CONCRETE MATHEMATICAL INCOMPLETENESS STATUS 3/6/18

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The book is expected to have three major parts:

CONCRETE MATHEMATICAL INCOMPLETENESS
abbreviated CMI

PART 1. BOOLEAN RELATION THEORY.
PART 2. EMULATION THEORY.
PART 3. INDUCTIVE EQUATION THEORY.

For BRT, see http://u.osu.edu/friedman.8/foundational-adventures/boolean-relation-theory-book/

Here we present most of the state of the art of Emulation Theory and Inductive Equation Theory as of 3/6/18.

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1. PRELIMINARIES

The definitions made in this section support the entire development in section 2, which is entirely in the contexts of (Q,<,0,1,...) and the (Q[0,n],<,0,1,...,n). Section 3 is in the same contexts, whereas sections 4,5 are in other contexts. Sections 3-6 require some mostly straightforward adaptations of the definitions given here.

DEFINITION 1.1. Z,Z^+,N,Q are, respectively, the sets of integers, positive integers, nonnegative integers, and rationals. For sets X, X* is the set of all nonempty finite sequences from X. lth(x) is the length of the nonempty finite sequence x. We use k,n,m,r,s,t for positive integers, with or without subscripts, unless indicated otherwise. We use p,q for rationals, with or without subscripts, unless indicated otherwise. $Q[(\alpha,\beta)]$ is the interval of rationals from α to β, with endpoint status indicated by [,). Here $\alpha,\beta \in Q \cup \{-\infty,\infty)$, where $[-\infty$, and $,\infty]$ are not used, since $-\infty,\infty$ are not rationals. For $x \in Qk$, max(x) is the greatest coordinate of x, and min(x) is the least coordinate of x. Let $S \subseteq Q^k$. $S|\leq p = \{x \in S: \max(x) \leq p\}$. S# is the least $E^k \supseteq S \cup \{0\}^k$.

DEFINITION 1.2. Let $S,S' \subseteq Q^k$ and $R \subseteq Q^{2k}$. $R(x,y) \leftrightarrow x\ R\ y \leftrightarrow (x,y) \in R$. R is upwards if and only if for all $x_1\ R\ x_2\ R\ ..\ R\ x_n$, some coordinate of x_1 is less than the corresponding coordinate of x_n. $ush(S) \subseteq Q^k$ is the result of adding 1 to all nonnegative coordinates of all elements of S.

DEFINITION 1.3. OE (read order equivalent) is the equivalence relation on Q* given by $x\ OE\ y \leftrightarrow lth(x) = lth(y) \wedge (\forall i,j)(1 \leq i,j \leq lth(x) \rightarrow (x_i < x_j \leftrightarrow y_i < y_j))$. The upper part of $x \in Q^*$ consists of the x_i such that every $x_j \geq x_i$ lies in N. The lower part of $x \in Q^*$ is the part of x not in the upper part of x. (Here the positions in x are incorporated in the upper and lower parts of x). OE↑ is the equivalence relation on Q* given by x OE↑ $y \leftrightarrow lth(x) = lth(y) \wedge x,y$ have the same upper part. OE↓ is the equivalence relation on Q* given by x OE↓ $y \leftrightarrow lth(x) = lth(y) \wedge x,y$ have the same lower part.

DEFINITION 1.4. Let E be an equivalence relation. $D \subseteq X$ is E invariant if and only if $(\forall x,y \in X)(x \, E \, y \rightarrow (x \in D \leftrightarrow y \in D))$. Let $X \subseteq Q^*$. $D \subseteq X$ is order invariant if and only if $D \subseteq X$ is OE invariant. Let $S \subseteq Q^k$ and $R \subseteq Q^{2k}$. $R[S] = \{y: (\exists x \in S)(x \, R \, y)$. $R_{<max}[S] = \{y: (\exists x \in S)(max(x) < max(y) \wedge x \, R \, y\}$. $R_{<min}[S] = \{y: (\exists x \in S)(min(x) < min(y) \wedge x \, R \, y)\}$.

Clearly $D \subseteq X$ is E invariant if and only if D is the union of equivalence classes of $E \cap X^2$.

