TANGIBLE MATHEMATICAL INCOMPLETENESS OF ZFC

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Abstract. We present the lead statements in the new Emulation Theory that provide the first tangible examples of the mathematical incompleteness of the usual ZFC axioms for mathematics. They are provable in SRP^+, an extension of ZFC by a large cardinal hypothesis well accepted by set theorists, but not in ZFC, or even SRP. We focus on three forms: Maximal Emulation Stability (MES), Maximal Duplication Stability (MDS), and Maximal Clique Stability (MCS). MES (MDS) asserts that every finite subset of $Q[0,k]^k$ (k-tuples of rationals in [0,k]) has a stable maximal emulator (duplicator). MCS asserts that every order invariant graph on $Q[0,k]^k$ has a stable maximal clique. Stability of $S \subseteq Q[0,k]^k$ here means that for $p < 1$, $(p,1,...,k-1) \in S \leftrightarrow (p,2,...,k+1) \in S$. In addition to MES, MDS, MCS being independent of ZFC, there are specific small k, unnatural finite subsets of $Q[0,k]^k$, and unnatural order invariant graphs on $Q[0,k]^k$ for which MES, MDS, MCS are independent of ZFC. Although graph theorists are expected to prefer MCS, the MES and MDS bypass the middleman of graphs and directly reflect the informal ideas of emulation and duplication. MES and MDS are so natural, transparent, concrete, elementary, interesting, memorable, flexible, teachable, and rich in varied intricate examples and weaker and stronger forms, that they merit being classified in the category of Everybody's Mathematics. MES, MDS, MCS are seen to be implicitly Π^0_1 via the Gödel Completeness Theorem, and all of the maximal objects asserted to exist can be taken to be recursive in the halting problem. Furthermore, all of these statements are equivalent to Con(SRP) over WKL_0.

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1. INTRODUCTION

Maximal Emulation Stability (MES) was presented in [Fr18a] to a software engineering audience. It is the first of the following three lead statements in the new Emulation Theory. Here $Q[p,q] = Q \cap [p,q]$.

MAXIMAL EMULATION STABILITY. MES. Every finite subset of $Q[0,k]^k$ has a stable maximal emulator.

MAXIMAL DUPLICATION STABILITY. MDS. Every finite subset of $Q[0,k]^k$ has a stable maximal duplicator.

MAXIMAL CLIQUE STABILITY. MCS. Every order invariant graph on $Q[0,k]^k$ has a stable maximal clique.

According to our conventions, MES, MDS, MCS are sentences universally quantify over k and therefore don't depend on k. This is indicated by the absence of "k" in the titles MES, MDS, MCS. However, in section 2.2 we introduce a single consolidated notation that introduces several parameters and allows either the fixing of or universal quantification over those parameters.

MCS may be preferable to an audience of graph theorists. However, MES and MDS may be preferable for other audiences. MES and MDS remove the "middleman" of graphs in our earlier

MCS. One indication of this is the comparison of their respective supporting Definitions 2.1.2, 2.1.3, 2.1.4 (over the common Definition 2.1.1 and stable). The respective word counts are 62, 64, 112.
As is clear from the supporting definitions, MES, MDS, MCS depends very much on the crucial notion of order equivalent $x,y \in Q^k$. Definition 2.1.2 is clear enough, but we also observe that $x,y \in Q^k$ are order equivalent if and only if there is an order automorphism of Q that sends x to y via the coordinate action.

For MES, MDS, MCS, we use the following default notion of stable. $S \subseteq Q[0,k]^k$ is stable if and only if

$$\text{1) for all } p < 1,\quad (p,1,\ldots,k-1) \in S \leftrightarrow (p,2,\ldots,k) \in S$$

MES, MDS, MCS are independent of ZFC. In fact, 1) is our default notion of stable, which is the weakest form of stability that we consider. This Stable is chosen for its memorable simplicity, and its deceptive weakness. The strongest Notion of Stability that we consider is Full Stability (along with its dual) presented in Definition 2.3.2, and see Appendix A. With all of these notions of stability, MES, MDS, MCS are independent of ZFC.

MES, MDS, MCS are provable in SRP^+ but not in SRP. Moreover, they are equivalent to Con(SRP) over WKL_0 (see Theorem 2.3.7). In fact, there are specific small k, artificial finite subsets of $Q[0,k]^k$, and artificial order invariant graphs on $Q[0,k]^k$, for which MES, MDS, MCS so restricted are independent of ZFC. For obtaining such independence from ZFC, we anticipate having control over k in [Fr19], hopefully with k = 8, and with an ambitious target of k = 3 (with k = 2 the statements are provable in ZFC). However, the specific finite subsets of $Q[0,k]^k$ and order invariant graphs on $Q[0,k]^k$ are not expected to be reasonably controlled. From the point of view of recursion theory, the stable maximal emulators and stable duplicators and stable maximal cliques can always be taken to be recursive in the halting problem, and uniformly so in the given k and set or order invariant graph. This is an a priori recursion theoretic result that depends only on the (ZFC unprovable) existence of the maximal emulator, maximal duplicator, or maximal clique.

We claim that MES and MDS are so natural, transparent, concrete, elementary, interesting, memorable, flexible, teachable, and rich in varied intricate examples and weaker and stronger forms, that they merit being classified in the category of Everybody's Mathematics. MCS has these same attributes also for audiences comfortable with graphs (a growing number of mathematical thinkers). A small sample of seven easy calculations of maximal emulators and maximal duplicators with k = 2 are given in section 2.1, as an indication of how beginning students can be engaged. This includes verifying that there is indeed a maximal emulator and maximal duplicator that is stable. Plans are under way to confirm this assessment through documented engagement with mathematically gifted high school students and undergraduate mathematics majors starting in 2019.

The definitions supporting MES, MDS, MCS are trivial by normal mathematical standards as can be seen from section 2.1. In each dimension k, they involve only the comparison of rational numbers ($p < q$), the concept of sets of rational tuples, and no other aspects of rational numbers such as addition, subtraction, multiplication, or divisibility. The numbers 0,...,k are used only as k+1 distinguished elements.

In fact, in each dimension k, MES, MDS, MCS asserts the satisfiability of a sentence in first order predicate calculus with <,=, effectively associated with k. Hence in each dimension k, MES, MDS, MCS are implicitly Π^0_1, and it follows that MES, MDS, MCS themselves are implicitly Π^0_1. Π^0_1 is at the lowest level of logical complexity above the (theoretically) finitely observable. Also this shows that MES, MDS, MCS enjoy a property analogous to that of valued physical theories: they are subject to refutation, in the sense that in principle, if they are false then they can be refuted - by computation in the case of this mathematics, and by experiments in the case of physics. See Theorems 2.1.8, 2.1.9 for a more detailed discussion.

In section 2.1, we work out some elementary implications between MES, MDS, MCS. (For Theorem 2.3.7, we present a non elementary proof of their equivalence with Con(SRP) over WKL_0). We give elementary proofs of MDS → MES and MCS → MES. However, there is a difficulty in giving elementary proofs of MES → MDS and MES → MCS. The latter needs to be addressed for the proper development of Emulation Theory.

We present a weakening of MCS called MCS(res), where the graphs are required to be "restricted", a rather mild condition.

MAXIMAL CLIQUE STABILITY(res). MCS(res). Every restricted order invariant graph on $Q[0,k]^k$ has a stable maximal clique.

We give an elementary proof of MES → MCS(res). This sets the stage for the promised reversal to appear in [Fr19] - the reversal that establishes MCS(res) → Con(SRP) in RCA_0.

We show that MES, MDS< MCS, MCS(res) are implicitly Π^0_1 over WKL_0 via the Gödel Completeness Theorem and Herbrand's Theorem (the latter is used for bringing ACA_0 down to WKL_0 here). See Definition 2.16 and Theorems 2.1.8, 2.1.9.

In section 2.2, we introduce our general framework for investigating wide ranging alternatives to our very basic MES, MDS, MCS, MCS(res). However, we experience a rapid proliferation of parameters, and we must use a powerful notation in order to keep this development under control. We use the consolidating notation

$$\bullet[R,k,n\infty,r\infty,ext^*,step^*]$$

which we presently explain. A completely rigorous definition is given in section 2.2. We move to the more general spaces $Q[0,n]^k$ instead of just the $Q[0,k]^k$. R is required to be a relation on Q^k. I.e., $R \subseteq Q^k \times Q^k = Q^{2k}$. The • indicates the choice from ME, MD, MC, MCres, as in emulation, duplication, graph/clique, restricted graph/clique. The k indicates the dimension in $Q[0,n]^k$. The n∞ indicates either the parameter n, which is the right endpoint in Q[0,n], or ∞, which indicates that we quantify over all n. It is important for the theory that we sometimes want to universally quantify over n, thereby putting more and more of R to use. (In sections 3.1,3.2,3.3, we always quantify over n). However, since we require $R \subseteq Q^{2k}$, it makes no sense to be quantifying over k.

The r∞ indicates either r as in r-emulator, r-duplicator, order invariant r-graph/r-clique, or ∞, which means that we quantify over all r. For emulator, duplicator, order invariant graph/clique, we use r = 2, indicating ordered pairs from $Q[0,n]^k$. For r-emulator, r-duplicator, order invariant r-graph/r-clique, we use ordered r-tuples from $Q[0,n]^k$. Thus the r-graph/r-clique often appears in the literature as r-hypergraph/r-hyperclique. Note that 2-graph/2-clique is the same as graph/clique.

Ext* means either ext (extended) or *, which is the default used in MES, MDS, MCS. Ext requires that the maximal emulator $S \subseteq Q[0,n]^k$ contain a given finite r-emulator (duplicator) of the given finite subset of $Q[0,n]^k$, or the maximal clique $S \subseteq Q[0,n]^k$ contain a given finite r-clique of the given order invariant r-graph on $Q[0,n]^k$. This is directly analogous to using the familiar "every clique extends to a maximal clique" instead of the familiar "there exists a maximal clique".

