COMPLETE DENSE LINEAR ORDERINGS

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Abstract. We explore conditions on complete dense linear orderings that imply they are separable, and hence isomorphic to a standard one; i.e., $[0,1] \subseteq \mathfrak{R}$ with zero or more endpoints removed. It is well known that the existence of an ordered group structure on a complete dense linear ordering is sufficient for separability. This brings in an algebraic consideration. We seek sufficient conditions of a more topological character. I^ω is an example of a complete dense linear ordering with strong homogeneity properties that is nonseparable. A continuous "between function" is sufficient for separability. A continuous strictly increasing binary function is also sufficient for separability. We also show that a pair of permutations with certain elementary conditions is sufficient for separability. In addition, we prove the uniqueness up to isomorphism of midpoint functions on [0,1] by the equation $M(x,M(y,z)) = M(M(x,y),M(x,z))$.

1. INTRODUCTION AND PRELIMINARIES

In this section we introduce the notation and terminology used throughout the paper.

DEFINITION 1.1. (D,<) is a linear ordering if and only if
i. D is a nonempty set.
ii. < is irreflexive and transitive.
iii. For all $x,y,z \in D$, $x < y \lor y < x \lor x = y$.

DEFINITION 1.2. $(D,<)$ is a dense linear ordering if and only if
i. $(D,<)$ is a linear ordering with at least two elements.
ii. $(\forall x,y)(x < y \rightarrow (\exists z)(x < z < y))$.

DEFINITION 1.3. Let $(D,<)$ be a linear ordering. x is an upper bound of $S \subseteq D$ if and only if $(\forall y \in S)(y \leq x)$. x is a lower bound of $S \subseteq D$ if and only if $(\forall y \in S)(y \geq x)$.

THEOREM 1.1. Let $(D,<)$ be a linear ordering. The following are equivalent.
i. Every nonempty subset of D with an upper bound has a least upper bound.
ii. Every nonempty subset of D with a lower bound has a greatest lower bound.
iii. Every nonempty subset of D with both an upper and lower bound has a least upper bound.
iv. Every nonempty subset of D with both an upper and lower bound has both a least upper bound and a greatest lower bound.

Proof: i implies ii. Suppose i. Let $A \subseteq D$ be nonempty with a lower bound. Let B be the set of all lower bounds of A. Then B is nonempty with an upper bound. Let x be the least upper bound of B. To see that x is a lower bound of A, let $y \in A$. Then all elements of B are $\leq y$, and so $x \leq y$. So x is a lower bound of A. Now let z be a lower bound of A. Then $z \in B$ and so $z \leq x$. Hence x is the greatest lower bound of A.

ii implies i. The same argument as above with "lower" and "upper" switched, and "least" and "greatest" switched, and $\leq$ and $\geq$ switched.

iii implies i. Assume iii. Let $A \subseteq D$ be nonempty with an upper bound. Let $x \in A$ and $A' = \{y \in A: y \leq x\}$. Then A' has a lower and upper bound. Let u be the least upper bound of A'. Then u is the least upper bound of A.

iv implies ii. As in the previous paragraph. QED

DEFINITION 1.4. $(D,<)$ is complete if and only if every nonempty subset of D with an upper bound has a least upper bound. Henceforth, D will always denote a complete dense linear ordering $(D,<)$. D may have any of the four endpoint situations.

THROUGHOUT THIS PAPER, D = (D,<) WILL ALWAYS DENOTE A COMPLETE DENSE LINEAR ORDERING, WITH ANY OF THE FOUR ENDOINT SITUATIONS.

DEFINITION 1.5. D is separable if and only if there is a countable $S \subseteq D$ such that $(\forall x,y \in D)(x < y \rightarrow (\exists z \in S)(x < z < y))$. D is standard if and only if D is the usual $[0,1] \subseteq \mathfrak{R}$ with the usual <, with zero or more endpoints removed. isomorphic to the usual unit interval $[0,1] \subseteq \mathfrak{R}$ with zero or more endpoints removed.

THEOREM 1.2. D is separable if and only if D is isomorphic to a standard D.

Proof: We give a brief sketch of this well known result. Obviously if D is standard then $Q \cap (0,1)$ is a countable subset of D with the required property. Now let $S \subseteq D$ have the required property. Let S' be S without any endpoints of D. Then (S',<) is a countable dense linear ordering without endpoints. Let h be an isomorphism from (S',<) onto $(Q \cap (0,1),<)$. Then h extends uniquely to an isomorphism from (D',<) onto ((0,1),<), where D' is D without its endpoints. Extend h further to a unique isomorphism from (D,<) onto (J,<), where J is (0,1) with zero or more endpoints adjoined. QED

The principal aim of the paper is to present simple sufficient conditions for D to be separable, which we now define. The most well known result of the kind is the following classic.

THEOREM 1.3. Suppose D has an ordered group structure (D,<,+). Then D is separable. In fact, (D,<,+) is isomorphic to $(\mathfrak{R},<,+)$.

A common reference is Otto Holder [Ho1901]. There considerably stronger results are obtained, where one does not start off with the assumption that the underlying < is a complete dense linear ordering. Rather, appropriate completeness is assumed for the entire structure. It might be interesting to further develop the approach taken here along the lines of [Ho1901] and later work based on semigroups.

DEFINITION 1.6. The left (right) endpoint of D is its least (greatest) element. These endpoints may or may not exist. An interval in D is a $J \subseteq D$ such that for all $x < y < z$, if $x,z \in J$ then $y \in J$. For any $S \subseteq D$, sup(S) (inf(S)) is the least upper bound (greatest lower bound) of S. These may or may not exist, according to the existence of upper (lower) bounds. If sup(S) (inf(S)) does not exist then we take $\sup(S) = \infty$ ($\inf(S) = -\infty$). We write J in the form $<\inf(J),\sup(J)>$, where $<$ is (or [, and $>$ is) or]. We require that if ∞ is used then it is used with $\infty)$ and D has no greatest element. We require that if $-\infty$ is used then it is used with $(-\infty$ and D has no least element. We also require that $\inf(J) \leq \sup(J)$. Thus $\emptyset$ can be written as $(c,c) = (c,c] = [c,c)$, and $\{c\}$ as $[c,c]$.

The usual topology on D is called the interval topology.

DEFINITION 1.7. The relatively open intervals in D are the intervals (a,b), $a,b \in D$, $[\min(D),b)$, $\min(D) < b$, $(a,\max(D)]$, $a < \max(D)$, and D. The open sets in D are the unions of relatively open intervals in D. The closed sets in D are the complements of the open sets in D. These open sets form the so called interval topology of D.

We avoid the phrases "open interval" and "open interval in D" because of Definition 1.7. We sometimes refer to (a,b) in D, which is $\{x \in D: a < x < b\}$. This is nonempty if and only if $a < b$.

