DISCRETE INDEPENDENCE RESULTS

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1. APPROXIMATE FIXED POINTS AND LARGE CARDINALS

Let P(Z) be the Cantor space of all subsets of N. We are interested in mappings □:P(Z) □ P(Z).

We say that □:P(Z) □ P(Z) is a contraction if and only if for all n ≥ 0 and A,B □ P(Z), if A,B agree on (-n,n) then □(A), □(B) agree on (-n-1,n+1).

THEOREM 1.1. Every contraction on P(Z) has a unique fixed point.

Let n,k ≥ 1. An approximate fixed point of □ of type (n,k) is a chain of sets A_1 □ ... □ A_n □ Z such that

for all 1 □ i □ n-1, □(A_{i+1}) and A_{i+1} agree on all sums and products of length k from A_i □ {0,±1}.

An infinite approximate fixed point of type (n,k) is an approximate fixed point of type (n,k) whose terms are infinite.

A bi-infinite approximate fixed point of type (n,k) is an approximate fixed point of type (n,k) whose terms are bi-infinite; i.e., contain infin-itely many positive and infin-itely many negative elements.

We would like to obtain a bi-infinite approximate fixed point theorem. I.e.,

PROTOTYPE. Every “suitable” □:P(Z) □ P(Z) has bi-infinite approximate fixed points of every type.

We present a weak condition on □ for this Prototype to hold. It turns out that we need large cardinals in order to prove that this condition works.

We say that □:P(Z) □ P(Z) is a compression on P(Z) if and only if there exists r ≥ 0 and t > 1 such that *if A,B agree on (-n,n) then □(A),□(B) agree on (-tn-1,tn+1).*

We say that □ is a decreasing compression on P(Z) iff □ is a compression on P(Z) satisfying

A □ B implies □(A) ⊇ □(B).

We say that □ is a uniformly decreasing compression on P(Z) iff □ is a decreasing compres-sion on P(Z) for which there exists r ≥ 0 such that

the value of □ at any set is equaled to the value of □ at some subset of cardinality □ r.

PROPOSITION 1.2. Every uniformly decreasing compres-sion on P(Z) mapping finite sets to cofinite sets has bi-infinite approximate fixed points of every type.

Here is a weak special case of Proposition 1.2.

PROPOSITION 1.3. Every uniformly decreasing compression on P(Z) mapping finite sets to cofinite sets has an infinite approximate fixed point of every type with a positive element.

We must use t > 1 in the definition of compression. If we use t = 1 then Propositions 1.2, 1.3 are false.

PROPOSITION 1.4. Let V be any countable set of infinte subsets of Z. Any uniformly decreasing contraction on P(Z) mapping finite sets to cofinite sets has approximate fixed points of every type whose first set meets every element of V.

PROPOSITION 1.5. Let t be a type. Any finite set of uniformly decreasing compressions on P(Z) mapping finite sets to cofinite sets have respective infinite approximate fixed points of type t with the same first set. Moreover, we can replace "infinite" by "bi-infinite."

PROPOSITION 1.6. Let t be a type. Any countable set of uniformly decreasing compressions on P(N) mapping finite sets to cofinite sets have respective infinite approximate fixed points of type t with almost equal first sets. Moreover, we can replace "infinite" by "bi-infinite."

THEOREM 1.7. Propositions 1.2 - 1.6 (all forms) are provably equivalent to the 1-consis-tency of MAH = ZFC + $\{$there exists a k-Mahlo cardinal$\}_k$ over ACA. In particular, they are independent of ZFC.

Theorem 1.7 holds even if we weaken the Propositions substantially. Here
are some simultaneous weakenings:
i) restrict to types of the form (3,k) and (n,2);
ii) restrict 1.5 to two uniformly decreasing compressions on P(Z).

Propositions 1.2 - 1.6 are provable for types (2,k) and (n,1).

2. DISJOINT COVERS

We use N for the set of all nonnegative integers. For $x \in N^k$, write max(x) for the maximum of the coordinates of x.