Step* indicates step maximal or *, which is just plain old maximal. In maximal, the rivals $S \subsetneq S' \subseteq Q[0,n]^k$ are ruled out. In step maximal, rivals $S \cap Q[0,i]^k \subsetneq S' \subseteq Q[0,i]^k$, $i \leq n$, are ruled out.

The r-emulator, r-duplicator, r-clique is required to be completely R invariant. See Definition 2.2.1.

In this notation, MES MDS, MCS, MCS(res) appear as

For all k, $ME[R_0[k],k,k,2,*,*]$
For all k, $MD[R_0[k],k,k,2,*,*]$
For all k, $MD[R_0[k],k,k,2,*,*]$
For all k, $MC(res)[R_0[k],k,k,2,*,*]$

respectively, where the equivalence relation $R_0[k] \subseteq Q^{2k}$ is chosen so that complete $R_0[k]$ invariance of $S \subseteq Q^k$ is equivalent to S being stable (as defined in 1) above, and Definition 2.1.5). A substantial list of natural alternative notions of stability appears in Appendix A, all of which can be viewed as complete invariance with respect to relations. The strongest of these relations is Critical Equivalence, crit[k], used in section 2.3, where complete invariance is call full invariance. See Definitions 3.1.2, 3.1.3.

In section 2.2, we establish the fundamental (easy) necessary condition for $ME[R,k,n,r,*,*]$, and for $MD[R,k,n,r,*,*]$, which is that R be order preserving on

$Q[0,n]^k$. I.e., $x,y \in Q]0,n]^k \wedge x\ R\ y \rightarrow x,y$ are order equivalent. See Theorem 2.2.15.

In section 2.3, we introduce Critical Equivalence on Q^k, which we write as $crit[k] \subseteq Q^{2k}$. crit[k] is very robust, and there are several definitions of it in this paper. The official one is Definitions 2.3.1, 3.1.2 (repeat), 3.1.3. But also see Theorem 3.1.5 and Appendix A. We fix on one convenient one for section 2.3. There are other definitions in Appendix A, and also in section 3.1. The corresponding stability notion, invariance with respect to Critical Equivalence - the crit[k] - is called Full Stability, in contrast to Stability. We relate $crit[k] \subseteq Q^{2k}$ to the equivalence relations $R_kA \subseteq Q[0,1]^{2k}$ from [Fr17]. crit[k] is the strongest equivalence relation on Q^k that we consider (along with its dual). We then focus on •[crit[k],k,n∞,r∞,ext*,step*]. So the strongest • statements that we consider in the paper are

For all k, ME[crit[k],k,∞,∞,ext,step]
For all k, MD[crit[k],k,∞,∞,ext,step]
For all k, MC[crit[k],k,∞,∞,ext,step]

and we give an elementary proof of equivalence. We conclude section 2.3 by proving ME[crit[k],k,∞,∞,ext,step] (and therefore all three) in SRP^+, and in fact, in WKL_0 + Con(SRP). We accomplish this using [Fr17]. We carefully adapt the proof of the statement FESU/2 in [Fr17] to ME[crit[k],k,∞,∞,ext,step].

In section 3, we give a complete characterization of certain "simple" equivalence relations $R \subseteq Q^{2k}$ that can be used in Emulation Theory, in certain ways. In sections 3.1, 3.2, 3.3, we focus on ME[R,k,∞,2,*,*]. We begin by showing that the only order invariant $R \subseteq Q^{2k}$ for which ME(R,k,∞,2,*,*) holds is trivial ($x\ R\ y \rightarrow x = y$). See Definition 2.1.4 and Theorem 3.1.10. So order invariance is far too restrictive.

The "simple" equivalence relations on Q^k that are handled here are those that are permutation invariant and order invariant/Z. Order invariant means involving < only (see Definition 2.??), and order invariant/Z means involving < and "being an integer". This allows a lot more coordinate free equivalence relations. See Definitions 3.1, 3.1.5, and Theorem 3.2.

In section 3.1, we focus on five natural equivalence relations on Q^k. These are triv[k], DZ[k], DQ\Z[k], crit[k], -crit[k], which are all order invariant/Z (triv[k] only is order invariant). Of these, DZ[k], DQ\Z[k], crit[k], -crit[k], are called the k-principal relations (Definition 3.1.4), and they are also permutation invariant. -crit[k] is the dual of crit[k] as discussed in section 3.

In section 3.2, we give a complete characterization of the permutation invariant/Z equivalence relations R on Q^2 such that ME[R,2,∞,2,*,*]. These are exactly those that are contained in at least one of the 2-principal relations DZ[2], DQ\Z[2], crit[2], -crit[2]. This characterization in section 3.2 is carried out in Z_3. See Theorem 3.2.21.

In section 3.3, we show (IN LATER DRAFT) that the characterization in section 3.2 does not extend to all k by giving a counterexample in dimension k = ? (SEE LATER DRAFT). We have encountered extensive difficulties in getting a complete characterization in higher dimensions along the lines of the one given in section 3.2. However, we succeed in section 3.3 by bringing in the notion of an initially fractional equivalence relation on Q^k. We also see that the resulting characterization theorem (universally quantifying over k) is equivalent to Con(SRP) over WKL_0. See Theorem 3.3.5.

In section 3.4, we take up ME[R,k,n,2,*,*], where R is an isometry between two integral line segments in $Q[0,n]^k$. POSTPONED LARGELY TO LATER DRAFT.

In section 4, we briefly discuss some far ranging ongoing and future projects.

Appendix A lists some natural alternative notions of stability and associated relations to be used for the complete invariance in Emulation Theory.

Appendix B compiles the formal systems used throughout the paper.

2. TANGIBLES

In section 2.1, we present our three lead statements of Emulation Theory.

MAXIMAL EMULATION STABILITY. MES. Every finite subset of $Q[0,k]^k$ has a stable maximal emulator.

MAXIMAL DUPLICATION STABILITY. MDS. Every finite subset of $Q[0,k]^k$ has a stable maximal duplicator.

MAXIMAL CLIQUE STABILITY. MCS. Every order invariant graph on $Q[0,k]^k$ has a stable maximal clique.

In section 2.1, we give elementary proofs of the implications MDS → MES and MCS → MES. We also give an elementary proof of MES → MCS(res).

MAXIMAL CLIQUE STABILITY(restricted). MCS(res). Every restricted order invariant graph on $Q[0,k]^k$ has a stable maximal clique.

where the restriction is: max(x) < min(y) → x,y are adjacent.

In section 2.3, we derive sharper versions of MES, MDS, MCS from WKL_0 + Con(SRP) by adapting the main proof in [Fr17]. The reversal MCS(res) → Con(SRP) over RCA_0 is promised for [Fr19]. Putting these developments together, we obtain the equivalence of MES, MDS, MCS, MCS(res), Con(SRP) over WKL_0.

In section 2.1, we work out seven easy examples of MES, MDS suitable for beginning students, we think suitable for gifted high school students, and certainly beginning undergraduates. We also show that MES, MDS, MCS, MCS(res) are implicitly Π^0_1 via the Gödel Completeness Theorem. Thus the lead statements in Emulation Theory are seen to be Π^0_1 (implicitly) in light of what essentially amounts to their logical form.

In section 2.2, we lay the groundwork for a more general theory where key parameters are identified and requirements are strengthened. In particular, the notion of stable was chosen for its memorable simplicity and deceptive weakness. In the more general theory, we use complete invariance with respect to a general relation (stable is of this form). We develop the flexible notation •(R,k,n∞,r∞,ext*,step*) to incorporate complete R invariance, r-emulators, r-duplicators, r-graphs/r-cliques, spaces Q[0,n]k, extension (analogous to the way "every clique can be extended to a maximal clique" sharpens "there is a maximal clique"), and step maximal (maximal in $Q[0,1]^k,...,Q[0,n]^k$ as opposed to just $Q[0,n]^k$). Here ∞ represents the option of universal quantification over n,r, and * represents the option of ignoring. ext(ension), step (maximal). We work out the trivial cases k = 1 ∨ n = 1 ∨ r = 1, and some elementary implications.

In section 2.3, we define the crucially important Critical Equivalence, written crit[k] on Q^k, with crit[k] ⊆ Q^{2k}. This is used for •[crit[k],k,∞,∞,ext,step], which is the strongest statement that we consider in the paper. Complete invariance with respect to Critical Equivalence is referred to as Full Stability. We prove Con(SRP) → (∀k)(•[crit[k],k,∞,∞,ext,step]) in WKL_0 using the so called Exotic Proof from [Fr17]. As mentioned earlier, this shows the equivalence of MES, MDS, MCS, MCS(res), Con(SRP) over WKL_0 using the promised reversal promised for [Fr19].

2.1. MES, MDS, MCS, MCS(res)

DEFINITION 2.1.1. N,Z,Z^+,Q is the set of all nonnegative integers, integers, positive integers, rationals, respectively. We use i,j,k,n,m,r,s,t with or without subscripts for positive integers unless otherwise indicated. We use p,q with or without subscripts for rationals unless otherwise indicated. Q[p,q] = Q ∩ [p,q]. $x,y \in Q^k$ are order equivalent if and only if for all $1 \le i,j \le k$, $x_i < x_j \leftrightarrow y_i < y_j$.

Evidently, order equivalence implies that for all $1 \le i,j \le k$, $(x_i = x_j \leftrightarrow y_i = y_j) \wedge (x_i \le x_j \leftrightarrow y_i \le y_j)$.

We emphasize that at numerous places in this paper, we rely on the conventions concerning variables in Definition 2.1.1.

EXAMPLE. (1,4,3),(2,8,3) are order equivalent, but (1,4,3),(2,8,2) are not.

The order equivalence relation on Q^k, or with Q replaced by any linearly ordered set of at least 2k elements, is a well studied concept. See, e.g, [Gr62]. Also the following is clarifying from the more abstract point of view.

THEOREM 2.1.1. $x,y \in Q^k$ are order equivalent if and only if there is an order automorphism of Q (automorphism of (Q,<)) which sends x to y (via the coordinate action).