DEFINITION 1.5. S is an emulation of $E \subseteq Q[0,n]^k$ if and only if $S \subseteq Q[0,n]^k$, and every element of S^2 is order equivalent to an element of E^2. S is a maximal emulation of $E \subseteq Q[0,n]^k$ if and only if S is an emulation of $E \subseteq Q[0,n]^k$, where no proper superset of S is an emulation of $E \subseteq Q[0,n]^k$.

Note that the ambient space of the given $E \subseteq Q[0,n]^k$, namely $Q[0,n]^k$, is important here and this determines the ambient space of all of the emulations of E (the same ambient space).

DEFINITION 1.6. Let $S,S' \subseteq Q^k$. The 2-sections of S are the sets $S_{p,q} = \{x: S(p,q,x)\} \subseteq Q^{k-2}$. If $k \leq 2$ then all 2-sections of S are $\emptyset$. The limited 2-sections of S are the sets $S_{p,q}|{\leq}r$. $S \supseteq^* S'$ if and only if $S \supseteq S'$ and every limited 2-section of S' is a limited 2-section of S.

$\supseteq^*$ is read "strongly contains". ush is read "upper shift".

2. IMPLICITLY FINITE

DEFINITION 2.1. A sentence φ in the language of set theory is implicitly Π^0_1 if and only if there is a Π^0_1 sentence ψ such that $\varphi \leftrightarrow \psi$ is provable in ZFC. More generally, a sentence φ in the language of set theory is implicitly Π^0_1 over a theory T if and only if there is a Π^0_1 sentence ψ such that $\varphi \leftrightarrow \psi$ is provable in T.

The infinitary sentences in CMI are usually Σ^1_1 (essentially Σ^1_1 if one is sensitive to outermost universal arithmetic quantifiers), and involve only the usual linear ordering of the rationals. Typically, the CMI statements are implicitly Π^0_1 over WKL_0.

Note that implicitly Π^0_1 statements are demonstrably falsifiable - i.e., a priori, we know if they are false then there are in principal refutable. This illustrates, in general terms, their concreteness. Demonstrable Falsifiability is a key concept in the philosophy of science.

2.1. MAXIMAL EMULATION SHIFT

MAXIMAL EMULATION SHIFT. MES. For (finite) subsets of $Q[0,k]^k$, some maximal emulation has for all $p < 1$, $S(p,1,...,k-1) \leftrightarrow S(p,2,...,k)$.

INVARIANT MAXIMAL EMULATION. IME. Every (finite) subset of $Q[0,n]^k$ has an OE↓ invariant maximal emulation.

THEOREM 2.1.1. In MES and IME, the statements with and without "finite" are provably equivalent in RCA_0. MES and IME are implicitly Π^0_1 over WKL_0 via the Gödel Completeness Theorem.

THEOREM 2.1.2. MES and IME are provably equivalent to Con(SRP) over WKL_0. The same results hold even if we add that the emulation is recursive in 0'.

In addition, results from

http://u.osu.edu/friedman.8/foundational-adventures/downloadable-manuscripts/
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will be incorporated into the book in the part on Emulation Theory.

2.2. INDUCTIVE UPPER SHIFT

INDUCTIVE UPPER SHIFT/max. IUS/max. For all order invariant $R \subseteq Q^{2k}$, there exists $S = S\#\backslash R_{<max}[S] \supseteq ush(S)$.

INDUCTIVE UPPER SHIFT/min. IUS/min. For all order invariant $R \subseteq Q^{2k}$, there exists $S = S\#\backslash R_{<min}[S] \supseteq ush(S)$.

We now greatly generalize IUS/max,min.

THEOREM 2.2.1. Let $R \subseteq Q^{2k}$. R is upwards if and only if its transitive closure is upwards. Assume R is order invariant. The transitive closure of R is order invariant. R is upwards if and only if the relevant statement in Definition 1.2 holds for all $1 \leq n \leq (8k)!$, and is therefore algorithmically determinable.

THEOREM 2.2.2. Let $R \subseteq Q^{2k}$ be upwards and $E \subseteq Q$ be well ordered. $R \cap E^{2k}$ is well founded in the sense that every nonempty subset of E^k has an R minimal element.

INDUCTIVE UPPER SHIFT/↑. IUS/↑. For all upwards order invariant $R \subseteq Q^{2k}$, there exists $S = S\#\backslash R[S] \supseteq ush(S)$.

THEOREM 2.2.3. IUS/↑, IUS/max, IUS/min are provably equivalent to Con(SRP) over WKL_0. The same results hold even if we add that the S and S# are recursive in 0'.