Let A,B,C be any sets. We say that A,B is a disjoint cover of C if and only if

i) A,B are disjoint;
ii) $C \subseteq A \cup B$.

Let $F:N^k \to N$. We say that F is strictly dominating iff for all $x \in N^k$, $F(x) > |x|$.

For $A \subseteq N$, write F[A] for the forward image of A^k under F.

The fundamental infinite disjoint cover theorem:

THEOREM 2.1. Let $F:N^k \to N$ be strictly dominating. There exists $A \subseteq N$ such that A,F[A] is a disjoint cover of N. A is unique and infinite.

We want to have some control over A. But since A is unique, we have no control. What about A,F[A] is a dijoint cover of A+A? Not very much control:

THEOREM 2.2. For all $k \geq 1$ there is a strictly dominating $F:N^k \to N$ such that the follow-ing holds. If A,F[A] is a disjoint cover of A+A then A is finite or eventually equaled to the multiples of a positive integer.

We thus are led to pairs $A \subseteq B \subseteq N$ such that B,F[B] is a disjoint cover of A+A. We wish to have some control over such A,B.

The following result indicates that we can find "large" A,B.

THEOREM 2.4. Let $k \geq 1$, $F:N^k \to N$ be strictly dominating, and V be a countable set of infinite subsets of N. There exist $A \subseteq B \subseteq N$ meeting every element of V, such that B,F[B] is a disjoint cover of A+A.

PROPOSITION 2.5. Let $k,r \geq 1$, $F:N^k \to N$ be strictly dominating, and V be a countable set of infinite subsets of N. There exist $A_1 \subseteq \ldots \subseteq A_r \subseteq N$ meeting every element of V, such that for all $1 \leq i \leq r-1$, A_{i+1},F[A_{i+1}] is a disjoint cover of $A_i + A_i$.

There are some important spec-ial cases of Proposition 2.5.

PROPOSITION 2.6. Let $k,r \geq 1$, $F:N^k \to N$ be strictly dominating, and E be an infinite subset of N. There exist $A_1 \subseteq \ldots \subseteq A_r \subseteq N$ with infinitely many elements in common with E, such that for all $1 \leq i \leq r-1$, A_{i+1},F[A_{i+1}] is a disjoint cover of $A_i + A_i$.

And also the case E = 2N+1.

PROPOSITION 2.7. Let $k,r \geq 1$, $F:N^k \to N$ be strictly dominating. There exist $A_1 \subseteq \ldots \subseteq A_r \subseteq N$ with infinitely many odd elements, such that for all $1 \leq i \leq r-1$, A_{i+1},F[A_{i+1}] is a disjoint cover of $A_i + A_i$.

A weakening:

PROPOSITION 2.8. Let $k,r \geq 1$, $F:N^k \to N$ be strictly dominating. There exist $A_1 \subseteq \ldots \subseteq A_r \subseteq N$ where A_r has infinitely many odd elements, such that for $1 \leq i \leq r-1$, A_{i+1},F[A_{i+1}] is a disjoint cover of $A_i + A_i$.

We can consider two strictly dominating functions.

PROPOSITION 2.9. Let $k,r \geq 1$ and $F,G:N^k \to N$ be strictly dominating. There exist infinite $A_1 \subseteq \ldots \subseteq A_r \subseteq N$ and

$B_1 \subseteq \ldots \subseteq B_r \subseteq N$ where for all $1 \leq i \leq r-1$, $A_{i+1},F[A_{i+1}]$ is a disjoint cover of A_i+A_i, $B_{i+1},G[B_{i+1}]$ is a disjoint cover of B_i+B_i, and $A_1 = B_1$.

THEOREM 2.10. Propositions 2.5 - 2.0 are provably equivalent to the 1-consistency of MAH = ZFC + {there exists an n-Mahlo cardinal}$_n$ over ACA. The forward direction is provable in RCA_0.