Starting in section 2.2, we will be working in the more general spaces $Q[0,n]^k$ rather than just the $Q[0,k]^k$.

DEFINITION 2.1.2. S is an emulator of $B \subseteq Q[0,n]^k$ if and only if $S \subseteq Q[0,n]^k$ and every element of S^2 is order equivalent to an element of B^2. S is a maximal emulator of E $\subseteq Q[0,n]^k$ if and only if S is an emulator of $B \subseteq Q[0,n]^k$ which is not a proper subset of any emulator of $B \subseteq Q[0,n]^k$.

DEFINITION 2.1.3. S is a duplicator of $B \subseteq Q[0,n]^k$ if and only if $S \subseteq Q[0,n]^k$ and every element of S^2 (B^2) is order equivalent to an element of B^2 (S^2). S is a maximal duplicator of $B \subseteq Q[0,n]^k$ if and only if S is a duplicator of $B \subseteq Q[0,n]^k$ which is not a proper subset of any duplicator of $B \subseteq Q[0,n]^k$.

Definitions 2.1.1, 2.1.2 support MES; 2.1.1, 2.1.3 support MDS.

DEFINITION 2.1.4. $X \subseteq Q[0,n]^r$ is order invariant if and only if for all order equivalent $x,y \in Q[0,n]^r$, $x \in X \leftrightarrow y \in X$. A graph on V is a pair G = (V,E), where $E \subseteq V^2$ is irreflexive and symmetric. G is an order invariant graph G on $Q[0,n]^k$ if and only if G is a graph on $Q[0,n]^k$ where $E \subseteq Q[0,n]^k \times Q[0,n]^k = Q[0,n]^{2k}$ is order invariant. A clique in G is a subset of V where any two distinct elements are E related. A maximal clique in G is a clique in G which is not a proper subset of any clique in G.

Definitions 2.1.1, 2.1.4 support MCS.

THEOREM 2.1.2. (RCA_0) Every subset of $Q[0,n]^k$ has a maximal emulator and a maximal duplicator. Every graph on $Q[0,n]^k$ has a maximal clique. These maximal emulators and duplicators can be taken to be (elementary) recursive. If the graph is (elementary) recursive then the maximal clique can be taken to be (elementary) recursive. Every subset of $Q[0,n]^k$ has the same emulators (duplicators) as some finite subset of $Q[0,n]^k$ whose cardinality is bounded by a polynomial in k!.

Proof: The last claim follows from the fact that there are only a polynomial in k! of equivalence classes under order equivalence on Q^k. Now start with a standard enumeration of $Q[0,n]^k$ and perform the obvious greedy algorithm, retaining a term in the enumeration if doing so maintains having an emulator (duplicator, clique), and skipping over that term otherwise. This yields a (elementary) recursive maximal emulator (duplicator, clique). QED

We now introduce our default notion of stability.

DEFINITION 2.1.5. $S \subseteq Q[0,k]^k$ is stable if and only if for all $p < 1$, $(p,1,...,k-1) \in S \leftrightarrow (p,2,...,k) \in S$.

EXAMPLE. Stable $S \subseteq Q[0,6]^6$ must have $(.99,1,2,3,4,5) \in S \leftrightarrow (.99,2,3,4,5,6) \in S$.

MES, MDS, MCS are based on this default notion, which is the weakest that we consider. There are many other natural and interesting choices for stability that we know lead to the same equivalence with Con(SRP) over WKL_0. Many such choices are listed in Appendix A.

We now give some examples in $Q[0,2]^2$ suitable for beginning students.

EXAMPLE 1. Emulators of $\emptyset \subseteq Q[0,2]^2$. These are $\emptyset$. Maximal emulator is $\emptyset$. Duplicators of $\emptyset \subseteq Q[0,2]^2$. These are $\emptyset$. Maximal duplicator is $\emptyset$.

EXAMPLE 2. Emulators of $\{x\} \subseteq Q[0,2]^2$. These are $\emptyset$, $\{y\}$, y order equivalent to x. Maximal emulators are these $\{y\}$. Duplicators of $\{x\} \subseteq Q[0,2]^2$. These are $\{y\}$, y order equivalent to x. Maximal duplicators are these $\{y\}$.

EXAMPLE 3. Emulators of $\{(0,0),(1,1)\} \subseteq Q[0,2]^2$ are the $S \subseteq Q[0,2]^2$ where all $(p,q) \in S$ have $p = q$. Maximal emulator is $\{(p,p): 0 \leq p \leq 2\}$. Duplicators of $\{(0,0),(1,1)\} \subseteq Q[0,2]^2$ are the $S \subseteq Q[0,2]^2$, $|S| \geq 2$, where all $(p,q) \in S$ have $p = q$. Maximal duplicator is $\{(p,p): 0 \leq p \leq 2\}$.

EXAMPLE 4. Emulators of $\{(0,1),(0,2)\} \subseteq Q[0,2]^2$ are the $S \subseteq Q[0,2]^2$ where all $(p,q),(r,s) \in S$ have $p < q \wedge p = r < s$. Maximal emulators are the $\{(p,q): p < q\}$, where $0 \leq p < 2$ is fixed. Duplicators of $\{(0,1),(0,2)\} \subseteq Q[0,2]^2$ are the $S \subseteq Q[0,2]^2$, $|S| \geq 2$, where all $(p,q),(r,s) \in S$ have $p < q \wedge p = r < s$. Maximal duplicators are the $\{(p,q): p < q\}$, where $0 \leq p < 2$ is fixed.

EXAMPLE 5. Emulators of $\{(0,1),(1,2)\} \subseteq Q[0,2]^2$ are $\emptyset$, $\{(p,q)\}$, $p < q$, and the $\{(r,s),(s,t)\} \subseteq Q[0,2]^2$, $r < s < t$. Maximal emulators are $\{(0,2)\}$ and these $\{(r,s),(s,t)\}$. Duplicators of $\{(0,1),(1,2)\} \subseteq Q[0,2]^2$ are the $\{(r,s),(s,t)\} \subseteq Q[0,2]^2$, $r < s < t$. Maximal duplicators are these $\{(r,s),(s,t)\}$. Note that one of the maximal emulators is not a duplicator.

EXAMPLE 6. Emulators of $\{(0,1),(3/2,2)\} \subseteq Q[0,2]^2$ are the $S \subseteq Q[0,2]^2$ where all distinct $(p,q),(r,s) \in S$ have $p < q < r < s \vee r < s < p < q$. Maximal emulators are these S with no gap. Duplicators of $\{(0,1),(3/2,2)\} \subseteq Q[0,2]^2$ are the $S \subseteq Q[0,2]^2$, $|S| \geq 2$, where all distinct $(p,q),(r,s) \in S$ have $p < q < r < s \vee r < s < p < q$. Maximal duplicators are these S with no gap.

EXAMPLE 7. Emulators of $\{(0,1),(2,2)\} \subseteq Q[0,2]^2$ are $\emptyset$, the $\{(p,q)\}$ for $p < q$, the $\{(p,p)\}$, and the $\{(p,q),(r,r)\} \subseteq$ Q[0,2]2, $p < q < r$. Maximal emulators are the $\{(p,q),(r,r)\} \subseteq$ Q[0,2]2, $p < q < r$. Duplicators of $\{(0,1),(2,2)\} \subseteq Q[0,2]^2$ are the $\{(p,q),(r,r)\} \subseteq Q[0,2]^2$, $p < q < r$. Maximal duplicators of $\{(0,1),(2,2)\} \subseteq Q[0,2]^2$ are the $\{(p,q),(r,r)\} \subseteq Q[0,2]^2$, $p < q < r$.

Rather than work with our default stability notion (Definition 2.1.5), we might as well work with a much stronger stability notion which is particularly simple when restricted to the present $Q[0,2]^2$. This is Full Stability, discussed in section 2.3 (Definition 2.3.2, Theorem 3.1.5, Appendix A) which occupies a special status in section 3. For now, we only need to know that $S \subseteq Q[0,2]^2$ is fully stable if and only if

i. For $p < 1$, $(p,1) \in S \leftrightarrow (p,2) \in S$.
ii. For $p < 1$, $(1,p) \in S \leftrightarrow (2,p) \in S$.

iii. $(1,1) \in S \leftrightarrow (2,2) \in S$.

Stable is just i. We can readily verify MES and MDS with Full Stability for our seven simple examples above on $Q[0,2]^2$:

EXAMPLE 1. $\emptyset$ is fully stable.

EXAMPLE 2. $\{y\}$ is fully stable if $\max(y) < 1$.

EXAMPLE 3. $\{(p,p): 0 \leq p \leq 2\}$ is fully stable.

EXAMPLE 4. $\{(1,q): 1 < q\}$ is fully stable.

EXAMPLE 5. $\{(r,s),(s,t)\} \subseteq Q[0,2]^2$, $r < s < t$, is fully stable if $r,s,t \notin \{1,2\}$.

EXAMPLE 6. Obviously there exist such S with no gaps where 1,2 do not appear as coordinates. These are fully stable.

EXAMPLE 7. $\{(1/3,1/2),(3/2,3/2)\} \subseteq Q[0,2]2$, $p < q < r$ is fully stable.

We now look at MES, MDS, MCS in dimension k, where k is fixed and not universally quantified.

THEOREM 2.1.3. (RCA_0) S is a duplicator of $E \subseteq Q[0,n]^k$ if and only if S is an emulator of $E \subseteq Q[0,n]^k$ and E is an emulator of $S \subseteq Q[0,n]^k$. If S is a duplicator of $E \subseteq Q[0,n]^k$ and $S' \supseteq S$ is an emulator of $E \subseteq Q[0,n]^k$, then S' is a duplicator of $E \subseteq Q[0,n]^k$.