THEOREM 1.4. An interval in D is open (i.e., an open set) if and only if it is a relatively open interval in D.

Proof: Let J be an interval in D. Suppose J is open. Then J is the union of relatively open intervals in D. Suppose $\min(J) = x \in D$. Let $x \in J' \subseteq J$, J' a relatively open interval in D. Then $x = \min(J') = \min(D)$. Similarly, if max(J) exists then $\max(J) = \max(D)$. Hence J is a relatively open interval in D.

Suppose J is a relatively open interval in D. Then J is the union of relatively open intervals in D - namely one, J. QED

DEFINITION 1.8. $S \subseteq D$ is dense if and only if S meets every nonempty open set in D.

THEOREM 1.5. D is separable (according to Definition 1.5) if and only if D has a countable dense subset.

Proof: Let D be separable. Let J be a nonempty open interval in D. Let $\{x < y\} \subseteq J$. Let $x < z < y$ an $z \in S$. Then $z \in S \cap (x,y)$.

Let $S \subseteq D$ be countable and dense. Let $x < y$. Then $S \cap (x,y) \neq \emptyset$, and so there exists $x < z < y$ with $z \in S$. Hence D is separable (under Definition 1.5). QED

We now present some important examples of D that are not separable.

DEFINITION 1.9. I^2 is ordered with the lexicographic ordering. I.e., $(x,y) <_{lex} (z,w) \leftrightarrow x < z \vee (x = z \wedge y < w)$. I^ω is ordered with the lexicographic ordering. I.e., $(a_1,a_2,\ldots) <_{lex} (b_1,b_2,\ldots) \leftrightarrow (\exists k \geq 0)((a_1,\ldots,a_k) = (b_1,\ldots,b_k) \wedge a_{k+1} < b_{k+1})$. I# is $\omega_1 \times [0,1)$ under the lexicographic ordering. I#' is I# with a right endpoint added.

THEOREM 1.6. I^2,I^ω,I#,I#' are complete dense linear orderings. I,I^2,I^ω,I#' have both endpoints, and I# has a left but not a right endpoint. I^2,I^ω,I#,I#' are not separable.

Proof: (0,0) and (1,1) are obviously the left and right endpoints of I^2. For density, let $(a,b) <_{lex} (c,d)$. If $a < c$ then $(a,b) <lex ((a+c)/2,0) < (c,d)$. If $a = c$ then $(a,b) <_{lex} (c,(b+d)/2)$. For completeness, let $S \subseteq I^2$ be nonempty. Let c be the sup of the first coordinates of the elements of S. If c is a strict sup then the sup of S is (c,0). If c is max(S) then the sup of S is (c,d), where d is the sup of $\{x: (c,x) \in S\}$. I^2 is not separable because the uncountably many nonempty open intervals ((a,0),(a,1)) are pairwise disjoint.

(0,0,...) and (1,1,...) are obviously the left and right endpoints of I^ω. For density, let $x <_{lex} y$. Write $x = (a_1,\ldots,a_k,b,\ldots)$, $y = (a_1,\ldots,a_k,c,\ldots)$, $k \geq 0$, $b < c$. Then $x <_{lex} (a_1,\ldots,a_k,(b+c)/2,0,0,\ldots) <_{lex} y$. For completeness, let $S \subseteq I^\omega$ be nonempty. Define a_i, $i \geq 1$, inductively as follows. a_1 is the sup of the first coordinates of elements of S. a_{i+1} is the sup of the elements of S that begin with $a_1,...,a_i$ if this sup is nonempty; undefined otherwise. If each a_i exists, then the sup of S is $(a_1,a_2,...)$. Suppose i is greatest such that a_i exists. Then the sup of S is $(a_1,..., i,0,0,...)$. Iω is not separable because the uncountably many nonempty open intervals $((a,0,0,...),(a,1,1,...))$ are pairwise disjoint.

(0,0) is obviously the left endpoint of I#, and I# has no right endpoint. For density, let (α,c) <lex (β,d). If $\alpha < \beta$ then (α,c) <lex $(\alpha,(c+1)/2)$ <lex (β,d). If $\alpha = \beta$ then (α,c) <lex $(\beta,(c+d)/2)$ <lex (β,d). For completeness, let $S \subseteq$ I# be nonempty with an upper bound (α,c). Let $\beta \leq \alpha$ be the least first coordinate of the elements of S. Let $d = \sup\{x: (\beta,x) \in S\}$. If $d < 1$ then the sup of S is (β,d). If $d = 1$ then the sup of S is $(\beta+1,0)$. I# is not separable because the uncountably many nonempty open intervals $((\alpha,0),(\alpha,1))$ are pairwise disjoint.

I#' is I# with the right endpoint adjoined, and so have both endpoints. I#' is not separable since I# is not separable. QED

DEFINITION 1.10. A permutation of D is a one-one map from D onto D. Let $f:X \to Y$, $X,Y \subseteq D$. f is increasing (decreasing) if and only if for all $x < y$ from dom(f), $f(x) \leq f(y)$ ($f(x) \geq f(y)$). f is monotone if and only if f is increasing or decreasing. f is strictly increasing (strictly decreasing) if and only if for all $x < y$ from dom(f), $f(x) < f(y)$ ($f(x) > f(y)$). f is strictly monotone if and only if f is strictly increasing or strictly decreasing.

We will use the following important open subset of D^2.

DEFINITION 1.11. $D^{2<} = \{x \in D^2: x_1 < x_2\}$.

THEOREM 1.7. $D^{2<}$ is an open subset of D^2 under the product topology.

Proof: Recall that the basic open sets in the product topology D^2 are the sets $A \times B$, where A,B are open sets in D. $D^{2<}$ is the union of basic open sets, since it is the union of all $J_1 \times J_2$, where J_1,J_2 are open intervals in D with every element of J_1 smaller than every element of J_2. QED

2. CONTINUOUS FUNCTIONS

We present some details of the essentially known theory of continuous functions on D that agrees with much of the usual theory of continuous functions on the standard D. Here D is always endowed with the interval topology of Definition 1.7, and D^2 is always endowed with the usual product topology. We use the usual notions of subspace topology, compact topology, connected topology, compact set, connected set, and continuous function from general topology. Specifically, a compact set in D is a set where the subspace topology is compact, and a connected set in D is a set where the subspace topology is connected. A continuous function f:X → D, where X ⊆ D, is a function that is continuous from the subspace topology X into D. Here f:X → D is continuous if and only if the inverse image of every open set in D is an open set in X; i.e., an open set in the subspace topology X of D. We use that the image of a continuous function on a compact space is compact, and the image of a continuous function on a connected space is connected.

THEOREM 2.1. Let J ⊆ D be an interval with at least two elements. The subspace topology of J is the same as the interval topology of J as a complete dense linear ordering.