3. DISJOINT COVERS OF A+A: SEMILINEAR FUNCTIONS

An integral semilinear func-tion on N is a function from some N^k into N whose graph is a semilinear subset of N^{k+1}. A semilinear subset of N^k is a Boolean combination of finitely many halfplanes given by integer coefficients.

We can consider Propositions 2.5 - 2.9 with "strictly dominating" replaced by "a strictly dominating integral semilinear function."

THEOREM 3.1. Propositions 2.5 - 2.9 so modified are provably equivalent to the consistency of MAH. The forward direction is provable in RCA_0.

PROPOSITION 3.2. Let $k,r \geq 1$, $F:N^k \rightarrow N$ be a strictly dominating integral semilinear function. There exist $A_1 \subseteq \ldots \subseteq A \subseteq N$ starting with an infinite geometric progression, such that for all $1 <= i \leq r-1$, $A_{i+1},F[A_{i+1}]$ is a disjoint cover of A_i+A_i.

THEOREM 3.3. Proposition 3.2 is provably equivalent to the consistency of MAH. The forward direction is provable in RCA_0.

4. DISJOINT COVERS OF A+A: SEMILINEAR FUNCTIONS, FINITE FORMS

Prop 3.2 in explicit form:

PROPOSITION 4.1. Let $k,r \geq 1$, $F:N^k \rightarrow N$ be a strictly dominating integral semilinear function, and t be sufficiently large. There exist $\{1,t,t^2,\ldots\} = A_1 \subseteq \ldots \subseteq A_r \subseteq N$ such that for $1 \leq i \leq r-1$, A_{i+1}, $F[A_{i+1}]$ is a disjoint cover of $A_i + A_i$.

PROPOSITION 4.2. Let $k,r \geq 1$, $F:N^k \rightarrow N$ be a strictly dominating integral semilinear function, and t,n be sufficiently large. There exist $\{1,t,t^2,\ldots, t^n\} = A_1 \subseteq \ldots \subseteq A_r \subseteq N$ where for $1 \leq i \leq r-1$, A_{i+1}, $F[A_{i+1}]$ is a disjoint cover of $A_i + A_i$.

Note that Proposition 4.2 is explicitly in Π^0_4 form.

We can put Proposition 4.2 in explicitly Π^0_1 form with the help of estimates. Here are the considerations that allow us to do this. By the "presentation", we will mean the presentation of the graph of F as a Boolean combination of finitely many halfplanes, and the parameters k,r.

i. We can use linear algebra to bound the universal quantifiers in the definition of strictly dominating, double exponentially in the presentation.

ii. We can hypothesize that t,n be double exponentially higher than the presentation. This requires a detailed understanding of the proof of 4.2 from Mahlo cardinals.

iii. We can bound the magnitudes of the elements of A_r double exponentially in the presentation and t,n.

THEOREM 4.3. Propositions 4.1, 4.2 are provably equivalent to the consistency of MAH. The forward direction is provable in RCA_0. For Proposition 4.2, the equivalence can be proved in EFA* = exponential function arithmetic with indefinitely iterated exponention, with the forward direction provable in EFA = exponential function arithmetic.

5. DISJOINT COVERS OF P[A]: POLYNOMIALS

We move to Z = the set of all integers.

Let $F:Z^k \to Z$ and $E \subseteq Z^k$. We say F is expansive on E iff there exists c > 1 where for all $x_1,\ldots,x_k \in E$,

$$|F(x_1,\ldots,x_k)| > c|x_1|,\ldots, c|x_k|.$$

We write $F_E[A]$ for $F[A^k \cap E]$.

A set is bi-infinite iff it has infinitely many positive and negative elements.

THEOREM 5.1. Let P be an integral polynomial that is expansive on a set E. There exists bi-infinite $A \subseteq B \subseteq Z$ such that $B,P_E[B]$ is a disjoint cover of P[A].

