CONCEPT CALCULUS

by
Harvey M. Friedman
Ohio State University
friedman@math.ohio-state.edu
http://www.math.ohio-state.edu/%7Efriedman/
Department of Philosophy
MIT
1-3PM
November 4, 2009

We have discovered an unexpectedly close connection between the logic of mathematical concepts and the logic of informal concepts from common sense thinking. Our results indicate that they are, in a certain precise sense, the same.

This connection is new and there is the promise of establishing similar connections involving a very wide range of informal concepts.

We call this development the **Concept Calculus**.

We begin with some background concerning the crucial notion of

interpretation between theories

that is used to state results in Concept Calculus. We then give a survey of major results in Concept Calculus.

In particular, we establish the **mutual interpretability** of formal systems for **set theory** and formal systems for a variety of **informal concepts from common sense thinking**.

INTERPRETATION POWER

The notion of interpretation plays a crucial role in Concept Calculus.

Interpretability between formal systems was first precisely defined by Alfred Tarski. We work in the usual framework of first order predicate calculus with equality.

An interpretation of S in T consists of

• A one place relation defined in T which is meant to carve out the domain of objects that S is referring to, from the point of view of T.
• A definition of the constants, relations, and functions in the language of S by formulas in the language of T, whose free variables are restricted to the domain of objects that S is referring to (in the sense of the previous bullet).
• It is required that every axiom of S, when translated into the language of T by means of i,ii, becomes a theorem of T.

In ii, we usually allow that the equality relation in S need not be interpreted as equality - but rather as an equivalence relation.

INTERPRETATION POWER

CAUTION: Interpretations do not necessarily preserve truth. They only preserve provability.

We give two illustrative examples. Let S consist of the axioms for strict linear order together with “there is a least element”.

- $\neg(x < x)$
- $x < y \wedge y < z \Rightarrow x < z$.
- $x < y \vee y < x \vee x = y$.
- $(\exists x)(\forall y)(x < y \vee x = y)$.

Let T consist of the axioms for strict linear order together with “there is a greatest element”. I.e.,

- $\neg(x < x)$
- $x < y \wedge y < z \Rightarrow x < z$.
- $x < y \vee y < x \vee x = y$.
- $(\exists x)(\forall y)(y < x \vee y = x)$.

INTERPRETATION POWER

S

- $\neg(x < x)$
- $x < y \land y < z \Rightarrow x < z$.
- $x < y \lor y < x \lor x = y$.
- $(\exists x)(\forall y)(x < y \lor x = y)$.

T

- $\neg(x < x)$
- $x < y \land y < z \Rightarrow x < z$.
- $x < y \lor y < x \lor x = y$.
- $(\exists x)(\forall y)(y < x \lor y = x)$.

CLAIM: S is interpretable in T and vice versa. Obvious interpretation of S in T: In T, take the objects of S to be everything (according to T). Define $x < y$ of S to be $y < x$ in T.

Interpretations of the axioms of S become

- $\neg(x < x)$.
- $y < x \land z < y \Rightarrow z < x$.
- $y < x \lor x < y \lor x = y$.
- $(\exists x)(\forall y)(y < x \lor y = x)$.

These are obviously provable in T.

INTERPRETATION POWER

We now present a much more sophisticated example.

PA = Peano Arithmetic, is a well known first order theory with equality, with symbols 0,S,+,•.

The axioms of PA consists of

- successor axioms
- defining equations for +,•
- the scheme of induction for all formulas in this language.

Now consider “finite set theory”. Ambiguous: could mean either

- ZFC without the axiom of infinity: ZFC\I; or
- ZFC with the axiom of infinity replaced by its negation; i.e., ZFC\I + ¬I.

THEOREM (well known). PA, ZFC\I, ZFC\I + ¬I are mutually interpretable.

PA in ZFC\I: nonnegative integers become finite von Neumann ordinals.

ZFC\I + ¬I in PA: Sets of ZFC\I + ¬I, are coded by the natural numbers in PA - in an admittedly ad hoc manner.

INTERPRETATION POWER

In many such examples of mutual interpretability, the considerably stronger relation of synonymy holds - strongest normal notion of synonymy is: having a common definitional extension. Notions of synonymy and other topics are treated in a forthcoming book with Albert Visser, entitled INTERPRETATIONS BETWEEN THEORIES.