2.3. INTERNAL INDUCTIVE UPPER SHIFT

INTERNAL INDUCTIVE UPPER SHIFT/max. IIUS/max. For all order invariant $R \subseteq Q^{2k}$, there exists $S = S\#\backslash R_{<max}[S] \supseteq^* ush(S)$.

INTERNAL INDUCTIVE UPPER SHIFT/min. IIUS/min. For all order invariant $R \subseteq Q^{2k}$, there exists $S = S\#\backslash R_{<min}[S] \supseteq^* ush(S)$.

THEOREM 2.3.1. IIUS/min is provably equivalent to Con(HUGE) over WKL_0. The same result holds even if we add that the S and S# are recursive in 0'. IIUS/max is refutable in RCA_0.

3. EXPLICITLY FINITE IN Q

Here we stay in Q but use only finite subsets of Q^k. In each case, the statements are explicitly Π^0_2, and become explicitly Π^0_1 when straightforward a priori upper bounds on the numerators and denominators are imposed. In sections 4,5, we move into the context of finite initial segments of Z^+, where the statements are explicitly Π^0_1 at the outset.

In section 3.1, we approximate the statements in section 2 by towers of finite sets.

In sections 3.2 - 3.4, we approximate the statements in section 2 by single finite sets.

3.1. TOWERS

It is obvious that S is a maximal emulation of $E \subseteq Q[0,n]^k$ if and only if S is an emulation of $E \subseteq Q[0,n]^k$, where if $S \cup \{x\}$ is an emulation of $E \subseteq Q[0,n]^k$, then $x \in S$.

DEFINITION 3.1.1. Let $S,S' \subseteq Q^k$. $S =_{\leq p} S' \leftrightarrow S|{\leq p} = S'|{\leq p}$. S is an S'-maximal emulation of $E \subseteq Q[0,n]^k$ if and only if S is an emulation of $E \subseteq Q[0,n]^k$, where if $S \cup \{x\}$, $x \in S'$, is an emulation of $E \subseteq Q[0,n]^k$, then $x \in S$.

MAXIMAL EMULATION SHIFT TOWER. MEST. For all finite $E \subseteq Q[0,n]^k$, there exist finite OE↓ invariant $S_1 \subseteq \ldots \subseteq S_k \subseteq Q[0,n]^k$, where each S_i is an S_{i+1}-maximal emulation of E.

INDUCTIVE UPPER SHIFT TOWER/max. IUST/max. For all order invariant $R \subseteq Q^{2k}$, there exist finite $S_1 \subseteq \ldots \subseteq S_k \subseteq Q^k$ with all $S_{i+1} = S_{i+1}\#\backslash R_{<max}[S_{i+2}] \supseteq ush(S_i)$.

INDUCTIVE UPPER SHIFT TOWER/min. IUST/min. For all order invariant $R \subseteq Q^{2k}$, there exist finite $S_1 \subseteq \ldots \subseteq S_k \subseteq Q^k$ with all $S_{i+1} = S_{i+1}\#\backslash R_{<min}[S_{i+2}] \supseteq ush(S_i)$.

INDUCTIVE UPPER SHIFT TOWER/↑. IUST/↑. For all upwards order invariant $R \subseteq Q^{2k}$, there exist finite $S_1 \subseteq \ldots \subseteq S_k \subseteq Q^k$ with all $S_{i+1} = S_{i+1}\#\backslash R[S_{i+2}] \supseteq ush(S_i)$.

INTERNAL INDUCTIVE UPPER SHIFT TOWER/min. IIUST/min. For all order invariant $R \subseteq Q^{2k}$, there exist finite $S_1 \subseteq \ldots \subseteq S_k \subseteq Q^k$ with all $S_{i+1} = S_{i+1}\#\backslash R_{<min}[S_{i+2}] \supseteq ush(S_i)$, where $ush(S_k)_{0,0} =_{\leq k/2} (S_k)_{-1,(2k+1)/4}$.

THEOREM 3.1.1. IUST/max, IUST/min, IUST/↑ are provably equivalent to Con(SRP) over EFA.

THEOREM 3.1.2. IIUST/min is provably equivalent to Con(HUGE) over EFA. IIUST/<max is refutable in EFA.