PROPOSITION 5.2. Let P be an integral polynomial that is expansive on a set E. There exists bi-infinite $A \subseteq B \subseteq C \subseteq Z$

such that B,P_E[B] is a dis-joint cover of P[A] and C, P_E[C] is a disjoint cover of P[B].

PROPOSITION 5.3. Let r ≥ 1 and P be an integral polynomial that is expansive on a set E. There exists bi-infinite $A_1 \subseteq \ldots \subseteq A_r \subseteq Z$ such that for all $1 \leq i \leq r-1$, A_{i+1},$P_E[A_{i+1}]$ is a disjoint cover of $P[A_i]$.

THEOREM 5.4. Propositions 5.2 and 5.3 are provably equiva-lent to the 1-consistency of MAH over ACA. The forward direction of this equivalence is provable in RCA_0.

6. DISJOINT COVERS OF P[A], POLYNOMIALS, FINITE FORMS

PROPOSITION 6.1. Let r ≥ 1 and P be an integral polynomial that is expansive on an intersection E of finitely many halfplanes, and t,n be sufficiently large. There exist finite sets $\{1,t,t^t,t^{t^2}), \ldots,t^{t^n})\} = A_1 \subseteq \ldots \subseteq A_r \subseteq Z$ such that for all $1 \leq i \leq n-1$, A_{i+1},$P_E[A_{i+1}]$ is a disjoint cover of $P[A_i]$.

THEOREM 6.2. Proposition 6.1 is provably equivalent to the 1-consistency of MAH over EFA*, with EFA for the forward direction. Moreover, these results hold if we set n = 3 or if we replace all n inclusions by proper inclusions.

7. DISJOINT COVERS OF T[A], POSITIVE LINEAR FUNCTIONS, FINITE FORMS

A positive integral linear function is a function from some Cartesian power of Z into Z which is given by a linear transformation whose coefficients are nonnegative integers.

PROPOSITION 7.1. Let r ≥ 1, T be a positive integral linear function that is strictly dom-inating on an integral semi-linear set E, and t,n be sufficiently large. There exist finite $\{1,t,t^2,\ldots,t^n\} \subseteq A_1 \subseteq \ldots \subseteq A_n \subseteq Z$ such that for all $1 \leq i \leq r-1$, A_{i+1}, $T_E[A_{i+1}]$ is a disjoint cover of $T[A_i]$.

THEOREM 7.2. 7.1 is provably equivalent to the consistency of MAH over EFA*, with EFA for the forward direction. More-over, these results hold if we set n = 3 or if we replace all n inclusions by proper inclusions.

We can follow the same procedure as indicated earlier to put Proposition 7.1 in explicitly Π^0_1 form, where Theorem 7.2 still holds.

8. CLASSIFICATION PROBLEMS

(one of many variants)

PROPOSITION. Let $r \geq 1$ and P be an integral polynomial that is expansive on a set E. There exists bi-infinite $A_1 \subsetneq \ldots \subsetneq A_r \subsetneq Z$ such that for all $1 \leq i \leq r-1$, $A_{i+1}, P_E[A_{i+1}]$ is a disjoint cover of $P[A_i]$.

FORM I. Let P be an integral polynomial that is expansive on a set E, $r \geq 1$. There exist bi-infinite $A_1 \subsetneq \ldots \subsetneq A_r \subsetneq Z$ where for $1 \leq i \leq r-1$, a specific Boolean equation holds among $A_{i+1}, P_E[A_{i+1}], P[A_i]$.

There are 256 instances of Form I up to Boolean equivalence of Boolean equations. Of course, there are far fewer than that many when one takes into account other formal aspects of the situation. The following result indicates that there are, in some sense, only three different kinds of instances of Form I.

THEOREM 8.1. Every instance of Form I is either provable in ACA, refutable in RCA_0, or equivalent to the 1-consistency of MAH over ACA. All three possibilities are realized.

THEOREM 8.2. Let T be an extension of ACA. The following are equivalent. i) every instance of Form I is either provable or refutable in T;
ii) T proves or refutes the 1-consistency of MAH.

