ELEMENTAL SENTENTIAL REFLECTION

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Abstract. "Sentential reflection" in the sense of [Fr03] is based on reflecting down from a category of classes. "Elemental sentential reflection" is based on reflecting down from a category of elemental classes. We present various forms of elemental sentential reflection, which are shown to interpret and be interpretable in certain set theories with large cardinal axioms.

1. Introduction.

As in [Fr03], we use "class" as a neutral term, without commitment to the developed notions of "set" and "class" that have become standard in set theory and mathematical logic. We use $\in$ for membership.

This framework supports interpretations of sentential reflection that may differ from conventional set theory or class theory. However, we do not pursue this direction here.

As in [Fr03], this framework is intended to accommodate objects that are not classes. Such nonclasses are treated as classes with no elements. Thus we are careful not to assume extensionality. In fact, we will not assume any form of extensionality.

As in [Fr03], all of our formal theories of classes are in the language $L(\in)$, which is the usual classical first order predicate calculus with only the binary relation symbol $\in$ (no equality).

As in [Fr03], we use "category of classes" or just "category" as a neutral term, not specifically related to category theory. They are given by a formula of $L(\in)$ with a distinguished free variable, with parameters allowed.

In [Fr03], the following two forms of "sentential reflection" are considered.

if a given sentence of L(□) holds in a given category then it holds in a subclass.

if a given sentence of L(□) holds in a given category then it holds in an inclusion subclass.

These are called SR(□) and SRIS(□), respectively. In [Fr03], they are shown to be mutually interpretable with Z2 and ZFC, respectively.

An elemental class is a class which is an element of a class. In some common formalizations of the usual theory of classes that correspond to NBG (the von Neumann Bernays Gödel theory of classes), the elemental classes are the sets, and the nonelemental classes are the proper classes.

In "elemental sentential reflection", we reflect from a category of elemental classes (i.e., a category all of whose elements are elemental), to an elemental subclass (of the category of elemental classes).

ESR(□) and ESRIS(□) are, respectively,

if a given sentence of L(□) holds in a given category of elemental classes then it holds in an elemental subclass.

if a given sentence of L(□) holds in a given category of elemental classes then it holds in an elemental inclusion subclass.

Just as in [Fr03] for SR(□) and SRIS(□), these are formalized as double schemes, with one schematic letter presenting the category, and one schematic letter presenting the sentence of L(□).

THEOREM 1.1. ESR(□) is mutually interpretable with Z_2. ESRIS(□) is mutually interpretable with ZFC.

Thus by merely making the move to elemental sentential reflection, we do not obtain anything new.

To take full advantage of elemental sentential reflection, we expand the language L(□) in a standard way to support quantification over subclasses of the domain.

For this purpose, we use the two sorted language L^2(□) with variables over objects and variables over classes of objects, whose atomic formulas are

x □ y
x □ A

where x,y are object variables and A is a class variable.

The object quantifiers are to range over the elements of the given category or class (the domain), and the class quantifiers are to range over all subclasses of the given category or class (the domain).

We emphasize that the language of all class theories considered here will still be L(□), despite the use of the "second order language" L^2(□).

ESR(□*) is

> if a given sentence of L^2(□) holds in a given category of elemental classes then it holds in an elemental subclass.

THEOREM 1.2. ESR(□*) is mutually interpretable with ZFC + {there exists a $□^1_n$ indescribable cardinal}$_n$.

We now move to some extremely strong forms of elemental sentential reflection. We define extensional equality as usual by x ≡ y □ (□z)(z □ x □ z □ y).

Let K be a category or class. We say that x is an extensionally proper subclass of K if and only if x is a subclass of K and (□y □ K)(□z ≡ y)(z □ x). Here x,y,z are distinct variables.

ESR(□*)# is

> if a given sentence of L^2(□) holds in a given category of elemental classes not forming an elemental class, then it holds in a nonelemental extensionally proper subclass.