Every theory is interpretable in every inconsistent theory. I.e., the most powerful level of interpretation power is inconsistency.

Fundamental fact: there is no maximal interpretation power - short of inconsistency.

THEOREM. (In ordinary predicate calculus with equality). Let S be a consistent recursively axiomatized theory. There exists a consistent finitely axiomatized theory T such that S is interpretable in T and T is not interpretable in S.

This is proved using Gödel's second incompleteness theorem. Consider T = EFA + Con(S), where EFA is exponential function arithmetic. S is interpretable in T by the formalized completeness theorem. If T is interpretable in S then EFA proves Con(S) implies Con(EFA + Con(S)). By Gödel's second incompleteness theorem, EFA + Con(S) is inconsistent, which is a contradiction.

INTERPRETATION POWER

COMPARABILITY(?). Let S,T be recursively axiomatized theories. Then S is interpretable in T or T is interpretable in S?

There are plenty of natural and interesting examples of incomparability for finitely axiomatized theories that are rather weak.

To avoid trivialities, an example of incomparability with only infinite models:

i) theory of discrete linear orderings without endpoints.
ii) theory of dense linear orderings without endpoints.

Neither is interpretable in the other.

THEOREM. Let S be a consistent recursively axiomatized theory. There exist consistent finitely axiomatized theories T_1,T_2, both in a single binary relation symbol, such that

- S is provable in T_1,T_2;
- T_1 not interpretable in T_2;
- T_2 is not interpretable in T_1.

INTERPRETATION POWER

COMPARABILITY(?). Let S,T be recursively axiomatized theories. Then S is interpretable in T or T is interpretable in S?

There are plenty of natural and interesting examples of incomparability for finitely axiomatized theories that are rather weak.

BUT, are there examples of incomparability between natural theories that are metamathematically strong? E.g., where PA is interpretable?

STARTLING OBSERVATION. ***Any two natural theories S,T, known to interpret PA, are known (with small numbers of exceptions) to have: S is interpretable in T or T is interpretable in S. The exceptions are believed to also have comparability.***

As a consequence, there has emerged a rather large linearly ordered table of "interpretation powers" represented by natural formal systems. Several natural systems may occupy the same position.

We call this growing table the ***Interpretation Hierarchy***. See my first Tarski lecture, on my website.

BETTER THAN
MUCH BETTER THAN

We use the informal notions: better than (>), and much better than (>>). These are binary relations. Passing from > to >> is an example of what we call ***concept amplification***. Equality is taken for granted.

We present some basic principles concerning Better Than and Much Better Than, that have a clear intuitive meaning, and inherent plausibility. Together, they form a formal system MBT (much better than), which is mutually interpretable with ZFC.

We need to consider properties of things. The properties that we consider are to be given by first order formulas. Their extensions are called "ranges of things".

When informally presenting axioms, we prefer to use "range of things" rather than "set of things", as we do not want to commit to set theory here.

BETTER THAN
MUCH BETTER THAN

We say that x is better than a given range of things iff it is better than all things in that range.

We say that x is exactly better than a given range of things iff it is better than all things in the range, and all things that something in the range is better than, and nothing else.

BASIC. Nothing is better than itself. If x is better than y and y is better than z, then x is better than z. If x is much better than y, then x is better than y. If x is much better than y and y is better than z, then x is much better than z. If x is better than y and y is much better than z, then x is much better than z. There is something that is much better than any given x,y. If x is much better than y, then x is much better than something better than y.

DIVERSE EXACTNESS. Let x be better than a given range of things. There is something that is exactly better than the given range of things, that x is not better than.

In Diverse Exactness, ranges of things are given by formulas in L(>,>>,=).

THE SYSTEM MBT (much better than)

BASIC + DIVERSE EXACTNESS + STRONG UNLIMITED IMPROVEMENT

DIVERSE EXACTNESS. Let x be better than a given range of things. There is something that is exactly better than the given range of things, that x is not better than.

UNLIMITED IMPROVEMENT. Assume that x is much better than and related to y by a given binary relation. Then arbitrarily good x are related to y by the given binary relation.

Instead, we use the following natural sharpening:

STRONG UNLIMITED IMPROVEMENT. Let x and a ternary relation be given. There are arbitrarily good y such that x,y are related, by the given ternary relation, to the same two things that x is much better than.

In UI and SUI, binary relations are given by formulas in L(>,=) with no side parameters. BUT, in Diverse Exactness, we use L(>,>>,=), with side parameters allowed.