Proof: Let J,D be as given. Then J is a complete dense linear ordering. This follows from the fact that the relatively open sets in J are exactly the intersection with J of the relatively open sets in D. QED

THEOREM 2.2. Let f:X → D, X ⊆ D. Then f is continuous if and only if the inverse image of every relatively open interval in D is the intersection of an open subset of D with X.

Proof; This follows immediately from the fact that the inverse image of a union is the union of the inverse images. QED

THEOREM 2.3. D is compact if and only if D has two endpoints. An interval J ⊆ D is compact if and only if J = ∅ or inf(J),sup(J) ∈ J.

Proof: Let D have left and right endpoints a < b. Let X be an open covering of (D,<); i.e., a covering of (D,<) by open sets. Let S = {x ∈ D: some finite subset of X covers [a,x]}. Note that a ∈ S and b is an upper bound of S. Let c be the least upper bound of S. Using an element of X that contains c, it is easy to see that c ∈ S. If c is not the right endpoint, then there exists d > c also lying in S. Hence c is the right endpoint, and so [a,b] = D has a finite subcover.

Now suppose D is missing the left endpoint. For each x ∈ D let J_x be a relatively open interval with x ∈ J_x. The J_x forms a cover of D. Clearly each J_x has no left endpoint. In any finite subcover of D, there is a least left endpoint. Thus there is no finite subcover of D. The analogous argument works if D is missing the right endpoint.

Let J be an interval in D. If |J| ≤ 1 then J is obviously compact, and inf(J),sup(J) ∈ J. Now assume |J| ≥ 2. Hence Theorem 2.1 applies. Now apply the first claim. QED

LEMMA 2.4. D is connected. Every interval in D is connected.

Proof: Let D = A ∪ B, where A,B are nonempty open sets in D and disjoint.

case 1. sup(A) ∈ A. Since A is the union of open intervals in D, sup(A) lies in one of those open intervals, and hence sup(A) = max(D). Suppose sup(B) ∈ B. Since B is the union of open intervals in D, sup(B) = max(D), which is impossible. Hence sup(B) ∉ B. Therefore sup(B) ∈ A. This contradicts that A,B are disjoint and nonempty.

case 2. sup(B) ∈ B. Symmetric with case 1.

case 3. sup(A) ∉ A. Then sup(A) ∈ B. This violates A,B are disjoint and open.

case 4. sup(B) ∉ A. Then sup(B) ∈ B. This violates A,B are disjoint and open.

Let J be an interval in D. If |J| ≤ 1 then J is obviously connected. If |J| ≥ 2, then Theorem 2.1 applies. Now apply the first claim. QED

THEOREM 2.5. The connected subsets of D are exactly the intervals in D.

Proof; Let $J \subseteq D$ be not an interval. Let $a < b < c$, $a,c \in J$, $b \notin J$. Then $\{x \in D: x < b\} \cap J$ and $\{x \in D: x > b\} \cap J$ are disjoint, nonempty, and open in J. Thus J is not connected. QED

THEOREM 2.6. (Intermediate/Maximum/Minimum Value Theorem) Let $f:J \rightarrow D$ be continuous, where J is an interval. rng(f) is an interval. If $\inf(J),\sup(J) \in J$ then rng(f) is an interval J' with $\inf(J'),\sup(J') \in J'$.

Proof: Let f,J,D be as given. We can assume $|J| \geq 2$. By Theorem 2.5, J is connected, and hence rng(f) is connected. By Theorem 2.5, rng(f) is an interval. Suppose $\inf(J),\sup(J) \in J$. By Theorem 2.3, J is compact, and so rng(f) is compact. By Theorem 2.3, rng(f) contains its inf and sup. QED

LEMMA 2.7. Let f be monotone, where $f:I \rightarrow J$ is onto, I,J intervals in D. Then the inverse image of every interval in J is an interval in I.

Proof: Let f,I,J be as given, and let J' be an interval in J. Let $x < y$ be from $f^{-1}(J')$. Let $x < z < y$. Then $f(x) < f(z) < f(y)$ or $f(x) > f(z) > f(y)$. Hence $f(z) \in J'$, and so $z \in f^{-1}(J')$. QED

LEMMA 2.8. Let f be monotone, where $f:I \rightarrow J$ is onto, I,J intervals in D with at least two elements.
i. The inverse image of every $(a,b) \subseteq J$ with at least two elements is a $(c,d) \subseteq I$ with at least two elements.
ii. The inverse image of every [min(J),b)] with at least two elements is a [min(I),d)] or [(c,max(I)] with at least two elements.
ii. The inverse image of every [(a,max(J)] with at least two elements is a [(c,max(I)] or [min(I),d)] with at least two elements.
iii. The inverse image of J is I.

Proof: Let f,I,J be as given, and $(a,b) \subseteq J$. By Lemma 2.7, $f^{-1}((a,b))$ is an interval in I.

case 1. $f^{-1}((a,b)) = [c,d)]$. Let $a < f(c) < b$. Let $a < f(x) < f(c) < f(y) < b$. where $x,y \in I$. Then $x,y \in f-1((a,b)) = [c,d)]$. If f is increasing then $x < c$, and if f is decreasing then $y < c$. This is a contradiction.

case 2. $f^{-1}((a,b)) = [(c,d]$. As in case 1.

For ii, let $f^{-1}([\min(J),b)] = [(c,d)]$. Let $f(x) = \min(J)$. If f is increasing then $f(y) = f(x)$ for all $y \leq x$ from I. Hence $[(c,d)] = [\min(I),d)]$. If f is decreasing then $f(y) = f(x)$ for all $y \geq x$ from I. Hence $[(c,d)] = [(c,\max(I)]$.

For iii, see ii. iii is immediate. QED

THEOREM 2.9. Let f be monotone where dom(f) is an interval in D. The following are equivalent.
i. f is continuous.
ii. rng(f) is an interval in D.

Proof: Let $f:I \rightarrow D$ be monotone, where I is an interval in D. Suppose f is continuous. By Theorem 2.5 and that continuous functions preserve connectivity, we see that rng(f) is an interval in D.

Now rng(f) is an interval J in D. Hence $f:I \rightarrow J$ is monotone and onto. By Lemma 2.8, the inverse image of every relatively open interval in J is a relatively open interval in I. Hence the inverse image of every open set in D is an open set in I. I.e., f is continuous. QED

LEMMA 2.10. Let $f:[a,b] \rightarrow D$, $[a,b]$ in D. The following are equivalent.
i. f is one-one and continuous.
ii. f is strictly monotone and continuous.
iii. f is strictly monotone onto an interval.

Proof: Let f,a,b,D be as given. ii $\leftrightarrow$ iii by Theorem 2.9. ii $\rightarrow$ i is trivial. Now assume f is one-one and continuous. Let $\mathrm{rng}(f) = [c,d]$. We can assume that $a < b$ and $c < d$.

