Equation for order invariant R? Since the number of order invariant R is roughly double exponential in k (and not t), we have definite prospects of a structure theory.

However, we know that these unique solutions have computational completeness properties which we will take up elsewhere. This precludes any structure theory for them. Thus we seek weak forms of the upper image equation.

3. STABILITY AND RICHNESS

We now work in $[kn]^k$, $n >> k$. This signifies the special significance of the numbers

$$n,2n,...,kn$$

We work with $A \subseteq [kn]^k$ and $R \subseteq [kn]^k \times [kn]^k$. Recall the definition of stable $A \subseteq [kn]^k$, which naturally reflects these numbers (Definition 1.3).

We would like to have

PROPOSITION. The solutions of the Upper Image Equation for order invariant $R \subseteq [kn]^k \times [kn]^k$, $n >> k$, are stable.

but this is refutable. Thus we seek the following:

THEMATIC. For all order invariant $R \subseteq [kn]^k \times [kn]^k$, $n >> k$, there exists stable A such that $A \cup. R_<[A]$ is "rich".

The remaining challenge is to find a suitable notion of "rich".

It appears that there is no suitable notion of "rich" - as an isolated subset of $[kn]^k$ - that can be used here.

However, consider this notion of rich *relative* to A:

DEFINITION 3.1. B is rich over $A \subseteq [kn]^k$ if and only if $A^2 \times [kn]^k$ and $A^2 \times B$ are order equivalent over $n,2n,...,kn$.

STABLE RICHNESS. SR. For all order invariant $R \subseteq [kn]^k \times [kn]^k$, $n >> k$, there exists nonempty stable $A \subseteq [kn]^k$ such that $A \cup. R_<[A]$ is rich over A. $n \geq (8k)!!$ suffices.

THEOREM 3.1. SR is provably equivalent to Con(SRP) over EFA.

There is an obvious general framework suggested by this definition of rich over A. Namely, we impose a finite set of conditions of the form

$$U_1 \times ... \times U_m \text{ and } V_1 \times ... \times V_m$$
$$\text{are order equivalent over } n,2n,...,kn$$

where the U's and V's are drawn from the three possibilities A,B,$[kn]^k$. It is necessary and sufficient that the A's are matched. Of course, this claim is itself provably equivalent to Con(SRP) over EFA.

4. PROOF SKETCH

Here we prove SR using SRP^+. Fix an order invariant $R \subseteq [kn]^k \times [kn]^k$, $n >> k$. Using large cardinal theory, we obtain the following.

1. Countable ordinals $\lambda_1 < \lambda_2 < ... < \lambda$, where λ is the limit of the λ_i.
2. R lifts to order invariant $R^* \subseteq \lambda^k \times \lambda^k$.
3. $S \subseteq \lambda^k$ is the unique solution to the Upper Image Equation with R*. I.e., $S \cup. R^*_<[S] = \lambda^k$.
4. $(0,...,0) \in S$.
5. S is stable in the following sense. For $\alpha < \lambda_1$, $S(\alpha,\lambda_1,...,\lambda_{k-1}) \leftrightarrow S(\alpha,\lambda_2,...,\lambda_k)$.

We now build a finite subset E_0 of λ_k+1, $|E_0| << n$. The idea is that $S \cap E_0^k$ is the small part of S that we need in order to witness the needed positive information related to the Upper Image Equation $S \cup. R^*_<[S] = \lambda^k$.

First we throw $0,\lambda_1,...,\lambda_k$ into E_0. Look at $(S^2 \times \lambda^k)|\leq\lambda_k$. The number of these elements under "order equivalence over $\lambda_1,...,\lambda_k$" is << n.

a. Choose one representative from each such equivalence class over $\lambda_1,...,\lambda_k$.
b. Let (y,z) be one of the representatives, $y \in S^2|\leq\lambda_k$, $z \in (\lambda_k+1)^k$.
c. Throw all ordinal coordinates of y,z into E_0.
d. Note that $z \in S \cup. R^*_<[S]$. If $z \in R^*_<[S]$, throw all ordinal coordinates of some w R* z, $w \in S$, max(w) < max(z), into E_0.

We now form the structure $(E_0,<,\lambda_1,...,\lambda_k,S|E_0)$, where $S|E_0$ is the k-ary predicate on $E_0 \subseteq \lambda_n+1$. This is a small structure since $|E_0| << n$.

We now expand this small structure to $(E_1,<,\lambda_1,...,\lambda_k,S|E_0)$ with finite $E_1 \subseteq \lambda_k+1$, as follows. Note that we are going to keep the same $S|E_0$ (and not use $S|E_1$).

We merely require that

1) each $|E_1 \cap \lambda_i| = in$.

This is easily arranged since $\lambda_1,...,\lambda_k$ are limit ordinals and $n >> k$.

We now verify that

2) $(E_1,<,\lambda_1,...,\lambda_k,S|E_0)$ has nonempty stable $S|E_0$, where $S|E_0$ $\cup$. $R^*_<[S|E_0]$ is rich over $S|E_0$

in the obvious sense within the finite domain E_1. Clearly $(0,...,0) \in S|E_0$.

Suppose $\alpha \in E_1 \cap \lambda_1$. Then $(\alpha,\lambda_1,...,\lambda_{k-1}) \in S|E_0 \leftrightarrow$ $(\alpha,\lambda_2,...,\lambda_k) \in S|E_0$ because this equivalence holds without $|E_0$ by 5 above.

$S|E_0 \cap R^*_<[S|E_0] = \emptyset$ because this is true without $|E_0$ by 3 above.

Finally, let $x \in (S|E_0)^2 \times E_1^k$. In the construction of E_0, we chose some (y,z) order equivalent to x over $\lambda_1,...,\lambda_k$, $y \in S^2|\leq\lambda_k$, $z \in (\lambda_k+1)^k$. By c above, $y \in (S|E_0)^2$, $z \in E_0^0$. If $z \in S$ then $z \in S|E_0$. Otherwise $z \in R^*_<[S]$, and we used d above. Therefore $z \in R^*_<[S|E_0]$.

We have now verified 2).

Using 1) above, let h be the unique strictly increasing isomorphism from $(E_1,<,\lambda_1,...,\lambda_k,S|E_0)$ onto $([kn],<,n,...,kn,A)$. Then $(0,...,0) \in A \subseteq [kn]^k$ is stable and $A \cup. R_<[A]$ is rich over $A \subseteq [kn]^k$ - as the stability and richness is preserved by isomorphism. QED

Obviously, we can use $A^k \times B$ instead of $A^2 \times B$ in Definition 5.1, and the same proof goes through.