CONCRETE MATHEMATICAL INCOMPLETENESS STATUS 2/8/18

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The book is expected to have three major parts:

CONCRETE MATHEMAITCAL INCOMPLETENESS

PART 1. BOOLEAN RELATION THEORY.
PART 2. EMULATION THEORY.
PART 3. INDUCTIVE EQUATION THEORY.

1. Emulation Theory/infinite/SRP.
2. Inductive Equation Theory/infinite/SRP.
3. Inductive Equation Theory/infinite/HUGE.
4. Inductive Equation Theory/finite/SRP.
5. Inductive Equation Theory/finite/HUGE.

I have recently been determined to move to functions rather than stick with sets. However, I have now realized how I can do more with sets than I had previously thought, and certain things are more naturally transparent with sets.

So I am now moving back to sets throughout.

1. EMULATION THEORY/INFINITE/SRP

LARGE CARDINAL PROPERTY. For every $R \subseteq \lambda^k$ there exists $0 < \alpha_1 < \ldots < \alpha_k$ such that for all $\beta < \alpha_1$, $R(\beta,\alpha_1,\ldots,\alpha_{k-1}) \leftrightarrow R(\beta,\alpha_2,\ldots,\alpha_k)$.

This large cardinal property corresponds exactly to the SRP hierarchy of large cardinals as $k \to \infty$. This kind of thing is very familiar in combinatorial set theory - although there are serious simplifying details here.

MAXIMAL EMULATION SHIFT. MES. For (finite) subsets of $Q[0,k]^k$, some maximal emulation has for all $p < 1$, $S(p,1,...,k-1) \leftrightarrow S(p,2,...,k)$.

It is easily seen that both forms of MES are equivalent (provably in RCA_0). It is easily seen, via Gödel's Completeness Theorem, that MES is provably equivalent, over WKL_0, to a Π^0_1 sentence, and therefore is PROVABLY FALSIFIABLE.

It is easy to see that every subset of Q^k has a maximal emulation. This is provable in RCA_0.

supporting definitions

Q(a,b] is the set of all rationals in the interval [a,b].

S is an emulation of $E \subseteq Q[0,k]^k$ if and only if $S \subseteq Q[0,k]^k$, and every $x \in S^2$ is order equivalent to some $y \in E^2$.

Maximal refers to maximality under inclusion.

THEOREM 1.1. MES is provably equivalent to Con(SRP) over WKL_0. This result hold even if we require that S be recursive in 0'.

2. INDUCTIVE EQUATION THEORY/INFINITE/SRP

INDUCTIVE UPPER SHIFT. IUS. Every order invariant $R \subseteq Q^{2k}$ has some $S = S\#\backslash R_<[S] \supseteq ush(S)$.

supporting definitions

Fix $R \subseteq Q^{2k}$. R is order invariant if and only if for all order equivalent $x,y \in Q^{2k}$, $x \in R \leftrightarrow y \in R$. $R_<[S]$ is the upper image of R on S, which is $\{y \in Q^k: (\exists x \in S)(\max(x) < \max(y) \wedge x\ R\ y)\}$. The notation R[S] requires that $S \subseteq Q^k$.

Let $S \subseteq Q^k$, S# is the least set $E^k \supseteq S \cup \{(0,...,0)\}$. The upper shift of S, ush(S), results from adding 1 to all nonnegative coordinates of S.

THEOREM 3.1. IUS is provably equivalent to Con(SRP) over WKL_0. This holds for fixed small dimension k. These results hold even if we require that S,S# be recursive in 0'.

3. SET EQUATION THEORY/FINITE/SRP

FINITE INDUCTIVE UPPER SHIFT. FIUS. Every order invariant $R \subseteq Q^{2k}$ has some nonempty finite $S_1 \subseteq \ldots \subseteq S_k \subseteq Q^k$, where each $S_{i+1} = S_{i+1}\#\backslash R_<[S_{i+2}] \supseteq ush(S_i)$.

FIUS is explicitly $\Pi^0{}_2$. There is an obvious double exponential bound on the numerators and denominators used in S_k as a function of k. This puts FIUS in explicitly $\Pi^0{}_1$ form.

THEOREM 3.1. FIUS is provably equivalent to Con(SRP) over EFA.

4. SET EQUATION THEORY/INFINITE/HUGE

INTERNAL INDUCTIVE UPPER SHIFT. IIUS. Every order invariant $R \subseteq Q^{2k}$ has some $S =_\leq S\#\backslash R_<[S]$ that strongly contains its upper shift.

supporting definitions

Let $A,B \subseteq Q^k$. $A =_\leq B$ if and only if A,B have the same elements with increasing ($\leq$) coordinates. The lower sections of A are the sets $A*p = \{x \in Q^{k-1}: \max(x) < p \wedge (p,x) \in A\}$.

A strongly contains B if and only if $A \supseteq B$ and every lower section of B is a lower section of A.

THEOREM 4.1. IIUS is provably equivalent to Con(HUGE) over WKL_0. This holds even for fixed small dimension k. These results hold even if we require that S be recursive in 0'.

5. SET EQUATION THEORY/FINITE/HUGE

FINITE INTERNAL INDUCTIVE UPPER SHIFT. FIIUS. Every order invariant $R \subseteq Q^{2k}$ has some nonempty finite $S_1 \subseteq \ldots \subseteq S_k \subseteq Q^k$, where for $1 \leq i \leq j \leq k$, $S_i =_\leq S_i\#\backslash R_<[S_{i+1}]$ and $ush(S_j)*i = S_j*(i+(1/2))$.

FIIUS is explicitly $\Pi^0{}_2$. There is an obvious double exponential bound on the numerators and denominators used in S_k as a function of k. This puts FIIUS in explicitly $\Pi^0{}_1$ form.

THEOREM 5.1. FIIUS is provably equivalent to Con(HUGE) over EFA.