THEOREM 1.3. ESR(□*)# is interpretable in ZF + "there is a nontrivial elementary embedding from some L(V(□+1)) into L(V(□+1))" and interprets ZF + "there is a nontrivial elementary embedding from some V(□+1) into V(□+1)".

It is well known that ZFC + "the existence of a measurable cardinal" can be interpreted in ZF + "there is a nontrivial elementary embedding from some V(□+1) into V(□+1)", or even just ZF + "there is a nontrivial elementary embedding from some V(□) into some transitive set". Such results are well known as far as the inner model theory has been developed, which means, e.g., that ESR(□*)# interprets ZFC + "there exists a proper class of Woodin cardinals".

Woodin has many results concerning the relationship between large cardinals above the reach of inner model theory with and without choice.

E.g., Woodin has proved the consistency of ZFC + "there is a nontrivial elementary embedding from some V(□+1) into V(□+1)" from NBG + "there is a nontrivial elementary embedding from V into V". Also Woodin has proved the consistency of ZFC + "there is a cardinal which is n-huge for all finite n" in ZF + "there is a nontrivial elementary embedding from some V(□+1) into V(□+1)".

Hence by Theorem 1.3 and results of Woodin, we see that ZFC + "there is a cardinal which is n-huge for all finite n" is interpretable in ESR(□*)#.

The consensus among set theorists is that ZFC + "there is a nontrivial elementary embedding from some L(V(□+1)) into itself" is consistent.

Finally, consider the following system ESR(□*)##.

> if a given sentence of L^2(□) holds in a given category of elemental classes not forming an elemental class, then it holds in an elemental subclass and in a nonelemental extensionally proper subclass.

THEOREM 1.4. ESR(□*)## is mutually interpretable with ZF + {there exists a $\square^1_n$ indescribable V(□) and a nontrivial elementary embedding from V(□) into V(□)$\}_n$. ESR(□*)## both interprets and proves the consistency of ZFC + "there is a nontrivial elementary embedding from some V(□+1) into V(□+1)".

The last conclusion in Theorem 1.4 uses the above cited results of Woodin.

For the formal presentations of ESR(□*), ESR(□*)#, ESR(□*)##, let □ be a formula of L(□), and let □ be a sentence of L2(□) with no variables in common with □. Let y be a variable in □. We want to define the formula □[□,y] of L(□). For the subclass variables Z in □, associate object variables Z′, so that the Z′ are distinct and do not appear in □,□ (and for technical reasons are not x,y). Let □[□,y] be the result of replacing all quantifiers (Qz) in □ by

(Qz|□[y/z])

and all quantifiers (QZ) in □ by

(QZ′|(□y □ Z′)(□))

and expanding the result to a formula of L(□).

Informally, □[□,y] is the (formula expressing the) result of relativizing all object quantifiers in □ to {y: □(y)} and all subclass quantifiers to subclasses of {y: □(y)}. In particular, □[y □ x,y] is the (formula expressing the) result of relativizing all object quantifiers in □ to (the elements of) x, and all subclass quantifiers to subclasses of x.

ESR(□*) is the formal system in L(□) whose nonlogical axioms are

((□y)(□ □ E(y)) □ □[□,y]) □
(□x)(E(x) □ (□y □ x)(□) □ □[y □ x,y])

where x,y are distinct variables, □ is a formula of L(□) in which x is not free, □ is a sentence of L^2(□) with no variables in common with □, y is a variable in □, E(x) is (□y)(x □ y), and E(y) is (□x)(y □ x).

We leave the formalizations of ESR(□*)# and ESR(□*)## to the reader.

2. ESR(□) and ESRIS(□). To be completed.

3. ESR(□*). To be completed.

4. ESR(□*)#. To be completed.

5. ESR(□*)##. To be completed.

REFERENCES

[Fr03] Sentential reflection, preprint, January 5, 2003.
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