Let $f(x) = c$ and $f(y) = d$. Then f maps $[x,y]$ onto $[c,d]$ or $[y,x]$ onto $[c,d]$. Since f is one-one, $[x,y]$ or $[y,x]$ is $[a,b]$.

case 1. $f(a) = c$ and $f(b) = d$. Suppose f is not strictly increasing. Let $a \leq a' < b' \leq b$ where $f(a') > f(b')$. Then a

$< a' < b' < b$. Also f maps $[b',b]$ onto an interval that includes $[f(b'),d]$, and therefore includes $f(a')$. This violates that f is one-one.

case 2. $f(a) = d$ and $f(b) = c$. Suppose f is not strictly decreasing. Let $a \leq a' < b' \leq b$ where $f(a') < f(b')$. Then $a < a' < b' < b$. Also f maps $[b',b]$ onto an interval that includes $[f(b),f(b')]$, and hence $f(a')$. This violates that f is one-one.

QED

THEOREM 2.11. Let $f:I \to D$, I an interval in D. The following are equivalent.
i. f is one-one and continuous.
ii. f is strictly monotone and continuous.
iii. f is strictly monotone onto an interval.

Proof: If $I = [a,b]$ then this is Lemma 2.10.

case 1. $I = [a,b)$, $a < b$. ii $\leftrightarrow$ iii by Theorem 2.9. ii $\to$ i is trivial. Assume f is one-one and continuous. Apply Lemma 2.10 to each $[a,c]$, $a \leq c < b$. Then f is strictly monotone on these $[a,c]$. Therefore f is strictly monotone on $[a,b)$. QED

THEOREM 2.12. (Inverse Function Theorem) Let $f:J \to D$ be one-one and continuous. Then $f^{-1}:J' \to J$ is one-one and continuous, where J' is the interval rng(f).

Proof: Let f,J,D be as given. Then rng(f) is an interval J'. By Theorem 2.11, f is strictly monotone. Hence f^{-1} is strictly monotone from J' onto J. By Theorem 2.11, f^{-1} is continuous. QED

THEOREM 2.13. Let $f:D \to D$ be a permutation. The following are equivalent.
i. f is a monotone permutation.
ii. f is a continuous permutation.

Proof: Let f be as given. Suppose f is monotone. Then f is strictly monotone and onto an interval. By Theorem 2.11, f is continuous. On the other hand, suppose f is continuous. By Theorem 2.11, f is strictly monotone. QED

It is often easier to think of "epsilon/delta" rather than general topology with its open sets.

THEOREM 2.14. Let D have no endpoints, $A \subseteq D$, and $f:A \to D$. The following are equivalent.
1. f is continuous.
2. For all $x \in A$ and (a,b) containing f(x), f maps some (c,d), containing x, into (a,b).

Proof: Let A,D,f be as given. Since D has no endpoints, there is only one notion of open interval in D. Suppose f is continuous. Let $a < f(x) < b$. Then $f^{-1}((a,b))$ is open in A, and hence is $V \cap A$, for some open V in D. Hence $x \in V$. Since V is the union of open intervals in D, let $x \in I$, where I is an open interval in D. We can shrink I to some $x \in (c,d) \subseteq I$ so that 2 holds.

Now suppose 2 holds. Let J be an open interval in D. We show that $f^{-1}(J)$ is open in A. It suffices to show that every $x \in f^{-1}(J)$ is

*) an element of an open interval in D that is mapped by f into J.

Let $f(x) \in J$. If 2 holds then x obeys *). QED

LEMMA 2.15. Let D have no endpoints, $f:[a,b] \to D$ be continuous, and $rng(f) \subseteq [a,b]$, $a < b$. Then f has a fixed point.

Proof: Let f,D,a,b be as given. Since D has no endpoints, there is only one notion of open interval in D. Let $S = \{x \in [a,b]: x < f(x)\}$. Let $c = sup(S)$. Then $a \leq c \leq b$.

case 1. $c < f(c)$. Hence $c < b$. Let $c < c' < f(c) < d$. Then f maps some open interval containing c into (c',d). Hence let $c < e < c' < f(e)$. This violates $c = sup(S)$.

case 2. $c > f(c)$. Hence $c > a$. Let $c > c' > f(c) > d$. Then f maps some open interval containing c into (d,c'). Hence let $c > e > c' > f(e)$. Then (e,c) is disjoint from S. This violates $c = sup(S)$.

So the only remaining possibility is $c = f(c)$. QED

THEOREM 2.16. (Fixed Point Theorem) Suppose D has both endpoints. Every continuous $f:D \to D$ has a fixed point.

Proof: Let D have endpoints $a < b$. Let $f:D \to D$ be continuous. We can trivially extend D to D' by adding stuff at the ends, so that D' has no endpoints. Then f remains continuous, and we can apply Lemma 2.15 to obtain a fixed point for f. QED

THEOREM 2.17. Every continuous $f:I\# \to I\#$ has a fixed point (even though I# has no right endpoint). This fails for I,I^2,I^∞ with one or two endpoints removed.

Proof: Let $f:I\# \to I\#$ be continuous. We claim that f is bounded above on each $[(0,0),x]$. Suppose this is false, and let $S = \{x\text{: f is bounded above on } [(0,0),x]\}$. Let $c = \sup(S)$. By the continuity of f, f is bounded on some nonempty open interval (d,e) containing c. But then f is bounded on $S \cup (d,e)$, and hence $e \in S$, contradicting the choice of c.

By iteration of f, using the claim, we construct intervals $[(0,0),(\alpha_1,0)) \subseteq [(0,0),(\alpha_2,0)) \subseteq \ldots$, where $\alpha_1 < \alpha_2 < \ldots$, and f maps each interval into the next. Hence f maps some $[(0,0),(\lambda,0)$ into $[(0,0),(\lambda,0)$, and by the continuity of f, maps some $[(0,0),(\lambda,0)]$ into $[(0,0),(\lambda,0)]$. Now apply Theorem 2.16 to the ordering $[(0,0),(\lambda,0)]$.

We now come to I. With one or two endpoints removed, rewrite as $\Re$, $[0,\infty)$, $(-\infty,0]$. Use $f(x) = x+1$, $f(x) = x+1$, $f(x) = x-1$, respectively.

We now consider I2. We first consider $I^2\backslash\{(1,1)\}$. Let f be constantly (1,1/2) on $[(0,0),(1,0)]$. Otherwise, $f(1,x) = (1,(x+1)/2)$. This works analogously for $I^2\backslash\{(0,0)\}$. The first f also works for I^2 with both endpoints removed.

The case of I^∞ is most informatively handled in section 3 where we discuss the homogeneity of I^∞. QED

3. HOMOGENEITY

We now show that I^∞ has some very good properties.

DEFINITION 3.1. D is homogenous if and only if

i. (D,<) and (D,>) are isomorphic.
ii. Any two intervals with both endpoints are isomorphic.