THEOREM 8.3. (First finite obstruction) Suppose a given instance of Form I holds with "bi-infinite" replaced by "arbitrarily large finite." Then it holds with "bi-infinite" replaced by "infinite."

PROPOSITION 8.4. (Second finite obstruction) Suppose a given instance of Form I holds with "bi-infinite" replaced by "infinite." Then it holds unmodified.

PROPOSITION 8.5. (Third finite obstruction) Suppose a given instance of Form I holds with "bi-infinite" replaced by "arbitrarily large finite." Then it holds unmodified.

Third finite obstruction = first and second ones.

THEOREM 8.6. Second and third finite obstruction are provably equivalent to the consistency of MAH over ACA, with RCA_0 for the forward direction.

We conjecture that these results hold for the much more comprehensive Form II below. However, too hard right now.

FORM II. Let P be an integral polynomial that is expansive on a set E, and r ≥ 1. There exists bi-infinite $A_1 \subseteq\neq \ldots \subseteq\neq A_r \subseteq\neq Z$ where for all $1 \leq i \leq r-1$, a specific Boolean equation holds among $A_i, A_{i+1}, P[A_i], P[A_{i+1}], P_E[A_i], P_E[A_{i+1}]$.

However, there is an interesting subcase of Form II that seems within reach.

FORM III. Let P be an integral polynomial that is expansive on a set E, and r ≥ 1. There exist bi-infinite $A_1 \subseteq\neq \ldots \subseteq\neq A_r \subseteq\neq Z$ such that for all $1 \leq i \leq r-1$, a specific set of disjoint cover conditions holds among $A_i, A_{i+1}, P[A_i], P[A_{i+1}], P_E[A_i], P_E[A_{i+1}]$.

9. POSET PRELIMINARIES

A poset is a reflexive, transitive, and antisymmetric relation $\leq$. The field of $\leq$, fld($\leq$), is {x: x $\leq$ x}. We write x < y iff x $\leq$ y and x ≠ y.

We say that x is a predecessor of y in $\leq$ iff x < y. We say that x is an immediate predecessor of y in $\leq$ iff

i) x < y;
ii) there is no z such that x < z < y.

A maximal point of $\leq$ is a field element x which is not a predecessor.

Write $\leq_x$ for $\leq \cap \{y: y \leq x\}^2$.

The height of a poset $\leq$ is the length of the longest finite sequence $x_1 < x_2 \ldots < x_n$. If there is no longest length, then the height is considered to be .

10. OPTIMIZED POSETS IN N^k

N = set of all nonnegative integers. Write PO(k) for the set of all posets $\leq$ such that

i) $\mathrm{fld}(\leq) \subseteq N^k$;
ii) x < y implies max(x) < max(y).

max(x) is the maximum of the coordinates of x, and so the second < in ii) is numerical.

We write FPO(k) for the set of all elements of PO(k) whose field is finite.

Write PO(k,r),FPO(k,r) for the set of all elements of PO(k) and FPO(k),resp., such that every field element has at most r immediate predecessors.

Let $F:\mathrm{FPO}(k,r) \to \mathbf{R}$. We define $F^*:\mathrm{PO}(k,r) \to \mathbf{R} \cup \{\pm\infty\}$ by

$$F^*(\leq) = \Sigma_{x \in \mathrm{fld}(\leq)\backslash\{0\}} F(\leq_x)/\max(x).$$

F* is undefined if sum doesn’t converge to an extended real number. F* may not extend F.

We say $\leq$ is F*-optimal if $F^*(\leq)$ is defined, and there is no $\leq'$ in PO(k,r) with the same field such that $F^*(\leq') > F^*(\leq)$.

THEOREM 10.1. Let $k,r \geq 1$ and $F:\mathrm{FPO}(k,r) \to \mathbf{R}$ have finite range. $\exists$ an F*-optimal poset whose field contains an infinite Cartesian power. If F is into $\mathbf{R}^{+0}$ then there exist F*-optimal posets whose field is any prescribed subset of N^k.