3.2. FINITE MAXIMAL EMULATION

DEFINITION 3.2.1. S is a weakly maximal emulation of $E \subseteq Q[0,n]^k$ if and only if S is an emulation of $E \subseteq Q[0,n]^k$, where if $S \cup \{x\}$ is an emulation of $E \subseteq Q[0,n]^k$, then x is OE↑ equivalent to an element of S.

FINITE MAXIMAL EMULATION SHIFT. FMES. For finite subsets of $Q[0,k]^k$, some finite weakly maximal emulation has for all $p < 1$, $S(p,1,...,k-1) \leftrightarrow S(p,2,...,k)$.

FINITE INVARIANT MAXIMAL EMULATION. FIME. Every finite subsets of $Q[0,n]^k$ has a finite OE↓ invariant weakly maximal emulation.

THEOREM 3.2.1. FMES and FIME are provably equivalent to Con(SRP) over EFA.

3.3. FINITE INDUCTIVE UPPER SHIFT

DEFINITION 3.3.1. Let E be an equivalence relation. A =E B if and only if every element of A is E equivalent to an element of B and vice versa.

FINITE INDUCTIVE UPPER SHIFT/max. FIUS/max. For all order invariant $R \subseteq Q^{2k}$, there exists finite S =OE↑ $S\#\backslash R_{<max}[S]$ with $S \supseteq ush(S)|{\leq}k$.

FINITE INDUCTIVE UPPER SHIFT/min. FIUS/min. For all order invariant $R \subseteq Q^{2k}$, there exists finite S =OE↑ $S\#\backslash R_{<min}[S]$ with $S \supseteq ush(S)|{\leq}k$.

FINITE INDUCTIVE UPPER SHIFT/↑. FIUS/↑. For all upwards order invariant $R \subseteq Q^{2k}$, there exists finite S =OE↑ $S\#\backslash R[S]$ with $S \supseteq ush(S)|{\leq}k$.

THEOREM 3.2.2. FIUS/max, FIUS/min, FIUS/↑ are provably equivalent to Con(SRP) over EFA.

3.4. FINITE INTERNAL INDUCTIVE UPPER SHIFT

DEFINITION 3.4.1. Let $S \subseteq Q^k$. $S' \supseteq^{**} S$ if and only if $S \subseteq S' \subseteq Q^k$ and $S'_{0,0} =_{\leq k/2} S_{-1,(2k+1)/4}$.

FINITE INTERNAL INDUCTIVE UPPER SHIFT/min. FIIUS/min. For all order invariant $R \subseteq Q^{2k}$, there exists finite S =OE↑ $S\#\backslash R_{<min}[S]$ with $S \supseteq^{**} ush(S)|{\leq}k$.

THEOREM 3.4.1. FIIUS is provably equivalent to Con(HUGE) over EFA.

4. TOWERS IN ($[2^r]$,<,1,2,4,...,2^r)

When we write $S \subseteq [2^r]^k$, we are declaring that we are in the context ($[2^r]$,<,1,2,4,...,2^r). Note that up to now, we have always been in the contexts $S \subseteq Q[0,n]^k$ and $S \subseteq Q^k$. This use of contexts allows us to reuse S# and other notation.

DEFINITION 4.1. [n] = {1,...,n}. Let $S \subseteq [2^r]^k$. S# is the least $E^k \supseteq S \cup \{1,2,4,...,2^r\}^k$. S is without n if and only if n is not a coordinate of any element of X.

INDUCTIVE TOWER/max,2^r. IT/max,2^r. For all order invariant R $\subseteq [2^r]^{2k}$, there exist $S_1 \subseteq ... \subseteq S_k \subseteq [2^r]^k$, where each $S_i = S_i\#\backslash R_{<max}[S_{i+1}]$ is without $2^{(8k)!}-1$.

INDUCTIVE TOWER/min,2^r. IT/min,2^r. For all order invariant R $\subseteq [2^r]^{2k}$, there exist $S_1 \subseteq ... \subseteq S_k \subseteq [2^r]^k$, where each $S_i = S_i\#\backslash R_{<max}[S_{i+1}]$ is without $2^{(8k)!}-1$.

INDUCTIVE TOWER/↑,2^r. IT/↑,2^r. For all upwards order invariant $R \subseteq [2^r]^{2k}$, there exist $S_1 \subseteq ... \subseteq S_k \subseteq [2^r]^k$, where each $S_i = S_i\#\backslash R[S_{i+1}]$ is without $2^{(8k)!}-1$.