THEOREM. MBT and ZFC are mutually interpretable.

COROLLARY. Con(MBT) ↔ Con(ZFC) is provable in EFA (a very weak fragment of Peano Arithmetic).

BETTER THAN
MUCH BETTER THAN

What happens to Russell's Paradox in this context? In sets, we start with

there is a set whose elements are exactly
the sets with a given property

and obtain a contradiction that Frege missed and Russell saw. The corresponding principle here is

there is something which is better than, exactly,
the things with a given property and
those things they are better than.

This immediately leads to a contradiction. Even the much weaker

there is something which is better than
the things with a given property

gives an immediate contradiction, because there cannot be anything which is better than all things - by irreflexivity.

THus Russell's Paradox now becomes entirely transparent and never would have trapped anyone: it disappears as a Paradox.

INTERPRETATION OF MBT IN ZFC

BASIC. ... There is something that is much better than any given x,y. If x is much better than y, then x is much better than something better than y.

DIVERSE EXACTNESS. Let x be better than a given range of things. There is something that is exactly better than the given range of things, that x is not better than.

STRONG UNLIMITED IMPROVEMENT. Let x and a ternary relation be given. There are arbitrarily good y such that x,y are related, by the given ternary relation, to the same two things that x is much better than.

We define pairs $(D_\alpha,>_\alpha)$, for all ordinals. Define $(D_0,>_0) = (\emptyset,\emptyset)$. Suppose $(D_\alpha,>_\alpha)$ has been defined, and is transitive and irreflexive. Define $(D_{\alpha+1},>_{\alpha+1})$ to extend $(D_\alpha,>_\alpha)$ by adding an exact upper bound of every $>_\alpha$ transitive subset of D - even if this subset of D already has an exact upper bound over $(D_\alpha,>_\alpha)$. For limit ordinals λ, define $D_\lambda = \mathbf{U}_{\beta<\lambda} D_\beta$, $>_\lambda = \mathbf{U}_{\beta<\lambda} >_\beta$.

Let S be an unbounded set of limit ordinals $< \lambda$. Define $x >>_S y$ if and only if

$$x,y \in D_\lambda \wedge (\exists \alpha,\beta \in S)(\alpha < \beta \wedge y \in D_\alpha \wedge (\forall w \in D_\beta)(x >_\lambda w)).$$

Then Basic + Diverse Exactness holds in $(D_\lambda,>_\lambda,>>_S)$.

If S has a familiar set theoretic property, then MBT holds:

INTERPRETATION OF MBT IN ZFC

BASIC. ... There is something that is much better than any given x,y. If x is much better than y, then x is much better than something better than y.

BASIC'. ... There is something that is much better than any given x,y. If x is much better than y,z then x is much better than something better than y,z.

DIVERSE EXACTNESS. Let x and a ternary relation be given. There are arbitrarily good y such that x,y are related, by the given ternary relation, to the same two things that x is much better than.

STRONG UNLIMITED IMPROVEMENT. Let x be much better than something, as well as everything x is related to by a given binary relation. Then arbitrarily good x are related to the same things that x is related to, by the given binary relation.

We define pairs $(D_\alpha,>_\alpha)$, for all ordinals.
Let S be an unbounded set of limit ordinals $< \lambda$. Define $x >>_S y$ if and only if

$$x,y \in D_\lambda \wedge (\exists\alpha,\beta \in S)(\alpha < \beta \wedge y \in D_\alpha \wedge (\forall w \in D_\beta)(x >_\lambda w)).$$

Suppose $S \subseteq \lambda$ has order type ω, and the $V(\alpha)$, $\alpha \in S$, form an elementary chain (under epsilon) in the usual sense of model theory. Then MBT = Basic + Diverse Exactness + Strong Unlimited Improvement holds in $(D_\lambda,>_\lambda,>>_S)$.

Actually, this needs a little more than ZFC. But there is a standard technical elaboration of the argument that shows that ZFC suffices.

WHAT CORRESPONDS TO Z?

BASIC'. ... There is something that is much better than any given x,y. If x is much better than y,z then x is much better than something better than y,z.

DIVERSE EXACTNESS. Let x be better than a given range of things. There is something that is exactly better than the given range of things, that x is not better than.