Proof: The first claim is immediate. Let S be a duplicator of $E \subseteq Q[0,n]^k$ and $S' \supseteq S$ be an emulator of $E \subseteq Q[0,n]^k$. We need to show that E is an emulator of $S' \subseteq Q[0,n]^k$. Every $x \in E^2$ is order equivalent to some $y \in S^2$, and hence some $y \in S'^2$. QED

LEMMA 2.1.4. RCA_0 proves MDS $\rightarrow$ MES.

Proof: Assume MDS, and let $B \subseteq Q[0,k]^k$ be finite. Let S be a maximal duplicator of $B \subseteq Q[0,k]^k$. Then S is an emulator of $B \subseteq Q[0,k]^k$. If S is a proper subset of some emulator S' of

$B \subseteq Q[0,k]^k$ then this S' is a duplicator of $B \subseteq Q[0,k]^k$, which is a contradiction. QED

It is not clear how to give an elementary proof of MES[k] → MDS[k]. In the Examples 1,2,3,4,6,7 in section 2.1, all maximal emulators are maximal duplicators. In Example 5, this is not the case.

LEMMA 2.1.5. RCA_0 proves MCS → MES.

Proof: Assume MCS, and let $B \subseteq Q[0,k]^k$ be finite. We construct a stable maximal emulator of $B \subseteq Q[0,k]^k$. If B = ∅ then B is a stable maximal emulator of B. If B = {x} then {x'} is a stable maximal emulator of B, where x,x' are order equivalent and $x' \in Q[0,1)^k$. Henceforth, we assume |B| ≥ 2.

Let G be the order invariant graph on $Q[0,k]^k$ whose edges are the $(x,y) \in Q[0,k]^k \times Q[0,k]^k$ with x ≠ y that are order equivalent to some (x',y') ∈ B × B. Hence G has at least one edge. Let X be the set of all $x \in Q[0,k]^k$ not order equivalent to any element of A. Then G and X are order invariant. Let G' be G together with all edges (x,y),(y,x), where x ≠ y and x ∈ X. Then G' is an order invariant graph on $Q[0,k]^k$.

We claim that if S is a clique in G' then S\X is a clique in G. Let S be a clique in G', and let x ≠ y be from S\X. Now (x,y) is an edge in G', and by the construction of G', (x,y) is an edge in G.

We claim that if S is a clique in G then S ∪ X is a clique in G'. Let S be a clique in G, and let x ≠ y be from S ∪ X. If x,y ∈ S then (x,y) is an edge in G and therefore in G'. Otherwise, (x,y) is an edge in G'.

We claim that if S is a maximal clique in G' then X ⊆≠ S and S\X is a maximal clique in G. To see this, let S be a maximal clique in G'. S ⊆ X is impossible since G has at least one edge. The insertion of elements of X cannot destroy cliquedom in G', so X ⊆≠ S. Now S\X is a clique in G since S is a clique in G'. Suppose S\X ∪ {x} is a clique in G. Then S\X ∪ {x} ∪ X is a clique in G'. Hence S ∪ {x} is a clique in G'. Therefore x ∈ S. Also from S\X ∪ {x} being a clique in G, and S\X ≠ ∅, we have x ∉ X. Hence x ∈ S\X.

Now let S ⊆ $Q[0,k]^k$ be a stable maximal clique in G'. We now show that S\X is a stable maximal emulator of A ⊆ $Q[0,k]^k$. Note that X ⊊ S and S\X is a maximal clique in G. We show the following.

1. S\X is an emulator of B ⊆ $Q[0,k]^k$. Let (x,y) ∈ S\X × S\X. If x ≠ y then (x,y) is an edge in G, and so (x,y) is order equivalent to an element of B^2. Suppose x = y. Since x ∉ X, x is order equivalent to an element of B. Hence (x,x) is order equivalent to an element of B^2.

2. S\X is a maximal emulator of B ⊆ $Q[0,k]^k$. Let S\X ∪. {x} be an emulator of B ⊆ $Q[0,k]^k$. We claim that S\X ∪. {x} is a clique in G. Let y,z ∈ S\X ∪. {x}, y ≠ z. Then (y,z) is order equivalent to an element of A^2. Since y ≠ z, (y,z) is an edge in G. We have a contradiction since S\X is a maximal clique in G.

3. S\X is stable. This follows from S being stable and X being order invariant. This is because equivalent tuples in the definition of stable are also order equivalent.

QED

There seem to be difficulties in giving an elementary proof of MES → MCS. We now weaken MCS as follows.

MAXIMAL CLIQUE STABILITY(res). MCS(res). Every order invariant graph on $Q[0,n]^k$ satisfying max(v) < min(w) → v,w are adjacent, has a stable maximal clique.

LEMMA 2.1.6. RCA_0 proves MES → MCS(res).

Proof: Assume MES. Let G be an order invariant graph on $Q[0,k]^k$, where max(v) < min(w) → v,w are adjacent. Let E be the edge set of G. If E = ∅ then obviously {(0,...,0)} is a stable maximal clique. So we assume E ≠ ∅.

We first construct a nonempty finite clique C in G such that every element of E is order equivalent to an element of C^2. Let $(x_1,y_1),(x_2,y_2),...,(x_n,y_n)$ enumerate a finite number of elements of E, without repetition, such that every element of E is order equivalent to some (x_i,y_i). Make copies of each $\{x_i,y_i\}$ in $Q[0,n]^k$ so that for all $1 \leq i < k$, the largest rational appearing in $\{x_i,y_i\}$ is less than the smallest rational appearing in $\{x_{i+1},y_{i+1}\}$. This results in the desired clique C in G.

Let S be a maximal emulator of C. We show that S is a maximal clique in G. Clearly $S \neq \emptyset$.

1. S is a clique in G. Let $x,y \in S$, $x \neq y$. Let (x,y) be order equivalent to $(z,w) \in C^2$, $z \neq w$. Then (x,y) is an edge in G.

2. S is a maximal clique in G. Let $S \cup. \{x\}$ be a clique in G. We claim that $S \cup. \{x\}$ is an emulator of C, which is a contradiction. Let $y,z \in S \cup. \{x\}$.

2.1. Assume $y,z \in S$. Since S is an emulator of C, (y,z) is order equivalent to an element of C^2.

2.2. Assume $y = x \neq z$. Then $z \in S$ and since $S \cup \{x\}$ is a clique in G, (y,z) is an edge in G, and therefore order equivalent to an element of C^2.

2.3. Assume $y = z = x$. Since $S \neq \emptyset$, let $w \in S$. Hence (x,w) is an edge in G, and so (x,w) is order equivalent to an element of C^2. In particular, (x,x) is order equivalent to an element of C^2.

QED

THEOREM 2.1.7. RCA_0 proves (MDS v MCS) → MES → MCS(res).

Proof: By Lemmas 2.1.4, 2.1.5, 2.1.6. QED

We now show that MES, MDS, MCS, MCS(res) are implicitly Π^0_1 over WKL_0. We view MES, MDS< MCS, MCS(res) as sentences in the usual language of second order arithmetic, $L(Z_2)$. It is convenient to assume that exponentiation is included in $L(Z_2)$.

DEFINITION 2.1.6. A Π^0_1 sentence φ is a sentence in $L(Z_2)$ with no set quantifiers and no set variables, beginning with zero or more universal numerical quantifiers followed by a formula all of whose quantifiers are bounded (<) by terms. Let K be a theory in $L(Z_2)$. A sentence φ is implicitly Π^0_1 over K if and only if there exists a Π^0_1 sentence ψ such that K proves $\varphi \leftrightarrow \psi$.

Π^0_1 is at the lowest level of logical complexity above the (theoretically) finitely observable.

Implicitly Π^0_1 sentences over trusted T enjoy a property analogous to that of valued physical theories: they are subject to refutation, in the sense that in principle, if they are false then they can be refuted - by computation in the case of this mathematics, and by experiments in the case of physics.

THEOREM 2.1.8. (EFA) Let K be a finitely axiomatized theory in $L(Z_2)$ extending EFA. φ be implicitly Π^0_1 over K. Then K proves ($\neg\varphi \rightarrow$ "T proves $\neg\varphi$").

THEOREM 2.1.9. MES, MDS, MCS, MCS(res) are implicitly Π^0_1 over WKL_0. WKL_0 proves ($\neg\varphi \rightarrow$ "WKL_0 proves $\neg\varphi$") for φ = MES, MDS, MCS, MCS(res).

Proof: For each of these four sentences A, we can effectively construct an infinite list of associated $\forall^*\exists^*$ sentence $B_1,B_2,\ldots$ in PC(=) = first order predicate calculus with equality, where "A $\leftrightarrow$ each B_i is satisfiable with each domain contained in N" is provable in WKL_0. This is almost sufficient, as one hopes to quote the Gödel Completeness Theorem that in PC(=), satisfiability (in domain contained in N) is equivalent to consistency, and consistency is obviously Π^0_1. That consistency implies satisfiability is clearly provable in WKL_0 by a standard proof of Completeness. However, that satisfiability implies consistency involves an arithmetic induction, and so the stronger ACA_0 suffices. To keep this within WKL_0, we go back and adjust B by adding new function symbols, so that B becomes purely universal, which is not a problem for $\forall^*\exists^*$ sentences using RCA_0. Then instead of quoting Gödel's Completeness Theorem, we can quote Herbrand's Theorem which is provable in WKL_0.

We provide some details for the construction of the $\forall^*\exists^*$ sentences for MES. We are given k and $E \subseteq Q[0,k]^k$. We start with the axioms for a dense linear ordering with left/right endpoints 0,k, using the binary relation symbol < and constants 0,1,...,k. We add the k-ary relation symbol P with the $\forall^*$ axiom asserting that P is an emulator of $E \subseteq Q[0,k]^k$. Here we don't actually put E in any way into this

∀* sentence, but rather reference the order types of the elements of E. Now stating that P is stable is another ∀* axiom involving the constants 1,...,k. Finally, there is the maximality of P. This is expressed by asserting that if we try to add a new tuple to P then we ruin that it is an emulator. The cause of the ruin is reflected by ∃*, and so the maximality is reflected by a ∀*∃* axiom. QED

Even though we have shown MES, MDS, MCS, MCS(res) to be implicitly Π^0_1 over WKL_0, we remark that the explicitly Π^0_1 equivalent is mathematically hopelessly artificial. So there remains the question of finding mathematically interesting Π^0_1 equivalents. This is ongoing research, mentioned in section 4.

