CONCRETE MATHEMATICAL INCOMPLETENESS HIGHLIGHTS 4/29/18

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Abstract. The Concrete Mathematical Incompleteness (CMI) project was founded in 1967, with the ambitious intention of touching mathematical thinkers at all levels in terms of their fundamental attitudes toward mathematical objectivity and reality. The initial impetus for CMI is of course the famous Gödel Incompleteness Theorems from the early 1930s which, at least initially, did touch mathematical thinkers at all levels in this way. Yet much more is needed to keep Incompleteness at the forefront of mathematical thought. We are at a seemingly advanced stage of Phase 1 CMI, where 1 refers to the use of essentially no mathematical machinery in the formulation of the Incompleteness. In fact, we use only sets of rational tuples and sometimes only finite sets of integer tuples, with only the usual ordering of the rationals and sometimes only the usual ordering of finite initial segments of the nonnegative integers. The plan has been to first go as deeply as we can (or know how to) in this pure context to uncover the necessarily new purely combinatorial structure. In latter Phases, the plan is to start involving very elemental structure such as addition on the nonnegative integers. We present four different Incompleteness results and variants in sections 2-5. All are implicitly Π^0_1, whereas some are explicitly finite and even explicitly Π^0_1. All correspond to the SRP hierarchy of large cardinals, except for one that corresponds to the much higher HUGE hierarchy of large cardinals. Each will resonate more or less according to the experience and orientation of various mathematical thinkers.

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1. INTRODUCTION

The Concrete Mathematical Incompleteness (CMI) project was founded by us when we left MIT with a Ph.D. in mathematics and accepted an Assistant Professorship in Philosophy at Stanford University in 1967. The full intention was to eventually touch mathematical thinkers at all levels in terms of their fundamental attitudes toward mathematical objectivity and reality.

The initial impetus for CMI is of course the famous Gödel Incompleteness Theorems from the early 1930s. This path breaking work of great general intellectual interest did generally touch mathematical thinkers at all levels in terms of their fundamental attitudes toward mathematical objectivity and reality, at the time. Yet much more is needed to keep Incompleteness at the forefront of mathematical thought.

After over 50 years of effort, we are now at a seemingly advanced stage of Phase 1 CMI, where 1 refers to the use of essentially no mathematical machinery in the formulation of the Incompleteness. In sections 2,3,5, we use only sets of rational tuples with only the usual ordering of the rationals. In section 4, we use only finite sets of integer tuples and only the usual ordering of finite initial segments of the nonnegative integers.

The plan has been to first go as deeply as we can (or know how to) in this pure context to uncover the necessarily new purely combinatorial structure. In latter Phases, the plan is to start involving very elemental structure such as addition on the nonnegative integers.

We present four different Incompleteness results and variants in sections 2-5. In sections 2,3,5, they are implicitly Π^0_1, and in section 4, they are explicitly Π^0_1. In sections 2,3,4, they correspond to the SRP hierarchy of large cardinals. In section 5, they correspond to the much higher HUGE hierarchy of large cardinals. I.e., these statements are provably equivalent to Con(SRP) or Con(HUGE).

Each of these independence results will resonate more or less according to the experience and orientation of various mathematical thinkers.

This manuscript is meant to be a Status Report. It treats only the clearest and most transparent highlights. It does not contain the kind of background information, examples, extended discussion, and the like, that may be required for resonance with many mathematical thinkers. This will be done as Phase 1 CMI comes to a (at least temporary) close. In fact, we are trying to limit the scope of the results in Phase 1 CMI with the intention of having intelligible write ups of the major proofs by the end of calendar year 2018.

2. MAXIMAL EMULATION

DEFINITION 2.1. Z,Q,Z^+,N is the set of all integers, rational, positive integers, nonnegative integers, respectively. We use n,m,r,s,t with or without subscripts for positive integers, unless otherwise indicated. We use p,q for rationals, with or without subscripts, unless indicated otherwise. Q[p,q] is the interval of rationals from p to q, inclusive.

DEFINITION 2.2. $x,y \in Q^k$ are order equivalent if and only if for all $1 \leq i,j \leq k$, $x_i < x_j \leftrightarrow y_i < y_j$.