Finite range means that the range is a finite set.

PROPOSITION 10.2. Let $k,r \geq 1$ and $F:\mathrm{FPO}(k,r) \to \mathbf{R}$ have finite range. There exists an F*-optimal poset of finite height whose field contains an
infinite Cartesian power.

I.e., whose field contains E^k for some infinite $E \subseteq N$.

THEOREM 10.3. Proposition 10.2 is provably equivalent to the 1-consistency of MAH over ACA. The forward direction is provable in RCA_0.

11. LARGER CARDINALS

We say that $x \in Z^{+k}$ is strict-ly increasing iff each coordinate is strictly less than the next.

For $x,y \in N^k$, we say that x is entirely lower than y iff every coordinate of x is strictly less than every coordinate of y.

PROPOSITION 11.1. Let $k,r \geq 1$ and $F:FPO(k,r) \to Z$ have finite range. There exists an F*-optimal poset whose maximal points comprise the strictly increasing k tuples from an infinite set, all of which have the same entirely lower predecessors.

And here is a sharper form.

PROPOSITION 11.2. Let $k,r \geq 1$ and $F:FPO(k,r) \to Z$ have finite range. There exists an F*-optimal poset whose maximal points comprise the strictly increasing k tuples from an infinite set, all of which have the same entirely lower predecessors and the same number of predecessors.

THEOREM 11.3. Propositions 11.1 and 11.2 are provably equivalent to the 1-consistency of SUB = ZFC + $\{$there exists an n-subtle cardinal$\}_n$ over ACA. The forward direction is provable in RCA_0.

12. FINITE FORMS

First we begin with two semi-finite forms. The hypotheses are infinite but the conclusion is finite.

Here we will not need the parameter r. For $F:FPO(k) \to \mathbf{R}$, we define F* as before. Also □ is an F*-optimal poset iff F*(□) is defined, and for no □' in PO(k) with the same field, is F*(□') > F*(□).

PROPOSITION 12.1. Let $k,p \geq 1$ and $F:FPO(k) \to Z$ have finite range. ∃ a finite F*-optimal poset whose maximal points comprise the strictly increasing k tuples from a p element set, all of which have the same entirely lower predecessors. And here is a sharper form.

PROPOSITION 12.2. Let $k,p \geq 1$ and $F:FPO(k) \to Z$ have finite range. There exists a finite F*-optimal poset whose maximal points comprise the strictly increasing k tuples from a p element set, all of which have the same entirely lower predecessors and the same number of predecessors.

We now give the obvious finite forms. Let PO(k:t) be the set of all elements of PO(k) whose field is $\subseteq$ {0,... ,t}^k. Let F:PO(k:t) $\rightarrow$ **R**. We define F* as before. Also $\leq$ is an F*-optimal poset iff $\leq$ $\in$ PO(k:t) and for no $\leq$' $\in$ PO(k:t) with the same field, is F*($\leq$') > F*($\leq$).

PROPOSITION 12.3. Let t >> k,r,p ≥ 1 and F:FPO(k:t) $\rightarrow$ {-r,...,r}. There exists an F*-optimal poset whose maximal points comprise the strictly increasing k tuples from a p element set, all of which have the same entirely lower predecessors.

And here is a sharper form.

PROPOSITION 12.4. Let t >> k,r,p ≥ 1 and F:FPO(k:t) $\rightarrow$ {-r,...,r}. There exists an F*-optimal poset whose maximal points comprise the strictly increasing k tuples from a p element set, all of which have the same entirely lower predecessors and the same number of predecessors.

THEOREM 12.5. Propositions 12.1 - 12.4 are provably equivalent to the 1-consistency of SUB = ZFC + {there exists an n-subtle cardinal}$_n$ over RCA_0. In the case of Propositions 12.3 and 12.4 we can use EFA*, with EFA for the forward direction.