THEOREM 3.1. Let D be homogenous. Let $a_1 < \ldots < a_k$ and $b_1 < \ldots < b_k$, where $k \geq 1$, and every endpoint of D is some a_i and some b_j. There exists a strictly increasing permutation of D that maps each a_i to b_i. Every $x \in D$ other than the left (right) endpoint is the inf (sup) of a strictly decreasing (increasing) ω sequence.

LEMMA 3.2. I^ω is isomorphic to its reverse. Every element of $I\omega$ that is not the left (right) endpoint is the sup (inf) of a strictly increasing (decreasing) sequence. In these strictly monotone sequences, we can require that the elements of I^ω are eventually in (0,1).

Proof: The last claim will be used in Lemma ? below.

For the first claim, map $(a_1,a_2,\ldots)$ to $(1-a_1,1-a_2,\ldots)$. Now let $(a_1,a_2,\ldots)$ be given, not the left endpoint.

case 1. The a's are eventually 0. Let i be greatest such that $a_i > 0$. Use $(a_1,\ldots,a_{i-1},x_1,x_1,,\ldots)$, $(a_1,\ldots,a_{i-1},x_2,x_2,\ldots)$, $(a_1,\ldots,a_{i-1},x_3,x_3,\ldots)$, ..., where $0 < x_1 < x_2 < x_3 \ldots$ converges to a_i.

case 2. The a's are not eventually 0. Use $(a_1,a_2/2,a_3/2,\ldots)$, $(a_1,a_2,a_3/2,a_4/2,\ldots)$, $(a_1,a_2,a_3,a_4/2,a_5/2,\ldots)$, ..., and delete repetitions.

Now let$(b_1,b_2,\ldots)$ be given, not the right endpoint. We can either proceed analogously or invoke the first claim. QED

LEMMA 3.3. Any two nondegenerate closed intervals in $I\omega$ are isomorphic.

Proof: Let x <lex y. Write $x = (a_1,\ldots,a_k,b,z_1,z_2,\ldots)$ and $y = (a_1,\ldots,a_k,c,w_1,w_2,\ldots)$, $k \geq 0$, and $b < c$, and assume that x,y are eventually in (0,1). We now partition [x,y] into a union of subintervals, ordered left to right, presented in five lines, as follows:

$\{x\} \cup$
$\ldots \cup (\{(a_1,\ldots,a_k,b,z_1)\} \times (z_2,1] \times I^\omega) \cup (\{(a_1,\ldots,a_k,b)\} \times (z_1,1] \times I\omega)$

$\cup$ $(\{(a_1,\ldots,a_k)\} \times (b,c) \times I^\omega)$
$\cup$ $(\{(a_1,\ldots,a_k,c)\} \times [0,w_1) \times I^\omega) \cup (\{(a_1,\ldots,a_k,c)\} \times [0,w_2) \times I^\omega) \cup \ldots$
$\cup$ $\{y\}$

Note that some of the terms in lines 2,4 may be empty. But since x,y are eventually in (0,1), all but finitely many terms in 2,4 are nonempty. Note also that the nonempty terms in line 2 have order type $(0,1] \times I^\omega$, the nonempty terms on line 4 have order type $[0,1) \times I^\omega$, and the term on line 3 has order type $(0,1) \times I^\omega$. So the order type of [x,y] is

$\theta = 1 + ((0,1] \times I^\omega) + ((0,1] \times I^\omega) + \ldots + ((0,1) \times I^\omega) + ([0,1) \times I^\omega) + ([0,1) \times I^\omega) + \ldots + 1.$

We are now ready to prove the claim. Let x <lex y and z <lex w. By Lemma 1.6 write

$x < \ldots u_{-1} < u_0 < u_1 < \ldots < y$
$z < \ldots v_{-1} < v_0 < v_1 < \ldots < w$

where x (z) is the inf of the u's (v's), y (w) is the sup of the u's (v's), and the u's,v's are each eventually in (0,1).

By the above, each interval $[u_i,u_{i+1}]$ is isomorphic to $[v_i,v_{i+1}]$. It follows that [x,y] is isomorphic to [z,w] as required. QED

THEOREM 3.4. I^ω is nonseparable and homogenous.

Proof: By Lemmas 3.2 and 3.3. QED

Theorem 3.4 indicates that separability is not going to follow from merely imposing some strong symmetry conditions.

The rest of this paper is devoted to conditions do imply separability, and hence isomorphic with [0,1], with or without endpoints.

4. BETWEEN FUNCTIONS

DEFINITION 4.1. D' is D without its endpoints. $A^{2<} = \{(x,y): x < y \text{ and } x,y \in A\}$, $A \subseteq D$.

From general topology, D^2 is given the product topology, and therefore we can consider continuous $f:X \to D$, where $X \subseteq D^2$.

LEMMA 4.1. Let $f:X \to D$ be continuous, $X \subseteq D'^2$. For all $f(x,y) \in (a,b)$ there exists $x \in (c,d)$, $y \in (c',d')$ such that f maps $(c,d) \times (c',d')$ into (a,b).

Proof: Let f,X,D be as given. Let $f(x,y) \in (a,b)$. Then $f^{-1}((a,b))$ is open in X. Write $f^{-1}((a,b)) = V \cap X$ where V is open in D^2. Then $(x,y) \in W \cap X$, where $W \subseteq V$ is a basic open set in D^2. Write $(x,y) \in W' \subseteq W \subseteq V$, where W' is a finite intersection of Cartesian products of relatively open intervals in D. Hence we can write $(x,y) \in (c,d) \times (c',d') \subseteq V$, using that $(x,y) \in D'^2$. QED

DEFINITION 4.2. BE is a between function for D if and only if $BE:D'^{2<} \to D$, where for all $x < y$ from D', $x < BE(x,y) < y$.

Evidently between functions map into D'.

LEMMA 4.2. Let D have a continuous between function. Every $[a,b] \subseteq D$, $a,b \in D'$, is separable.

Proof: Let D have continuous between function $BE:D'^{2<} \to D$, and $a < b$ be from D'. Let S be the least set containing a,b and closed under f. We will prove that S is dense in [a,b]. Suppose not, and let $(a',b') \subseteq [a,b]$ be disjoint from S, with $a' < b'$. Let (α,β) be a maximal open subinterval of (a,b) containing (a',b') that is disjoint from S. Clearly $a \leq \alpha < \beta \leq b$.

We now apply Lemma 4.1 to $BE(\alpha,\beta)$, which lies in the open interval (α,β). Let I be a nonempty open subinterval of (α,β). Let J be an open interval about α and J' be an open interval about β such that f maps $J \times J'$ into I. Clearly there are elements of J and elements of J' that lie in S because of the choice of α,β. Therefore f maps a pair of elements of S into I, and so S meets I. This is a contradiction. QED

LEMMA 4.3. Let D have no endpoints, with a continuous between function. D has a strictly increasing sequence with no upper bound.