2.2. •[R,k,n∞,r∞,ext*,step*].

We now develop a general theory where we extend MES, MDS, MCS, MCS(res) through the addition of several parameters. We work in the spaces $Q[0,n]^k$ rather than just the spaces $Q[0,k]^k$, adding the additional parameter n.

DEFINITION 2.2.1. S ⊆ X is completely R invariant if and only if R is a set of ordered pairs, and for all x,y ∈ X, x R y → (x ∈ S ↔ y ∈ S).

Note the important use of the ambient space X for S, which regulates the condition using "x,y ∈ X". There is a fair amount of general nonsense that is well known about complete invariance.

DEFINITION 2.2.2. A relation on X is a subset of X^2. An equivalence relation on X is a reflexive, symmetric, transitive relation on X. In particular, X^2 is required to be contained in any equivalence relation on X.

The most immediately transparent case of completely invariance is covered as follows.

THEOREM 2.2.1. Let R be an equivalence relation on X. The following are equivalent.
i. S ⊆ X is completely R invariant.
ii. Every x R y with x ∈ S has y ∈ S.
iii. S is the union of equivalence classes of R.

Proof: Assume i. Let x R y, x ∈ S. Since R is on X, x,y ∈ X. Hence x ∈ S ↔ y ∈ S. So y ∈ S. Assume ii. By ii, every equivalence class of R is contained or disjoint from S. Therefore S is the union of the equivalence classes of R that are contained in S. Assume iii. Let S ⊆ X and x R y. Then S contains the equivalence class of x or is disjoint from it. In the first case, x,y ∈ S. In the second case, x,y ∉ S. QED

DEFINITION 2.2.3. Let R be an arbitrary set of ordered pairs. EQR(R) is the least equivalence relation on the set of coordinates of elements of R (fld(R)) that contains R. EQR(R,X) is the least equivalence relation on X that contains R ∩ X^2.

THEOREM 2.2.2. The following are equivalent.
i. S ⊆ X is completely R invariant.
ii. S ⊆ X is completely R ∩ X^2 invariant.
iii. S ⊆ X is completely EQR(R) invariant.
iv. S ⊆ X is completely EQR(R,X) invariant.

Proof: Assume ii. Then ii is immediate. Let x,y ∈ X, x R y. Then x R ∩ X^2 y, and so x ∈ S ↔ y ∈ S. Hence i. Assume i. Let V = {(x,y) ∈ EQR(R) ∩ X^2: x ∈ S ↔ y ∈ S}. We claim that V is an equivalence relation on fld(R). Suppose x ∈ fld(R) ∩ X. Then (x,x) ∈ V. Obviously V is symmetric and transitive. Hence V is an equivalence relation on fld(R) ∩ X^2. Therefore EQR(R) ∩ X^2 ⊆ V. Hence S ⊆ X is completely EQR(R) invariant. Hence iii. Assume iii. Then obviously iv. Assume iv. Let x,y ∈ X, x R y. Then x EQR(R,X) y, and so x ∈ S ↔ y ∈ S. Hence i. QED

We view the complete R invariance of S ⊆ X as a notion of stability for S ⊆ X. These notions of stability generalize the very special and elegant stable used in MES, MDS, MCS, MCS(res).

DEFINITION 2.2.4. S is an r-emulator of B ⊆ $Q[0,n]^k$ if and only if every element of S^r is order equivalent to an element of B^r. S is an r-duplicator of B ⊆ $Q[0,n]^k$ if and only if every element of S^r (B^r) is order equivalent to an element of B^r (S^r).

Evidently 2-emulators and 2-duplicators are the same as emulators and duplicators.

DEFINITION 2.2.5. An r-graph on $Q[0,n]^k$ is a $G = (Q[0,n]^k,E)$ where $E \subseteq (Q[0,n]^k)^r$, E is symmetric (invariant under all r! permutations), and every element of E has its r coordinates distinct. An r-clique in G is a subset of $Q[0,n]^k$ where any r-tuple of distinct elements lies in E. E is the k-edge set. A k-graph is order invariant if and only if its $E \subseteq Q[0,n]^{kr}$ is order invariant. An r-graph on $Q[0,n]^k$ is restricted if and only if any $(x_1,...,x_r) \in Q[0,n]^{kr}$ with $x_1,...,x_r$ distinct and $(\exists i,j)(\max(x_i) < \min(x_j)$, lies in E.

Evidently the (order invariant) 2-graphs on $Q[0,n]^k$ are the same as the (order invariant) graphs on $Q[0,n]^k$, the 2-edge set is the edge set, and a restricted 2-graph on $Q[0,n]^k$ is a restricted graph on $Q[0,n]^k$.

DEFINITION 2.2.6. S is a step maximal r-emulator (r-duplicator) of E ⊆ Q[0,n]k if and only if for all i ≤ n, S ∩ Q[0,i]k is an r-emulator (r-duplicator) of E ⊆ Q[0,n]k which is not a proper subset of any r-emulator (r-duplicator) S' ⊆ Q[0,i]k of E ⊆ Q[0,n]k. S is a step maximal r-clique of graph G on Q[0,n]k if and only if for all i ≤ n, S is an r-clique of graph G on Q[0,n]k which is not a proper subset of any r-clique S' ⊆ Q[0,i]k of graph G on Q[0,n]k.

Obviously step maximal implies maximal.

We now introduce notation for our statements in the following table, where we assume that R ⊆ Q* × Q*.

•[R,k,n,r,ext*,step*] TABLE

ME(R,k,n,r,*,*). Every finite subset of $Q[0,n]^k$ has a completely R invariant maximal r-emulator.
MD(R,k,n,r,*,*). Every finite subset of $Q[0,n]^k$ has a completely R invariant maximal r-duplicator.
MC(R,k,n,r,*,*). Every order invariant r-graph on $Q[0,n]^k$ has a completely R invariant maximal r-clique.
MCres(R,k,n,r,*,*). Every restricted order invariant r-graph on $Q[0,n]^k$ has a completely R invariant maximal r-clique.

ME(R,k,n,r,ext,*). Let A ⊆ Q[0,1)^k be a finite r-emulator of finite B ⊆ Q[0,n]^k. Some completely R invariant maximal r-emulator of B contains A.

MD(R,k,n,r,ext,*). Let A ⊆ Q[0,1)^k be a finite r-duplicator of finite B ⊆ Q[0,n]^k. Some completely R invariant maximal r-duplicator of B contains A.

MC(R,k,n,r,ext,*). Let A ⊆ Q[0,1)^k be a finite r-clique in an order invariant r-graph G on Q[0,n]^k. Some completely R invariant maximal r-clique in G contains A.

MCres(R,k,n,r,ext,*). Let A ⊆ Q[0,1)^k be a finite r-clique in a restricted order invariant r-graph G on Q[0,n]^k. Some completely R invariant maximal r-clique in G contains A.

ME(R,k,n,r,*,step). Every finite subset of Q[0,n]^k has a completely R invariant step maximal r-emulator.

MD(R,k,n,r,*,step). Every finite subset of Q[0,n]^k has a completely R invariant step maximal r-duplicator.

MC(R,k,n,r,*,step). Every order invariant r-graph on Q[0,n]^k has a completely R invariant step maximal r-clique.

MCres(R,k,n,r,*,step). Every restricted order invariant r-graph on Q[0,n]^k has a completely R invariant step maximal r-clique.

ME(R,k,n,r,ext,step). Let A ⊆ Q[0,1)^k be a finite r-emulator of finite B ⊆ Q[0,n]^k. Some completely R invariant step maximal r-emulator of B contains A.

MD(R,k,n,r,ext,step). Let A ⊆ Q[0,1)^k be a finite r-emulator of finite B ⊆ Q[0,n]^k. Some completely R invariant step maximal r-emulator of B contains A.

MC(R,k,n,r,ext,step). Let A ⊆ Q[0,1)^k be a finite r-clique in an order invariant r-graph G on Q[0,n]^k. Some completely R invariant step maximal r-clique in G contains A.

MCres(R,k,n,r,ext,step). Let A ⊆ Q[0,1)^k be a finite r-clique in a restricted order invariant r-graph G on Q[0,n]^k. Some completely R invariant step maximal r-clique in G contains A.

We also want to universally quantify zero or more of n,r.

•(R,k,n,∞,ext*,step*). (∀r)(•(R,k,n,r,ext*,step*).
•(R,k,∞,r,ext*,step*). (∀n)(•(R,k,n,r,ext*,step*).
•(R,k,∞,∞,ext*,step*). (∀n,r)(•(R,k,n,r,ext*,step*).

Thus our original MES, MDS, MCS, MCS(res) of section 2.1, correspond to

$$(\forall k)(\mathrm{ME}[R_0[k],k,\infty,2,*,*]$$

$(\forall k)(MD[R_0[k],k,\infty,2,*,*])$
$(\forall k)(MC[R_0[k],k,\infty,2,*,*)$
$(\forall k)(MCres[R_0[k],k,\infty,2,*,*])$,

for an appropriate choice of $R_0[k] \subseteq Q^{2k}$, which we make now.

DEFINITION 2.2.7. $R_0[k] \subseteq Q^{2k}$ is given by $x R y \leftrightarrow (\exists p < 1)(x = (p,1,...,k-1) \wedge y = (p,2,...,k))$.

THEOREM 2.2.4. $S \subseteq Q[0,k]^k$ is stable if and only if S is completely $R_0[k]$ invariant.