THEOREM 4.1. IT/max,2^r. IT/min,2^r, IT/↑,2^r are provably equivalent to Con(MAH) over ACA'.

5. IN ([k(8k)!],<,(8k)!,2(8k)!,...,k(8k)!)

We now work in the context ([k(8k)!],<,(8k)!,2(8k)!,...,k(8k)!). We use (8k)! in the statement headers to indicate this context.

5.1. TOWERS

DEFINITION 5.1.1. Let $S \subseteq [k(8k)!]^k$. S# is the least $E^k \supseteq S \cup \{(8k)!\}^k$. ush(S) is the result of adding (8k)! to all coordinates of elements of S that are in [(8k)!,(k-1)(8k)!].

INDUCTIVE TOWER/max,(8k)!. IT/max,(8k)!. For all order invariant $R \subseteq [k(8k)!]^{2k}$, there exist $S_1 \subseteq \ldots \subseteq S_k \subseteq [k(8k)!]^k$, where each $S_{i+1} = S_{i+1}\#\backslash R_{<max}[S_{i+2}] \supseteq ush(S_i)$.

INDUCTIVE TOWER/(8k)!,min. IT/min,(8k)!. Every order invariant $R \subseteq [k(8k)!]^{2k}$, there exist $S_1 \subseteq \ldots \subseteq S_k \subseteq [k(8k)!]^k$, where each $S_{i+1} = S_{i+1}\#\backslash R_{<min}[S_{i+2}] \supseteq ush(S_i)$.

INDUCTIVE TOWER/(8k)!,↑. IT/↑,(8k)!. Every upwards order invariant $R \subseteq [k(8k)!]^{2k}$, there exist finite $S_1 \subseteq \ldots \subseteq S_k \subseteq [k(8k)!]^k$, where each $S_{i+1} = S_{i+1}\#\backslash R[S_{i+2}] \supseteq ush(S_i)$.

INTERNAL INDUCTIVE TOWER/min,(8k)!. IIT/min,(8k)!. Every order invariant $R \subseteq [k(8k)!]^{2k}$ has some finite $S_1 \subseteq \ldots \subseteq S_k \subseteq [k(8k)!]^k$, where each $S_{i+1} = S_{i+1}\#\backslash R_{<min}[S_{i+2}] \supseteq ush(S_i)$, and $ush(S_k)_{(8k)!,(8k)!} =_{\leq(k/2)(8k)!} (S_k)_{1,(2k+1)/4)(8k)!}$.

THEOREM 5.1.1. IT/max,(8k)!, IT/min,(8k)!, IT/↑,(8k)! are provably equivalent to Con(SRP) over EFA.

THEOREM 5.1.2. IIT/min,(8k)! is provably equivalent to Con(HUGE) over EFA.

5.2. MAXIMAL EMULATION

DEFINITION 5.2.1. S is an emulation of $E \subseteq [k(8k)!]^k$ if and only if $S \subseteq [k(8k)!]^k$ and every element of S^2 is order equivalent to some element of E^2. The upper part of $x \in [k(8k)!]^k$ consists of the x_i such that every $x_j \geq x_i$ lies in $\{(8k)!,2(8k)!,\ldots,k(8k)!\}$. The lower part of $x \in [k(8k)!]^k$ is the part of x not in the upper part of x. OE↑ is the equivalence relation on $[k(8k)!]^k$ given by x OE↑ y ↔ x,y have the same upper part. OE↓ is the equivalence relation on $[k(8k)!]^k$ given by x OE↓ y ↔ x,y have the same lower part.

DEFINITION 5.2.2. S is a weakly maximal emulation of $E \subseteq [k(8k)!]^k$ if and only if S is an emulation of $E \subseteq [k(8k)!]^k$, where if $S \cup \{x\}$ is an emulation of $E \subseteq [k(8k)!]^k$, then x is OE↑ equivalent to an element of S.

MAXIMAL EMULATION SHIFT/(8k)!. MES/(8k)!. For subsets of $[k(8k)!]^k$, some weakly maximal emulation has for all p < (8k)!, S(p,(8k)!,...,(k-1)(8k)!) ↔ S(p,2(8k)!,...,k(8k)!).