Proof: Let D have no endpoints, with continuous between function $BE:D'^{2<} \rightarrow D$. Suppose there is no strictly increasing sequence without an upper bound. Let $c[\alpha]$, $\alpha < \omega 1$, be strictly increasing. We can assume that for all $\lambda < \omega 1$, $c[\lambda] = \sup\alpha<\lambda c[\alpha]$. I.e., $\{c[\alpha]: \alpha < \omega 1\}$ is closed in D. Here and henceforth, λ always denotes a limit ordinal.

For $\alpha < \lambda < \omega 1$, let $g(\alpha,\lambda)$ be the least β such that $BE(c[\alpha],c[\lambda]) \leq c[\beta]$. Obviously each $g(\alpha,\lambda) \in (\alpha,\lambda)$.

By Fodor's theorem on $\omega 1$, for all $\alpha < \omega_1$, there exists $\alpha' > \alpha$ such that for uncountably many λ, $g(\alpha,\lambda) = \alpha'$. Hence for all $\alpha < \omega 1$, there exists $c[\alpha'] > c[\alpha]$ such that for uncountably many $c[\lambda]$, $BE(c[\alpha],c[\lambda]) < c[\alpha']$.

Using the continuity of BE, we see that for all $\alpha < \omega_1$, there exists $\alpha' > \alpha$ such that $\{c[\lambda]: BE(c[\alpha],c[\lambda]) \leq c[\alpha']\}$ is uncountable and closed in D. Hence for all $\alpha < \omega 1$, there exists $\alpha' > \alpha$ such that $\{\lambda: BE(c[\alpha],c[\lambda]) \leq c[\alpha']\}$ is closed and unbounded in ω_1.

It is now clear by induction that there are $\tau_0 < \tau_1 < \ldots < \omega_1$ and sets $A_0 \supseteq A_1 \supseteq \ldots$ closed and unbounded in ω_1 such that the following holds.

i. $\tau_0 = 0$.
ii. For all $i \geq 0$ and $\lambda \in A_i$, $M(c[\tau_{2i}],c[\lambda]) \leq c[\tau_{2i+1}]$.
iii. For all $i \geq 1$, $\tau_{2i} = \min(A_{i-1})$.

Let κ be a limit ordinal in $\cap_i A_i$ that is greater than $\sup_i \tau_i$. Then for all $i \geq 0$, $f(c[\tau_{2i}],c[\kappa]) \leq c[\tau_{2i+1}]$. By the continuity of BE, $BE(\sup i\tau i,c[\kappa]) \leq \sup_i \tau_i$. This contradicts that BE is a between function. QED

THEOREM 4.4. Every D with a continuous between function is separable, and therefore isomorphic to a standard D.

Proof: Let D have a continuous between function. By Lemma 4.3, D' has a strictly increasing sequence with no upper bound. By reversing the order or trivially redoing the proof, D' also has a strictly decreasing sequence with no lower bound. By Lemma 4.1, the intervals between the terms in these sequences are separable. Hence all of D' is separable. Hence D is separable. QED

This proof essentially appeared in [Fr05].

5. STRICTLY INCREASING BINARY FUNCTIONS

DEFINITION 5.1. Let $f:D'^2 \to D$. f is strictly increasing if and only if for all $x,y,y' \in D'$, $y < y' \to f(x,y) < f(x,y') \wedge f(y,x) < f(y',x)$. f is regular if and only if for all $x \in D'$, f(x,y) and f(y,x) are strictly increasing from D' onto intervals in D.

THEOREM 5.1. $f:D'^2 \to D$ is regular if and only if it is continuous strictly increasing.

Proof: By Theorem 2.11. QED

LEMMA 5.2. Let $f:D'^2 \to D$ be continuous and strictly increasing. Let $g:D'^2 \to D$ be such that for all $x,y \in D'$, $g(x,y) = z$ if and only if $f(x,y) = f(z,z)$. g is well defined, continuous, and $g|D'^{2<}$ is a continuous between function on D.

Proof: Let D,f be as given. Let $x,y \in D'$. Since f(x,y) is between f(x,x) and f(y,y), by Theorem 2.11, there exists z between x,y such that $f(z,z) = f(x,y)$, and z is obviously unique. It now suffices to prove that g is continuous. Let $f(x,y) = f(z,z)$ and $a < z < b$, where a,b in D'. By the continuity of f, let $c < x < d$ and $e < z < f$, $c,d,e,f \in D'$, where f maps $(c,d) \times (e,f)$ into $(f(a,a),f(b,b))$. Then g maps $(c,d) \times (e,f)$ into (a,b). QED

THEOREM 5.3. Every D with a continuous strictly increasing $f:D'^2 \to D$ (or regular $f:D'^2 \to D$) is separable, and therefore isomorphic to a standard D.

Proof: By Theorem 4.4 and Lemma 5.2. QED

6. MIDPOINT FUNCTIONS

We have kept the domains of the hypothesized functions in sections 4,5 limited to D'^2 as that is all that is needed to imply separability. Now that this is accomplished, we work with functions from all of D^2 into D which have rather obvious properties.

DEFINITION 6.1. M is a midpoint function on D if and only if $M:D^2$ into D, M(x,x) = x, M(x,y) = M(y,x), and for all x, M(x,y) is strictly increasing and onto an interval.

THEOREM 6.1. Let M be a midpoint function on D. M is continuous, with $x < y \rightarrow x < M(x,y) < y$. $M|D'^2$ is a regular function on D. $M|D'^{2<}$ is a continuous between function on D. In particular, D is separable.

Proof: Left to the reader. QED

DEFINITION 6.2. Two midpoint functions M,M' on D are isomorphic if and only if the structure (D,<,M), (D,<,M') are isomorphic.

Are midpoint functions on [0,1] isomorphic? If not, can be put a condition on them so that any two obeying that condition are isomorphic?

There are a couple of laws that the regular midpoint function (x+y)/2 on [0,1] obey:

P1. M(x,M(y,z)) = M(M(x,y),M(x,z)).
P2. Let a < b < c < d < e < f, M(a,c) = b, M(d,f) = e. Either M(a,f), M(b,e), M(c,d) are distinct, or they are the same.
Let a < b < c < d, M(a,c) = b and M(b,d) = c. Then M(a,d) = M(b,c).

Obviously P1 is mathematically much simpler, being just an equation. However both parts of P2 are easier to "see".

THEOREM 6.2. Let M(x,y) = (x+y)/2 on $[0,1]^2$. M is a regular midpoint function on [0,1] obeying P1,P2.