Proof: $S \subseteq Q[0,k]^k$ is completely R0[k] invariant if and only if for all $x,y \in Q[0,k]^k$, $x R_0[k] y \rightarrow (x \in S \leftrightarrow y \in S)$. Suppose $(\forall x,y \in Q[0,k]^k)(x R_0[k] y \rightarrow (x \in S \leftrightarrow y \in S))$. Let $p < 1$. Setting $x = (p,1,...,k-1)$, $y = (p,2,...,k)$, we have $(p,1,...,k-1) \in S \leftrightarrow (p,2,...,k) \in S$. Conversely, suppose $S \subseteq Q[0,k]^k$ is stable. Let $x R_0[k] y$. Write $x = (p,1,...,k-1)$, $y = (p,2,...,k)$, $p < 1$. Then $x \in S \leftrightarrow y \in S$. QED

Our full notation here is represented by

$\bullet[R,k,n\infty,r\infty,ext*,step*]$

where we get to choose $R \subseteq Q^{2k}$, positive integer n or ∞, positive integer r or ∞, ext or *, step or *. After these five choices, we arrive at a single mathematical statement.

The choice of ∞'s indicate universal quantification. The choice of *'s indicate the ignoring of ext or step. Ext has the effect analogous to using "every clique extends to a maximal clique" in place of "there exists a maximal clique". Step has the effect of using step maximal instead of maximal, which means using $Q[0,1]^k,...,Q[0,n]^k$ instead of just $Q[0,n]^k$.

We now want to address the trivial cases. Before doing so, we introduce a condition on R that comes up in the theory from time to time.

DEFINITION 2.2.8. $R \subseteq Q^{2k}$ is regular if and only if $(\forall x,y \in Q[0,1)^k)(x R y \rightarrow x = y)$. $R \subseteq Q^{2k}$ is order preserving if and only if $x R y \rightarrow x,y$ are order equivalent.

LEMMA 2.2.7. (RCA_0) Let R ⊆ Q^{2k} be regular.
i. ME(R,1,n,r,ext,step).
ii. MD(R,1,n,r,ext,step).
iii. MC(R,1,n,r,ext,step).

Proof: Let A ⊆ Q[0,1) be a finite r-emulator of finite B ⊆ Q[0,n].

case 1. |B| < r. Then |A| < r. Let A ⊆ S ⊆ Q[0,1), |S| = |B|. Then S is a step maximal r-emulator of B. Furthermore, S ⊆ Q[0,n] is completely R invariant, as R is regular.

case 2. |B| ≥ r. Let A ⊆ S = Q[0,n]. Then S is a step maximal r-emulator of B. Obviously S ⊆ Q[0,n] is completely R invariant.

ii is proved the same way.

Let A ⊆ Q[0,1) be a finite clique in the order invariant r-graph G on Q[0,n].

case 1. G has no edges. Then |A| < r. Let A ⊆ S ⊆ $Q[0,1)^k$, |S| = r-1. Then S is a step maximal r-clique in G. S ⊆ Q[0,n] is completely R invariant.

case 2. G has an edge. Let A ⊆ S = Q[0,n]. Then S is a step maximal r-clique in G. Obviously S ⊆ Q[0,n] is completely R invariant.

QED

LEMMA 2.2.8. (RCA_0) Let R ⊆ Q^{2k} be regular.
i. ME(R,k,1,r,ext,step).
ii. MD(R,k,1,r,ext,step).
iii. MC(R,k,1,r,ext,step).

Proof: Let A ⊆ $Q[0,1)^k$ be a finite r-emulator of finite B ⊆ $Q[0,1]^k$. Let S be a maximal r-emulator of B ⊆ $Q[0,1]^k$ containing A. Then S is also step maximal. S ⊆ $Q[0,1]^k$ is completely R invariant, as R is regular.

Let A ⊆ $Q[0,1)^k$ be a finite r-duplicator of finite B ⊆ $Q[0,1]^k$. Let S be a maximal r-duplicator of B ⊆ $Q[0,1]^k$

containing A. Then S is also step maximal. $S \subseteq Q[0,1]^k$ is completely R invariant, as R is regular.

Let $A \subseteq Q[0,1)^k$ be a finite clique in the order invariant graph G on $Q[0,1]^k$. Let S be a maximal clique in G containing A. Then S is also step maximal. $S \subseteq Q[0,1]^k$ is completely R invariant, as R is regular. QED

LEMMA 2.2.9. (RCA_0) Let $R \subseteq Q^{2k}$ be order preserving on $Q[0,n]^k$.
i. ME(R,k,n,1,ext,step).
ii. MD(R,k,n,1,ext,step).
iii. MC(R,k,n,1,ext,step).

Proof: Let $A \subseteq Q[0,1)k$ be a finite 1-emulator of finite $B \subseteq Q[0,n]^k$. Let $S = \{x \in Q[0,n]^k$: x is order equivalent to an element of A$\}$. Then S is a step maximal 1-emulator of $B \subseteq Q[0,n]^k$. S is order invariant, and hence completely R invariant, since R is order preserving.

Let $A \subseteq Q[0,1)^k$ be a finite 1-duplicator of finite $B \subseteq Q[0,n]^k$. Let $S = \{x \in Q[0,n]^k$: x is order equivalent to an element of A$\}$. Then S is a step maximal 1-duplicator of $B \subseteq Q[0,n]^k$. $S \subseteq Q[0,n]^k$ is order invariant, and hence completely R invariant, since R is order preserving.

Let $A \subseteq Q[0,n]^k$ be a finite 1-clique of order invariant 1-graph G on $Q[0,n]^k$.

case 1. G has no 1-edge. Then $A = \emptyset$ and $\emptyset$ is a step maximal 1-clique that is completely R invariant.

case 2. G has a 1-edge. Then A is a set of 1-edges in G. Let $S = \{x \in Q[0,n]^k$: x is a 1-edge in G$\}$. Then S is a step maximal 1-clique in G. $S \subseteq Q[0,n]^k$ is order invariant, and hence completely R invariant, since R is order preserving.

QED

THEOREM 2.2.10. (RCA_0) If $R \subseteq Q^2$ is regular, then $\bullet$(R,1,∞,∞,ext,step). If $R \subseteq Q^{2k}$ is regular, then $\bullet$(R,k,1,∞,ext,step). If $R \subseteq Q^{2k}$ is order preserving on $Q[0,n]^k$ then $\bullet$(R,k,∞,1,ext,sep).

Proof: By Lemmas 2.2.7, 2.2.8, 2.2.9. QED

We now establish some elementary implications.

LEMMA 2.2.11. (RCA_0) MD(R,k,n,r,ext*,step*) → ME(R,k,n,r,ext*,step*).

Proof: See the proof of Lemma 2.1.4. The idea is that r-duplicators are r-emulators, and every r-emulator containing an r-duplicator is an r-duplicator. Complete R invariance passes through. QED

LEMMA 2.2.12. (RCA_0) Let $R \subseteq Q^* \times Q^*$ be order preserving on $Q[0,\infty)^k$. MC(R,k,n,2,ext*,step*) → ME(R,k,n,2,ext*,step*).

Proof: By adaptation of the proof of Lemma 2.1.5. The hypothesis on R is used to handle the order types that are missing in the given finite $A \subseteq Q[0,n]^k$. QED

Some difficulties arise with Lemma 2.2.12 if r > 2.

LEMMA 2.2.13. (RCA_0) ME(R,k,n,r,ext*,step*) → MCres(R,k,n,r,ext*,step*).

Proof: FILLED IN IN NEXT DRAFT. QED

THEOREM 2.2.14. (RCA_0) MD(R,k,n,r,ext*,step*) → ME(R,k,n,r,ext*,step*). ME(R,k,n,r,ext*,step*) → MCres(R,k,n,r,ext*,step*). Let $R \subseteq Q^* \times Q^*$ be order preserving on $Q[0,\infty)^k$. MC(R,k,n,2,ext*,step*) → ME(R,k,n,2,ext*,step*).

Proof: By Lemmas 2.2.11, 2.2.12, 2.2.13. QED

THEOREM 2.2.15. (RCA_0) ME(R,k,n,r,*,*) → R is order preserving on $Q[0,n]^k$. MD[R,k,n,r,*,*] → R is order preserving on $Q[0,n]^k$.

Proof: Assume ME(R,k,n,r,*,*). Let $x,y \in Q[0,n]^k$, x R y. We show that x,y are order equivalent.

Let $B = \{z \in Q[0,n]^k$: x,z are order equivalent}, and choose finite $B' \subseteq B$ such that B'^r and B^r contain the same order types. Let S be a completely R invariant maximal r-emulator of $B' \subseteq Q[0,n]^k$. Then $S = \{z \in Q[0,n]^k$: x,z are order equivalent}. Since S is completely R invariant, $x \in S \leftrightarrow y \in S$. Hence $y \in S$, and so x,y are order equivalent.

The same argument works for MD. QED

2.3. •[crit[k],k,∞,∞,ext,step].

Critical Equivalence is essentially the strongest equivalence relation that we consider (see section 3.1 for clarification). We write crit[k] for critical equivalence on Q^k.

There are several definitions of the robust crit[k] $\subseteq Q^{2k}$ that we give in this paper. The one we are about to give is the official definition, and is most convenient for the purpose of this section 2.3. We give another definition in Appendix A, and also in section 3.1.

DEFINITION 2.3.1. x crit[k] y if and only if
i. x,y are order equivalent elements of Qk.
ii. min(x) = min(y).
iii. If $x_i \neq y_i$ then every $x_j \geq x_i$ and every $y_j \geq y_s$ are integers.

THEOREM 2.3.1. (EFA) Let x crit[k]]. $(\forall i,j)(x_i \neq y_i \rightarrow x_i,y_i \in Z)$. $(\forall i)(x_i \in Z \leftrightarrow y_i \in Z)$. crit[k] is an equivalence relation on Q^k.