INVARIANT MAXIMAL EMULATION/(8k)!. IME/(8k)!. Every subset of $[k(8k)!]^k$ has a finite OE↓ invariant weakly maximal emulation.

THEOREM 5.2.1. FMES and FIME are provably equivalent to Con(SRP) over EFA.

5.3. INDUCTIVE UPPER SHIFT

INDUCTIVE UPPER SHIFT/max.(8k)!. IUS/max,(8k)!. For all order invariant R ⊆ $[k(8k)!]^{2k}$, there exists S =OE↑ $S\#\backslash R_{<max}[S]$, S ⊇ ush(S).

INDUCTIVE UPPER SHIFT/min,(8k)!. IUS/min,(8k)!. For all order invariant R ⊆ $[k(8k)!]^{2k}$, there exists S =OE↑ $S\#\backslash R_{<min}[S]$, S ⊇ ush(S).

INDUCTIVE UPPER SHIFT/↑,(8k)!. IUS/↑,(8k)!. For all upwards order invariant R ⊆ $[k(8k)!]^{2k}$, there exists S =OE↑ S#\R[S], S ⊇ ush(S).

THEOREM 5.3.1. IUS/max,(8k)!, IUS/min,(8k)!, IUS/↑,(8k)! are provably equivalent to Con(SRP) over EFA.

5.4. INTERNAL INDUCTIVE UPPER SHIFT

DEFINITION 5.4.1. Let S ⊆ $[k(8k)!]^k$. S' ⊇* S if and only if S ⊆ S' ⊆ $[k(8k)!]^k$ and $S'_{(8k)!,(8k)!} =_{\leq(k/2)(8k)!} S_{1,((2k+1)/4)(8k)!}$.

INTERNAL INDUCTIVE UPPER SHIFT/min,(8k)!. IIUS/min,(8k)!. Every order invariant R ⊆ Q^{2k} has some finite S =OE↑ $S\#\backslash R_{<min}[S]$, S ⊇* ush(S).

THEOREM 5.4.1. IIUS/(8k)! is provably equivalent to Con(HUGE) over EFA.

6. SOME TEMPLATES

DEFINITION 6.1. A Template is an algorithmically presented set of mathematical sentences. A Template is resolved by a pair of formal systems K,K' if and only if every instance of the Template is either provable in K or refutable in K'. A Template is polynomially resolved by a pair of formal systems K,K' if and only if every instance of the Template is either provable in K or refutable in K', where the number of symbols in the proofs and refutations are bounded by a polynomial in the size of the Template instance.

In order to be practically usable, formal systems need to directly accommodate the introduction of new symbols by explicit definition, and other related devices. Most of the usual formalizations given in mathematical logic do not directly accommodate such devices. Nevertheless, the addition of these devices is polynomially eliminable, as seen in http://www.andrew.cmu.edu/user/avigad/Papers/definitions.pdf Hence our notion above of polynomially resolvable is appropriate, at least from a theoretical standpoint. We naturally want to control the degree and ultimately coefficients, but that is a much more detailed matter beyond the scope of this abstract.

THEOREM 6.1. If a Template is resolved by a pair of systems extending EFA then its set of true instances is recursive. If a Template is polynomially resolved by a pair of reasonable true systems extending EFA then its set of true sentences is in NP ∩ co-NP.

DEFINITION 6.2. SRP = ZFC + {(∃λ)(λ has k-SRP): k ≥ 1}. SRP^+ = ZFC + (∀k)(∃λ)(λ has k-SRP). SRP[n] = ZFC + (∃λ)(λ has n-SRP).

We present Templates involving the Implicitly Finite statements from section 2 only. We will take other Templates up at a later date.

TEMPLATE 1. Let k and φ be a quantifier free sentence in k-ary S and constants for every rational in Q[0,1] be given. For finite subsets of $Q[0,1]^k$, some maximal emulation has φ holding universally over Q[0,1].

Obviously MES for any fixed k is a special case of Template 1.

CONJECTURE 1. Template 1 is polynomially resolved by SRP,RCA_0.

Even for one variable, Template 1 is not resolved by any SRP[n],SRP, assuming SRP is 1-consistent.

TEMPLATE 2. Let α(S,S#,R[S],ush(S)) be a formal Boolean combination of S,S#,R[S],ush(S) with formal universal set Q^k. For all upwards order invariant R $\subseteq Q^{2k}$, there exists S such that α(S,S#,R[S],ush(S)) holds.