STRONG DIVERSE EXACTNESS. Let x be much better than something better than a given range of things. Then x is better than some, but not all, things exactly better than the given range of things.

VERY STRONG DIVERSE EXACTNESS. Let x be much better than something better than a given range of things. THen x is much better than some, but not all, things exactly better than the given range of things.

SUPER STRONG DIVERSE EXACTNESS. Let x be much better than something, and a given range of things. Then x is better than some, but not all, things exactly better than the given range of things.

Use L(>,>>,=), with side parameters allowed.

THEOREM. The following are provable in Basic. SSDE → SDE → DE. VSDE → SDE → DE.

THEOREM. B' + VSDE + SSDE is interpretable in Z. Z is interpretable in B + SDE. Hence B(B') + SDE, B(B') + VSDE + SSDE are both mutually interpretable in Z.

ALTERNATIVE CORRESPONDING TO ZFC

The system MBT =

Basic + Diverse Exactness + Strong Unlimited Improvement

is mutually interpretable with ZFC.

MBT proves Strong Diverse Exactness. Hence

Basic + Strong Diverse Exactness + Unlimited Improvement

is a fragment of MBT.

THEOREM. Basic + Strong Diverse Exactness + Unlimited Improvement is mutually interpretable with ZFC.

STAR AXIOM

STAR. There is a Star. I.e., something that is better than something, and much better than everything that it is better than.

MBT + Star is not interpretable in ZF(C). It interprets a pretty significant large cardinal (at least indescribable cardinals), and is interpretable using subtle cardinals.

VARYING QUANTITY
COMMON SCALE

We now consider a single varying quantity - where the time and quantity scales are the same, and are linearly ordered.

  This is common in ordinary physical science, where the time scale and the quantity scale may both be modeled as nonnegative real numbers.

The associated language has >,>>,=,F, where F is a unary function.

When thinking of time, >,>> is later than and much later than.
When thinking of quantity, >,>> is greater than and much greater than.

BASIC. Nothing is larger than itself. If x is larger than y and y is larger than z, then x is larger than z. If x is much larger than y, then x is larger than y. If x is much larger than y and y is larger than z, then x is much larger than z. If x is larger than y and y is much larger than z, then x is much larger than z. There is something that is much larger than any given x,y. For any $x \neq y$, x is larger than y or y is larger than x. If y is much larger than x, then y is much larger than something larger than x.

VARYING QUANTITY
COMMON SCALE

BOUNDED RANGES. If x is much larger than a range of values, then that range of values is the actual range of values over some interval with right endpoint smaller than x.

Here we use L(>,>>,=,F) to present the bounded range of values.

AMPLIFICATION. If a value is related, in a given way, to a much large value, then the value is related to arbitrarily large values.

Here we use L(>,=,F) to present the relation, with no side parameters.

THEOREM. Basic + Bounded Ranges is mutually interpretable with ZC.

THEOREM. Basic + Bounded Ranges + Amplification is mutually interpretable with ZFC.

VARYING BIT
FLASHING LIGHT

We now use a bit varying over time. Physically, this is like a flashing light. This corresponds to having a time scale with a unary predicate.

In order to get logical power out of this particularly elemental situation, we need to use forward translations of time.

We think of b+c so that the amount of time from b to b+c is the same as the amount of time before c.

BASIC. As before.

BOUNDED TIME TRANSLATION. Let t be much later that a given range of times. There is a translation time c < t such that a time r lies in the range of times if and only the bit at time r+c is 1. Use L(>,>>,F,+).

AMPLIFICATION. Same as before. Use L(>,F,+).

THEOREM. Mutual interpretability with ZC and ZFC, as before.

PERSISTENTLY VARYING BIT
FLASHING LIGHT WITH PERSISTENCE

A reasonable objection can be raised about the Varying Bit: a varying bit must have persistence. I.e., if the bit is 1 then it remains 1 for a while, and if the bit is 0 then it remains 0 for a while. Define a persistent range of times in the obvious way.

BASIC. Same as before.

PERSISTENT BOUNDED TIME TRANSLATION. Same as before, but only for a persistent range of times.

AMPLIFICATION. Same as before.

We obtain the same results, (mutual interpretability with ZC and ZFC), but we need to add two additional axioms.

ADDITION. $y < z \Rightarrow x+y < x+z$.

ORDER COMPLETENESS. Every nonempty range of times with an upper bound has a least upper bound.

Use the full language for Order Completeness.