Proof: If $x_i \neq y_i$ then all $x_i \geq x_i$ and $y_i \geq y_i$ lie in Z. Now suppose $x_i \in Z$. If $y_i \notin Z$ then $x_i \neq y_i$, and so $y_i \in Z$, contradiction. This establishes $x_i \in Z \rightarrow y_i \in Z$, and the same argument switched establishes $y_i \in Z \rightarrow x_i \in Z$. Now clearly critical equivalence is reflexive and symmetric. Suppose $x,y \in Q^k$ and $y,z \in Q^k$ are critically equivalent. For i, clearly x,y,z are order equivalent. For ii, x = z is immediate. For iii, let $x_i \neq z_i$.

case 1. $x_i \neq y_i \neq z_i$. Then all $x_j \geq x_i$ and $y_j \geq y_i$ and $z_j \geq z_i$ lie in Z.

case 2. $x_i = y_i \neq z_i$. Then all $y_j \geq y_i$ and $z_j \geq z_i$ lie in Z. Let $x_j \geq x_i$. If $x_j = y_j$ then $x_j \in Z$. If $x_j \neq y_j$ then $x_j \in Z$.

case 3. $x_i \neq y_i = z_i$. Symmetric with case 2.

QED

We quote some excerpts from [Fr17] in italics to highlight its source.

DEFINITION 3.4.1. [Fr17]. Let A ⊆ Q[0,1]. The relation $R_k(A)$ ⊆ $Q[0,1]^k \times Q[0,1]^k$ is given by $R_k(A)(x,y)$ if and only if
i. x,y are order equivalent.
ii. If $x_i \neq y_i$ then all $x_j \geq x_i$ and $y_j \geq y_i$ lie in A.

LEMMA 3.4.4. [Fr17]. (RCA_0) Let $R_k(A)(x,y)$. $(\forall i,j)(x_i \neq y_i \rightarrow x_i,y_i \in A)$. $(\forall i)(x_i \in A \leftrightarrow y_i \in A)$.

THEOREM 3.4.5. [Fr17]. (RCA_0) Each $R_k(A)$ is an equivalence relation on $Q[0,1]^k$.

MAXIMAL EMULATION USE DEFINITION. MEU/DEF. (adapted from [Fr17]). R ⊆ $Q[0,1]^k \times Q[0,1]^k$ is ME usable if and only if for all subsets of $Q[0,1]^k$, some maximal emulator contains its R image.

MAXIMAL EMULATION SMALL USE/2. MESU/2. [Fr17]. For finite A ⊆ Q(0,1], $R_k(A)$ ⊆ $Q[0,1]^k$ x $Q[0,1]^k$ is ME usable.

THEOREM 3.5.9. [Fr17]. MESU/2 for dimension k = 2 is provable in ZFC. In fact, Z and even Z_3 suffices.

THEOREM 3.5.13. (excerpt from [Fr17]). The following are provable in EFA.
ii. ZFC proves that for all k ≥ 1, if there is a max(k-1,0)-subtle cardinal then MESU/2 holds for dimension k+1.

We now adapt this development from [Fr17], for present purposes. We are now going to fix A = {1/n,...,n/n} for $R_k(A)$, and modify MESU/2 from [Fr17], into three statements, bringing in r-emulators, r-duplicators, r-graphs/r-cliques, extensions, and step maximality. Then we are going to adapt the proofs from [Fr17] of Theorems 3.5.9, 3.5.13, to handle these three statements in place of MESU/2 from [Fr17].

MESU/2/emulators. Let A ⊆ $Q[0,1/n)^k$ be a finite r-emulator of finite B ⊆ $Q[0,1]^k$. Some completely $R_k(1/n,...,n/n)$ invariant step maximal r-emulator of B ⊆ $Q[0,1]^k$ contains A. Steps at 1/n,...,n/n.

MESU/2/duplicators. Let A ⊆ Q[0,1/n)^k be a finite r-duplicator of finite B ⊆ Q[0,1]^k. Some completely R_k(1/n,...,n/n) invariant step maximal r-duplicator of B ⊆ Q[0,1]^k contains A. Steps at 1/n,...,n/n.

MESU/2/cliques. Let A ⊆ Q[0,1/n)^k be a finite r-clique in the order invariant graph G on Q[0,1]^k. Some completely R_k(1/n,...,n/n) invariant step maximal r-clique of G contains A. Steps at 1/n,...,n/n.

LEMMA 2.3.2. MESU/2/emulators, MESU/2/duplicators, MESU/2/cliques can be used instead of MESU/2, in Theorems 3.5.9, 3.5.13 from [Fr17].

Proof: We adapt the proofs of Theorems 3.5.9 and 3.5.13 from [Fr17] as follows. We will focus on MESU/2/emulators. It suffices to

i. Fix k,n,r, k ≥ 2, finite A ⊆ Q[0,1/n)^k, finite B ⊆ Q[0,1]^k, A an r-emulator of E ⊆ Q[0,1]^k.
ii. Tweak the proof in [Fr17] of MESU/2 for k and A = {1/n,...,n/n} so that it produces a step maximal r-emulator S of B ⊆ Q[0,1]^k containing A, where the steps are located at 1/n,...,n/n.

Now let's look at the proof of MESU/2 = "R_k(1/n,...,n/n) ⊆ Q[0,1]^{2k} is ME usable", in section 3.5, [Fr17], with the k,n,r,A of i above fixed. So far, the only change we have made is to fix A = {1/n,...,n/n}. We will make some minor changes at the places indicated.

That proof takes place in the transfinite space T[κ] = κ+1 × Q[0,1) where κ is a max(k-2,0)-subtle cardinal, and in T[κ]^k. Here a 0-subtle cardinal is an uncountable regular cardinal, and we use $\kappa = \omega_1$ in case k = 2. That proof starts with the specification of the dense linear ordering with left endpoint only, $<_{T[\kappa]}$ of T[κ], which is firstly by the ordinal first coordinate, and secondly by the second coordinate with the usual ordering of rational numbers. The notion of emulator is based on $<_{T[\kappa]}$. Here we use r-emulators instead of emulators = 2-emulators.

But to support the crucial transfinite recursion, we needed a suitable well ordering of T[κ] and a lifting of it to a well ordering of $T[\kappa]^k$. Accordingly, <* is the well ordering of $T[\kappa]$ first by the ordinal first coordinate, and second by an effective enumeration of Q[0,1), starting with 0, given in advance. <* is lifted to <** on $T[\kappa]^k$ by ordering first by the <* maximum of the k components, and then by the lexicographic ordering of $T[\kappa]^k$ via <*. We then perform the crucial greedy construction of a preferred maximal r-emulator of the given finite $B \subseteq Q[0,1]^k$, based on the well ordering <** of $T[\kappa]^k$. This results in GE(E,κ).

We make a small modification in GE(E,κ), which we call GE(E,κ;B). Note the initial segment {0} × Q[0,1) of both $<_{T[\kappa]}$ and <*, and the initial segment $(\{0\} \times Q[0,1))^k$ of <**. We start the transfinite construction of GE(E,κ;B) by setting the intersection with $(\{0\} \times Q[0,1))^k$ to be $\{((0,q_1),...,(0,q_k)): (q_1,...,q_k) \in B\}$. This is an r-emulator of $B \subseteq Q[0,1]^k$ as required. We then move on in the usual greedy manner to the k-tuples >** than this initial segment $(\{0\} \times Q[0,1))^k$. Because this construction is greedy, we obtain the step maximality condition, where we can take the steps to be determined by the points ((0,0),...,(0,0),(α,0)), $0 < \alpha \leq \kappa$, a lot more steps than we will ever make use of.

These changes are very minor in the scheme of things, changing emulators to r-emulators, and enforcing a tiny beginning of greedy r-emulator, that the rest of the proof is not affected. We obtain a step maximal r-emulator of $E \subseteq Q[0,1]^k$ that includes A, and which is $R_k(\{1/n,...,n/n\})$ invariant, where 1/n,...,n/n are the locations of steps.

MESU/2/duplicators and MESU/2/cliques are handled in the same way. QED

We have now finished using [Fr17] and can now proceed in a self contained fashion from Theorem 2.3.2.

MESU/2/emulators*. Let $A \subseteq Q[0,1)^k$ be a finite r-emulator of finite $B \subseteq Q[0,n]^k$. Some completely $nR_k(1/n,...,n/n)$ invariant step maximal r-emulator of $B \subseteq Q[0,n]^k$ contains A. Steps at 1,...,n.

MESU/2/duplicators*. Let $A \subseteq Q[0,1)^k$ be a finite r-duplicator of finite $B \subseteq Q[0,n]^k$. Some completely

$nR_k(1/n,...,n/n)$ invariant step maximal r-duplicator of B ⊆ $Q[0,n]^k$ contains A. Steps at 1,...,n.

MESU/2/cliques*. Let A ⊆ $Q[0,1)^k$ be a finite r-clique in the order invariant graph G on $Q[0,n]^k$. Some completely $nR_k(1/n,...,n/n)$ invariant step maximal r-clique of G contains A. Steps at 1,...,n.

LEMMA 2.3.3. MESU/2/emulators*, MESU/2/duplicators*, MESU/2/cliques* for $k = 2$ are provable in Z3. For $k \geq 3$, they are provable in ZFC + "there exists a (k-2)-subtle cardinal".

Proof: This is simply a matter of multiplying through by n, exploiting that multiplication by n is an order preserving bijection from [0,1] onto [0,n]. QED

LEMMA 2.3.4. (EFA) $nR_k(1/n,...,n/n) \subseteq Q[0,n]^k$ is defined by x $nR_k(1/n,...,n/n)$ y if and only if
i. x,y are order equivalent.
ii. If $x_i \neq y_i$ then all $x_j \geq x_i$ and $y_j \geq y_i$ lie in {1,...,n}.

Proof: Let $x,y \in Q[0,n]^k$.

$$x\ nR_k(1/n,...,n/n)\ y \leftrightarrow$$
$$x/n\ R_k(1/n,...,n/n)\ y/n \leftrightarrow$$
$$(x/n,y/n \text{ are order equivalent} \wedge (\text{if } x_i/n \neq y_i/n \text{ then all } x_j/n \geq x_i/n \text{ and } y_j/n \geq y_i/n \text{ lie in } \{1/n,...,n/n\}) \leftrightarrow$$
$$(x,y \text{ are order equivalent} \wedge (\text{if } x_i \neq y_i \text{ then all } x_j \geq x_i \text{ and } y_j \geq y_i \text{ lie in } \{1,...,n\}).$$

QED

LEMMA 2.3.5. For all $x,y \in Q[0,n]^k$, x crit[k] y → x $nR_k(1/n,...,n/n)$ y. If $S \subseteq Q[0,n]^k$ is completely $nR_k(1/n,...,n/n)$ invariant then $S \subseteq Q[0,n]^k$ is completely crit[k] invariant.