DEFINITION 2.3. S is an emulator of $E \subseteq Q[0,k]^k$ if and only if $S \subseteq Q[0,k]^k$ and every element of S^2 is order equivalent to some element of E^2. S is a maximal emulator of $E \subseteq Q[0,k]^k$ if and only if S is an emulator of $E \subseteq Q[0,k]^k$ and no proper superset of S is an emulator of $E \subseteq Q[0,k]^k$.

MAXIMAL EMULATION. ME. Every finite subset of $Q[0,k]^k$ has a maximal emulator.

We seek maximal emulators with some invariance properties. The weakest of these is stability:

DEFINITION 2.4. $S \subseteq Q[0,k]^k$ is stable if and only if for all $p < 1$, $(p,1,...,k-1) \in S \leftrightarrow (p,2,...,k) \in S$.

We also use the following much stronger invariance condition.

DEFINITION 2.5. The upper part of $x \in Q^k$ consists of the x_i such that every $x_j \geq x_i$ is a positive integer. The lower part of $x \in Q^k$ consists of the x_i not in the upper part of $x \in Q^k$. $S \subseteq Q[0,k]^k$ is lower invariant if and only if for all order equivalent $x,y \in Q[0,k]^k$ with the same lower part, $x \in S \leftrightarrow y \in S$. These upper and lower parts carry along the positions of their coordinates from the original x.

MAXIMAL EMULATION STABILITY. MES. Every finite subset of $Q[0,k]^k$ has a stable maximal emulator.

MAXIMAL EMULATION LOWER. MEL. Every finite subset of $Q[0,k]^k$ has a lower invariant maximal emulator.

By Theorem 2.1, MEL implies MES.

THEOREM 2.2. ME is provable in EFA. MES, MEL are provably equivalent to Con(SRP) over WKL_0.

MES, MEL are implicitly Π^0_1 via Gödel's Completeness Theorem.

3. INDUCTIVE UPPER SHIFT

DEFINITION 3.1. $S \subseteq Q^k$ is order invariant if and only if for all order equivalent $x,y \in Q^k$, $x \in S \leftrightarrow y \in S$. $R \subseteq Q^k \times Q^k$ is order invariant if and only if R is order invariant as a subset of Q^{2k}. $R[S] = \{y: (\exists x \in S)(x\ R\ y)\}$. $R_<[S] = \{y: (\exists x \in S)(\max(x) < \max(y) \wedge x\ R\ y)\}$. S# is the least $V^k \supseteq S \cup \{0\}^k$.

$R_<[S]$ is called the upper image of R on S.

THEOREM 3.1. For all $R \subseteq Q^k \times Q^k$ there exists $S = S\#\backslash R_<[S]$.

Theorem 3.1 is trivially proved using $S \subseteq \{0\}^k$.

DEFINITION 3.2. The upper shift of $S \subseteq Q^k$, ush(S), is obtained by adding 1 to all nonnegative coordinates of elements of S.

INDUCTIVE UPPER SHIFT. IUS. For all order invariant $R \subseteq Q^k \times Q^k$ there exists $S = S\#\backslash R_<[S] \supseteq \text{ush}(S)$.

THEOREM 3.2. IUS is provably equivalent to Con(SRP) over WKL_0.

4. FINITE UPPER IMAGE

We now work in the [n] = {0,...,n}, and we use Definition 3.1 with Q replaced by [n] (except we don't use # here). In this section we obtain explicitly finite and explicitly Π^0_1 statements.

FINITE UPPER IMAGE EQUATION. For all R ⊆ $[n]^k \times [n]^k$ there exists S ⊆ $[n]^k$ such that $R_<[S]$ = S'. In fact, S is unique.

DEFINITION 4.1. S ⊆ $[n]^k$ is stable if and only if for all p < n/k, (n,n/2,...,n/(k-1),p) ∈ S ↔ (n/2,n/3,...,n/k,p) ∈ S. Here n/i is understood to be the floor of the rational n/i.

FINITE UPPER IMAGE STABILITY/1. FUIS/1. For all order invariant R ⊆ $[n]^k \times [n]^k$ there exists stable S ⊆ $[n]^k$ such that $R_<[S]$ = S'.

FUIS/1 is refutable, and so we need an approximate form of the upper image equation R<[S] = S'.

DEFINNITION 4.2. A,B ⊆ $[n]^k$ are order equivalent over n,n/2,...,n/k if and only if every (x,n,n/2,...,n/k), x ∈ A, is order equivalent to some (y,n,n/2,...,n/k), y ∈ B, and vice versa.