Proof: For P1, note that the left side is (x+((y+z)/2))/2 and the right side is ((x+y)/2+(x+z)/2)/2, which are both (2x+y+z)/4. For P2, assume hypotheses. Then we are looking at the three points (a+f)/2, (b+e)/2, (c+d)/2. It suffices to show that (b+e)/2 is the average of (a+f)/2 and (c+d)/2. To see this, we need to show that b+e = (a+f+c+d)/2. By the hypotheses, a+c = 2b and d+f = 2e. So we need to verify that b+e = (2b+2e)/2, which is evident.

Finally suppose a < b < c < d, a-b = b-c, b-c = c-d. Then a+d = b+c. QED

LEMMA 6.3. f:[0,1] → [0,1] be continuous and strictly increasing, with 0 < x ≤ 1 → 0 < f(x) < x. Define g:$[0,1]^2$ → $[0,1]^2$ except at (0,0) by M(x,y) = f(x)(1-(y/x))+y if 0 ≤ y ≤ x; M(y,x) otherwise. Then M is a midpoint function on $[0,1]^2$.

Proof: Let f,g be as given. M(x,x) = x by inspection. Note that M(x,y) = M(min(x,y),max(x,y)). Therefore M(x,y) = M(y,x). For fixed y, M(x,y) is strictly increasing in x ≥ y by inspection. For fixed y, M(x,y) is strictly increasing in y ≤ x because f(y)(1-(x/y))+x is strictly increasing in y ≤ x also by inspection. Hence for fixed y, M(x,y) is strictly increasing. Hence for fixed y, M(y,x) is strictly increasing. Hence M is strictly increasing.

For y < x, f(x)(1-(y/x))+y < x(1-(y/x))+y = x. Also for y < x, f(x)(1-(y/x))+y > y since f(x) > 0. So for y < x, y < M(x,y) < x. It is also clear that M is continuous on $[0,1]^2_{\geq}$, and on $[0,1]^2_{\leq}$, and therefore on [0,1]2. Therefore M maps onto an interval if we fix any argument. This verifies that M is a midpoint function on [0,1]2. QED

LEMMA 6.4. Let f be as in Lemma 6.3 with f(1/2) = 1/3, f(3/4) = 2/5, f(1) = 1/2. Then M(1/2,1) = 3/4, M(0,3/4) = 2/5. Also M(0,1/2) = 1/3, M(0,1) = 1/2, M(1/3,1/2) = 4/9. M(0,M(1/2,1) ≠ M(M(0,1/2),M(0,1)).

THEOREM 6.5. There is a midpoint function on [0,1] that does not obey P1. It must not be isomorphic to the midpoint function (x+y)/2. Furthermore, we can take the midpoint function to be piecewise linear.

Proof: By Lemmas 6.2, 6.3, 6.4. QED

THEOREM 6.6. Any midpoint function on [0,1] satisfying P1 is isomorphic to (x+y)/2. The same holds for P2.

Proof: Let M be a midpoint function on [0,1] satisfying P1. For each k ≥ 0, we define 0 = x[k,0] < ... < x[k,2^k] = 1 as follows. Set 0 = x[0,0] < x[0,1] = 1. Suppose 0 = x[k,0] < ... < x[k,2^k] = 1 has been defined. Set x[k+1,0],...,x[k+1,2^{k+1}] to be

x[k,0],M(x[k,0],x[k,1]),x[k,1],M(x[k,1],x[k,2]),x[k,2],...,
x[k,2^k-1],M(x[k,2^k-1],x[k,2^k]),x[k,2^k] = 1.

Let k ≥ 0. We claim that for all 0 ≤ i ≤ 2^k-2, M(x[k,i],x[k,i+2]) = x[k,i+1]. This is proved by induction on k as follows. This is clearly true for k = 0. Suppose this is true for fixed k ≥ 0, and let 0 ≤ i ≤ 2^{k+1}-2. M(x[k+1,i],x[k+1,i+2]) = x[k+1,i+1] is immediate by construction if i is even. Now assume i is odd. We want M(x[k+1,i],x[k+1,i+i+2) = x[k+1,i]. The left side is M(M(x[k,(i-1)/2],x[k,(i+1)/2]),M(x[k,(i+1)/2],x[k,(i+3)/2])). By P1, this is M(x[k,(i+1)/2],M(x[k,(i-1)/2],x[k,(i+3)/2])) = x[k,j+1], using the induction hypothesis.

We now claim that for all 0 ≤ i ≤ j ≤ 2^k, M(x[k,i],x[k,j]) = x[k+1,i+j]. The basis case k = 0 is trivial. Now fix k ≥ 0 and assume that for all 0 ≤ i ≤ j ≤ 2^k, M(x[k,i],x[k,j]) = x[k+1,i+j]. We now verify that for all 0 ≤ i ≤ j ≤ 2^{k+1}, M(x[k+1,i],x[k+1,j]) = x[k+2,i+j]. Let 0 ≤ i ≤ j ≤ 2^{k+1}.

First suppose i,j are even. By the induction hypothesis, M(x[k+1,i],x[k+1,j]) = M(x[k,i/2],x[k,j/2]) = (x[k+1,(i+j)/2] = x[k+2,i+j) as required.

Now suppose i is even and j is odd. By P1 and the induction hypothesis, M(x[k+1,i],x[k+1,j]) = M(x[k,i/2],M(x[k,(j-1)/2],x[k,(j+1)/2]) = M(M(x[k,i/2],x[k,(j-1)/2]),M(x[k,i/2],x[k,(j+1)/2])) = M(x[k+1,(i+j-1)/2],x[k+1,(i+j+1)/2]), which is x[k+2,i+j) by the previous claim.

Next suppose i is odd, j is even. By P1 and the induction hypothesis, M(x[k+1,i],x[k+1,j]) = M(M(x[k,(i-1)/2],x[k,(i+1)/2]),x[k,j/2]) = M(M(x[k,(i-1)/2],x[k,j/2]),M(x[k,(i+1)/2],x[k,j/2])) = M(x[k+1,(i+j-1)/2],x[k+1,(i+j+1)/2]), which is x[k+2,i+j) by the previous claim..

Finally suppose i,j are odd. By P1 and the induction hypothesis, M(x[k+1,i],x[k+1,j]) = M(x[k+1,i],M(x[k,(j-1)/2]),x[k,(j+1)/2])) = M(M(x[k+1,i],x[k,(j-1)/2]),M(x[k+1,i],x[k,(j+1)/2])) = M(M(x[k+1,i],x[k+1,j-1]),M(x[k+1,i],x[k+1,j+1])). Applying the previous paragraph, since i is odd and j-1 is even, this is

M(x[k+2,i+j-1],x[k+2,i+j+1]), which is M(x[k+2,i+j]) by the previous claim. Thus the present claim is established.

It is now immediate that M at any two terms in x[k,0],...,x[k,2^k] is among x[k+1,0],...,x[k+1,2^{k+1}]. So the union S of all of the x[k,i], over k,i, is closed under M and contains 0,1. Therefore by the proof of Lemma 4.3, S is dense in [0,1].