Obviously IUS/↑, which quantifies over k, is a special case of Template 2.

CONJECTURE 2. Template 2 is polynomially resolved by SRP^+,RCA_0.

There are 2^{16} = 65,536 instances of Template 2. We know that Template 2 is not resolved by SRP,SRP, assuming SRP is 1-consistent.

We plan to first handle the far easier Template 3, with 256 instances:

TEMPALTE 3. Let α(S,S#,R[S]) be a formal Boolean combination of S,S#,R[S] with formal universal set Q^k. For all upwards order invariant R $\subseteq Q^{2k}$, there exists S such that α(S,S#,R[S]) holds.

showing that it is polynomially resolved by EFA,EFA, with EFA viewed as a weak finite set theory.

Stronger Templates than Template 2 arise from Templating ush. Note that ush:Q* → Q* is the lifting to Q* of the one dimensional ush:Q → Q. Note that ush:Q → Q is a rational piecewise linear function, of which there are countably many. Thus ush:Q* → Q* is what we call a rational piecewise linear lifting.

TEMPLATE 4. Let H:Q* → Q* be a rational piecewise linear lifting. For all upwards order invariant R $\subseteq Q^{2k}$, there exists S = S#\R[S] $\supseteq$ H[S].

CONJECTURE 4. Template 4 is polynomially resolved by SRP^+,RCA_0.

We know that Template 4 is not resolved by SRP,SRP assuming SRP is 1-consistent.

TEMPLATE 5. Let α(S,S#,R[S],H[S']) be a formal Boolean combination of S,S#,R[S],H[S'] with formal universal set Q^k. Let H:Q* → Q* be a rational piecewise linear lifting. For all upwards order invariant R ⊆ Q^{2k}, there exists S such that α(S,S#,R[S],H[S]) holds.

CONJECTURE 5. Template 5 is polynomially resolved by SRP^+,RCA_0.

We know that Template 5 is not resolved by SRP,SRP assuming SRP is 1-consistent.

It also makes sense to template "upwards order invariant" at various levels of detail. However, we will not take this up here.

TEMPLATE 6. Let p,q,r,p',q',r' be given. For all order invariant R ⊆ Q^{2k}, there exists S = S#\$R_{<min}$[S] ⊇ ush(S), where $ush(S)_{p,q}|\leq r = S_{p',q'}|\leq r'$.

CONJECTURE 6. Template 5 is polynomially resolved by $HUGE^+$,RCA_0.

We know that Template 6 is not resolved by HUGE,HUGE, assuming HUGE is 1-consistent.

7. FORMAL SYSTEMS USED

EFA Exponential function arithmetic.

RCA_0 Recursive comprehension axiom naught.

WKL_0 Weak Konig's Lemma naught.

ACA_0 Arithmetic comprehension axiom naught.

ACA' ACA_0 + $(\forall k)(\forall x \subseteq \omega)$(the k-th Turing jump of x exists).

ZF(C) Zermelo Frankel set theory (with the axiom of choice).

SMAH[k] ZFC + $(\exists\lambda)(\lambda$ is strongly k-Mahlo), k fixed.

SMAH ZFC + {$(\exists\lambda)(\lambda$ is strongly k-Mahlo): $k \geq 1$}.

SMAH$^+$ ZFC + $(\forall k)(\exists\lambda)$($\lambda$ is strongly k-Mahlo).

SRP[k] ZFC + $(\exists\lambda)$(λ has the k-SRP), for fixed k.

SRP ZFC + {$(\exists\lambda)$(λ has the k-SRP): $k \geq 1$}.

SRP$^+$ ZFC + $(\forall k)(\exists\lambda)$($\lambda$ has the k-SRP).

HUGE[k] ZFC + $(\exists\lambda)$(λ is k-HUGE), for fixed k.

HUGE ZFC + {$(\exists\lambda)$(λ is k-huge): $k \geq 1$}.

HUGE$^+$ ZFC + $(\forall k)(\exists\lambda)$($\lambda$ is k-huge).

λ is k-huge if and only if there exists an elementary embedding $j:V(\alpha) \to V(\beta)$ with critical point λ such that $\alpha = j^k(\lambda)$. (This hierarchy differs in inessential ways from the more standard hierarchies in terms of global elementary embeddings). For more about huge cardinals, see [Ka94], p. 331.