FINITE UPPER IMAGE STABILITY/2. FUIS/2. For all order invariant R ⊆ $[n]^k \times [n]^k$, n >> k, there exists stable S ⊆ $[n]^k$ such that $R_<[S]$ and S' are order equivalent over n,n/2,...,n/k. In fact, n ≥ (8k)!! suffices.

FINITE UPPER IMAGE STABILITY/3. FUIS/3. For all order invariant R ⊆ $[n]^k \times [n]^k$, n >> k, there exists stable S ⊆ $[n]^k$ such that $R_<[S] \times S^2$ and $S' \times S^2$ are order equivalent over n,n/2,...,n/k. In fact, n ≥ (8k)!! suffices.

THEOREM 4.1. FUIE and FUIS/2 are provable in EFA. FUIS/3 is provably equivalent to Con(SRP) over EFA.

Note that FUIS/3 is explicitly Π^0_1 using (8k)!!.

5. INTERNAL INDUCTIVE UPPER SHIFT

We begin with a modification of the upper image.

DEFINITION 5.1. Let $R \subseteq Q^k \times Q^k$, $S \subseteq Q^k$. $R_{<min}[S] = \{y: (\exists x \in S)(\min(x) < \min(y) \wedge x R y)\}$.

INDUCTIVE UPPER SHIFT/min. IUS/min. For all order invariant $R \subseteq Q^k \times Q^k$ there exists $S = S\#\backslash R_{<min}[S] \supseteq ush(S)$.

We now use a strong form of inclusion, $\supseteq^*$ between subsets of Q^{k+1}.

DEFINITION 5.1. Let $S,S' \subseteq Q^{k+1}$. The 1-sections of S are the sets $S_x = \{p: S(x,p)\} \subseteq Q$, $x \in Q^k$. The limited 1-sections of S are the sets $S_x|{\le}r$, $x \in Q^k$, $r \in Q$. $S \supseteq^* S'$ if and only if $S \supseteq S'$ and every limited 1-section of S' is a limited 1-section of S.

INTERNAL INDUCTIVE UPPER SHIFT. IIUS. For all order invariant $R \subseteq Q^{k+2} \times Q^{k+2}$ there exists $S = S\#\backslash R_{<min}[S] \supseteq^* ush(S)$.

THEOREM 5.1. IUS/min is provably equivalent to Con(SRP) over WKL_0. IIUS is provably equivalent to Con(HUGE) over WKL_0.

6. FORMAL SYSTEMS USED

EFA Exponential function arithmetic.

RCA_0 Recursive comprehension axiom naught.

WKL_0 Weak Konig's Lemma naught.

ZF(C) Zermelo Frankel set theory (with the axiom of choice).

SRP[k] ZFC + $(\exists\lambda)$(λ has the k-SRP), for fixed k.

SRP ZFC + $\{(\exists\lambda)(\lambda$ has the k-SRP): $k \ge 1\}$.

SRP^+ ZFC + $(\forall k)(\exists\lambda)$($\lambda$ has the k-SRP).

HUGE[k] ZFC + $(\exists\lambda)$(λ is k-HUGE), for fixed k.

HUGE ZFC + $\{(\exists\lambda)(\lambda$ is k-huge$): k \geq 1\}$.

$HUGE^+$ ZFC + $(\forall k)(\exists\lambda)(\lambda$ is k-huge$)$.

λ is k-huge if and only if there exists an elementary embedding $j:V(\alpha) \to V(\beta)$ with critical point λ such that $\alpha = j^k(\lambda)$. (This hierarchy differs in inessential ways from the more standard hierarchies in terms of global elementary embeddings).

7. PRINCIPLE OF INVARIANT MAXIMALITY

One main theme has emerged clearly in section 2 (and also in section 3).

PRINCIPLE OF MAXIMALITY. Every appropriately constrained family of mathematical structures has an appropriately maximal element.

PRINICPLE OF INVARIANT MAXIMALITY. Every appropriately constrained family of mathematical structures has an appropriately invariant maximal element.

In section 2, the structures are the subsets of $Q[0,k]^k$, and the constraints are given by finite subsets of $Q[0,k]^k$. The constrained families consist of the emulators S of the various finite subsets of $Q[0,k]^k$. Maximality is under inclusion. The invariance properties in section 2 are stability and the stronger lower invariance.

We are looking at the use of the Principle of Invariant Maximality as an impetus for Phase 2 CMI.