We claim that x[k,i] = x[k',i'] if and only if $i/2^k = i'/2^{k'}$. This is clear if k = k'. The claim follows from the law x[k,i] = x[k+1,2i].

We claim that x[k,i] < x[k',i'] if and only if $i/2^k < i'/2^{k'}$. If k = k' then this is clear. Suppose k < k', and rewrite the inequality as x[k,i] = x[k',i*] < x[k',i'], where $i/2^k = i^*/2^{k'}$. This inequality is equivalent to i* < i' which in turn is equivalent to $i^*/2^{k'} < i'/2^{k'}$ or $i/2^k < i'/2^{k'}$ as required.

We claim that M(x[k,i],x[k',i']) = x[k''+1,i''] if and only if $i/2^k + i'/2^{k'} = i''/2^{k''}$. To see this, we can rewrite the equation in the form M(x[k*,j],x[k*,j']) = x[k*,j''], where $j/2^{k^*} = i/2^k$, $j'/2^{k^*} = i'/2^{k'}$, and $j''/2^{k^*} = i''/2^{k''+1}$. By the above, this equation is equivalent to x[k*+1,j+j'] = x[k*+1,2j'']. It remains to show that j+j' = 2j''. Now $(j+j')/2^{k^*} = i/2^k + i'/2^{k'} = i''/2^{k''}$. Also $2j''/2^{k^*} = i''/2^{k''}$. Hence j+j' = 2j''.

This establishes an isomorphism from (S,<,M) onto (DY,<,(x+y)/2), where DY is the set of all dyadic rationals in [0,1]. Since S is dense in [0,1] and M is continuous, ([0,1],<,M) is isomorphic to ([0,1],<,(x+y)/2).

We now come to P2. We first show, using P2, that for $0 \le i \le 2^k-2$, M(x[k,i],x[k,i+2]) = x[k,i+1]. This is clearly true for k = 0. Assume true for fixed k ≥ 0. For even $0 \le i \le 2^{k+1}-2$, M(x[k+1,i],x[k+1,i+2]) = x[k+1,i+1] is by construction. Now let $0 \le i \le 2^{k+1}-2$ be odd. We want M(x[k+1,i],x[k+1,i+2]) = x[k+1,i+1]. Note that $0 \le i-1 < i+3 \le 2^{k+1}$. Look at

x[k+1,i-1] x[k+1,i] x[k+1,i+1] x[k+1,i+1] x[k+1,i+2] x[k+1,i+3].

M of the first and third terms is the second term, and M of the fourth and sixth terms is the fifth term, by construction. Also M of the first and sixth terms is M at the third and fourth terms by halving and using the induction hypothesis. Hence by P2, M of the second and fifth terms is x[k+1,i+1] as required.

We now claim that for all $0 \le i \le 2^k-3$, M(x[k,i+1],x[k,i+2] = M(x[k,i],x[k,i+3]). Let $0 \le i \le 2^k-3$, and look at

x[k,i] x[k,i+1] x[k,i+2] x[k,i+3]

Clearly M at the first and third terms is the second term, and M at the second and fourth terms is the third term. By P2, M at the first and fourth terms is M at the second and third terms.

Now we show that for $1 < i,j < 2^k$, M(x[k,i],x[k,j]) = M(x[k,i-1],x[k,j+1). Let i,j,k be as given. We define numbers a,b as follows. If i-j is even then a = b = (i+j)/2. Otherwise a = (i+j-1)/2, b - (i+j+1)/2. Look at the statements

M(x[k,a],x[k,b]) = M(x[k,a-1],x[k,b+1])
M(x[k,a-1],x[k,b+1]) = M(x[k,a-2],x[k,b+2])
...
M(x[k,i+1],x[k,j-1]) = M(x[k,i],x[k,j])
M(x[k,i],x[k,j]) = M(x[k,i-1],x[k,j+1)

It is clear that if any one of these lines, other than the last line, then the next line holds. This is because we can list the six distinct numbers used in the two lines, in strictly increasing order, and apply the first part of P2. Also, the first line comes directly from the first part of P2, since a = b or a+1 = b.

Thus we have M(x[k,i],x[k,j]) = M(x[k,a],x[k,b]) = x[k+1,a+b] = x[k+1,i+j] by construction, since a = b or a+1 = b.

We now argue exactly as we did from P1. QED

7. PERMUTATION SYSTEMS

DEFINITION 6.1. f,g form a permutation system if and only if
i. f,g are increasing and strictly dominating permutations.

ii. For all $x < y$ there exists k such that $fg^k x < g^k y$.

The idea is that for any $x < y$, some $g^k x < g^k y$ are far apart, as measured by f.

LEMMA 7.1. There is a permutation system on $\Re$.

Proof: On $(0,\infty)$, $f(x) = 2x$, $g(x) = 2x + x^2$ forms a permutation system. QED

LEMMA 7.2. Any D with a permutation system is separable.

Proof: Let D have the permutation system f,g. Let $\inf(D) < x < y < \sup(D)$. Let S be the least set closed under f,g,f^{-1},g^{-1}. We show that S is dense.

Suppose S is not dense, and let (a,b) be a maximal open interval disjoint from S. Let $fg^k a < g^k b$.

We claim that there is a greatest $n \geq 1$ such that $f^n g^k a < g^k b$. If there is no greatest such n then $\sup_n f^n g^k a$ is a fixed point of f, contrary to condition ii.

Let $n \geq 1$ be greatest such that $f^n g^k a < g^k b$. Then $g^{-k} f^n g^k a < b$. Since f is strictly dominating, $g^k a < f^n g^k a$, and so $a < g^{-k} f^n g^k a < b$. By the continuity of $f,g,g-1$, let $c < a$ be such that for all $x \in (c,a)$, $a < g^{-k} g^n g^k x < b$. By the maximality of (a,b), choose $x \in S \cap (c,a)$. Then $a < g^{-k} g^n g^k x < b$, and $g^{-k} g^n g^k x \in S$. This contradicts $S \cap (a,b) = \emptyset$. QED

THEOREM 7.3. D has a permutation system if and only if D is isomorphic to $\Re$.

Proof: Suppose D has a permutation system. By Lemma 7.2, D is standard. However, D cannot have a least or greatest element for otherwise there is no increasing and strictly dominating permutation. The converse is by Lemma 7.1. QED

REFERENCES

[Fr05] H. Friedman, Characterization of R/Simple proof, http://www.cs.nyu.edu/pipermail/fom/2005-February/008773.html, FOM email list, February 15, 2005.

[Ho1901] O. Hölder, 1901, "Die Quantität und die Lehre vom Mass," *Berichte über die Verhandlungen der königlich* *sächsischen Gesellschaft der Wissenschaften zu Leipzig, Mathematisch - Physische Classe* 53, pp. 1–64.