Proof: Fix k,n. We use the characterization of $nR_k(1/n,...,n/n)$ provided by Lemma 2.3.4. Thus we show that crit[k] satisfies i,ii of Lemma 2.3.4, for all $x,y \in Q[0,n]^k$.

Let $x,y \in Q[0,n]^k$, x crit[k] y. Then x,y are order equivalent. Suppose $x_i \neq y_i$. Then all $x_j \geq x_i$ and $y_j \geq y_i$ lie in Z. Hence all $x_j \geq x_i$ and $y_j \geq y_i$ lie in {0,...,n}.

We claim that all $x_j \geq x_i$ and $y_j \geq y_i$ lie in {1,...,n}. To rule out 0, suppose $0 \geq x_i \vee 0 \geq y_i$. Then $x_i = 0 \vee y_i = 0$, in which case by the definition of crit[k], min(x) = min(y) = 0. Since $x_i = 0 \vee y_i = 0$, we have $x_i = \min(x) \vee y_i = \min(y)$. Since x,y are order equivalent, $x_i = \min(x) \wedge y_i = \min(y)$. Hence $x_i = y_i = 0$, which is a contradiction.

Suppose $S \subseteq Q[0,n]^k$ is completely $nR_k(1/n,...,n/n)$ invariant. Let $x,y \in Q[0,n]^k$, x crit[k] y. Then x $nR_k(1/n,...,n/n)$ y, and so $x \in S \leftrightarrow y \in S$. QED

DEFINITION 2.3.2. $S \subseteq Q[0,n]^k$ is fully stable if and only if S is completely invariant with respect to crit[k].

When we use the ME, MD, MC notation with no mention of R, the default R is crit[k], or full stability.

ME[k,n,r,ext,step]. Let $A \subseteq Q[0,1)^k$ be a finite r-emulator of finite $B \subseteq Q[0,n]^k$. Some fully stable step maximal r-emulator of $B \subseteq Q[0,n]^k$ contains A.

MD[k,n,r,ext,step]. Let $A \subseteq Q[0,1)^k$ be a finite r-duplicator of finite $B \subseteq Q[0,n]^k$. Some fully stable step maximal r-duplicator of $B \subseteq Q[0,n]^k$ contains A.

MC[k,n,r,ext,step]. Let $A \subseteq Q[0,1)^k$ be a finite r-clique in the order invariant graph G on $Q[0,n]^k$. Some fully stable step maximal r-clique of G contains A.

LEMMA 2.3.6. (RCA0) MESU/2/emulators*, MESU/2/duplicators*, MESU/2/cliques* implies ME[k,n,r,ext,step], MD[k,n,r,ext,step], MC[k,n,r,ext,step], respectively.

Proof: Immediate from Lemma 2.3.5. QED

THEOREM 2.3.7. MES, MDS, MCS, $(\forall k)$(ME[k,∞,∞,ext,step]), $(\forall k)$(MD[k,∞,∞,ext,step]), $(\forall k)$(MC[k,n,r,ext,step])
i. are equivalent to Con(SRP) over WKL_0.
ii. are provable for fixed k = 2 in Z3.

iii. are provable for fixed k ≥ 3 in ZFC + "there exists a (k-2)-subtle cardinal".
iv. are provable in SRP^+ but not in SRP, assuming SRP is consistent.

Proof: By Lemma 2.3.3, we have ii,iii. We also have the forward direction of i and iv before "but". A proof of MCS(res) → Con(SRP) in RCA_0 is promised in [Fr19]. Since any of these three statements imply MES(res) over RCA_0, (Theorem 2.2.14), we see that any of these three statements imply Con(SRP) over RCA_0. Hence if any of them were provable SRP then SRP would prove Con(SRP), which, by Gödel's Second Incompleteness Theorem, yields that SRP is inconsistent. QED

3. USABILITY CHARACTERIZATIONS

In Emulation Theory, we seek to understand of just which $R \subseteq Q^{2k}$ we can use in •[R,k,n∞,r∞,ext*,step*]. In sections 3.1 - 3.3 we focus on ME[R,k,∞,2,*,*]. We give this an abbreviated name: k-usable.

R IS k-USABLE. Every finite subset of $Q[0,n]^k$ has a completely R invariant maximal emulator.

Of course we intend this for $R \subseteq Q^k \times Q^k = Q^{2k}$ only, but there is no need to enter that condition in the above definition, given the notions of complete invariance and maximal emulator. Obviously R is k-usable if and only if R ∩ Q2k is k-usable.

Because of the absence of n in the name k-usable, our conventions signal that n is universally quantified. This is an important kind of uniformity we are requiring for a singe $R \subseteq Q^{2k}$ which we considerably exploit.

We have already seen in Theorem 2.3.7 that crit[k] is k-usable (using large cardinals). Therefore every R ⊆ crit[k] is k-usable. Appendix A has a nice supply of very natural R ⊆ crit[k].

GOAL. Determine which "simple" $R \subseteq Q^{2k}$ are k-usable.

Here we have succeeded with a very specific natural notion of "simple" here. It is a daunting challenge to (incrementally) enlarge (or even alter) our space of "simple R" but that is for the future.

Note that for most of the examples of R from Appendix A, R involves only < and "being an integer". Some of them go beyond this and use constants or +1 (on the integers only). But here we handle only the R involving < and "being an integer" in the precise sense below.

Let's take a step back and discuss which $R \subseteq Q^{2k}$ involve only <. The standard way to interpret "involve only <" is in terms of definability without constants (parameters), although allowing constants (parameters) is also important. Definable without parameters is generally referred to as 0-definable, whereas definable with parameters is generally referred to as definable.

THEOREM 3.1. (RCA_0) Let $D \subseteq Q^k$. The following are equivalent.
i. D is quantifier free 0-definable over (Q,<).
ii. D is 0-definable over (Q,<).
iii. D is order invariant.

Proof: This is very well known from elementary model theory, and the well known quantifier elimination over (Q,<). QED

At the end of section 3.1, we prove the following.

THEOREM 3.1.10. (RCA_0) Let $R \subseteq Q^{2k}$ be order invariant. R is k-usable if and only if $R \subseteq triv[k] = \{(x,x): x \in Q^k\}$.

So according to Theorem 3.1.10, using only < for R results exclusively in trivialities.

Now what about using only < and "being an integer"? Note that crit[k] is of this form.

DEFINITION 3.1. $x,y \in Q^k$ are order,Z equivalent if and only if x,y are order equivalent and for all $i \le k$, $x_i \in Z \leftrightarrow y_i \in Z$. $E \subseteq Q^k$ is order,Z invariant if and only if for all order,Z equivalent $x,y \in Q^k$, $x \in E \leftrightarrow y \in E$.

From the well known abstract nonsense about complete invariance (see Theorems 2.2.1, 2.2.2), $E \subseteq Q^k$ is order,Z invariant if and only if E is completely invariant with respect to order,Z equivalence if and only if E is a union of equivalence classes under order,Z equivalence.

Note that, just as is the case with order equivalence, the number of equivalence classes under order,Z equivalence on Q^k is finite.

THEOREM 3.2. (RCA_0) Let $E \subseteq Q^k$. The following are equivalent.
i. E is quantifier free 0-definable over (Q,<,Z).
ii. E is order,Z invariant.
crit[k] $\subseteq Q^{2k}$ is order,Z invariant.

Proof: Fix $E \subseteq Q^k$. Suppose i, and write $D = \{(v_1,...,v_k): \varphi\}$, where φ is a propositional (boolean) combination of atomic formulas $v_i < v_j$ and $v_i \in Z$. Suppose $(p_1,...,p_k)$ and $(q_1,...,q_k)$ are order equivalent/Z. then the truth values of all of those atomic formulas are the same whether you use the p's or use the q's. Hence the truth value of the entire propositional combination is the same whether you use the p's or use the q's. Therefore $(p_1,...,p_k) \in E \leftrightarrow (q_1,...,q_k) \in E$, and ii is established.

Suppose ii. Let $K_1,...,K_t$ list the equivalence classes under order,Z equivalence that are contained in E. Now consider the formula

$$\varphi = (v_1,...,v_k) \in K_1 \vee ... \vee (v_1,...,v_k) \in K_t$$

Each $(v_1,...,v_k) \in K_i$ is an obvious conjunction of atomic formulas $v_i < v_j$, $v_i \in Z$, and their negations. φ obviously defines E by ii, and is clearly a quantifier free 0-definition.

Looking at Definition 2.3.1 of crit[k], it is obvious that we have a massive quantifier free 0-definition of just the right kind. QED

What about $D \subseteq Q^k$ that are 0-definable over (Q,<,Z)? Unlike the situation with (Q,<), the 0-definable sets over (Q,<,Z) are far more inclusive than the quantifier free 0-definable sets over (Q,<,Z). For instance, $\lceil x \rceil = \lceil y \rceil + 3$ is 0-definable over (Q,<,Z) but not quantifier free 0-definable over (Q,<,Z).

In sections 3.1 - 3.3 we focus on $R \subseteq Q^{2k}$ that are permutation order,Z invariant. Section 3.1 identifies four equivalence relations on Q^k that have a special status throughout section 3. We identify some key properties that these four equivalence relations have: permutation invariance and order,Z invariance.

In section 3.2, we characterize all permutation order,Z invariant transitive relations in Q^2 that are 2-usable. This characterization is carried out in Z_3.

In section 3.3, we characterize all permutation order,Z invariant initially fractional transitive relations in Q^k that are k-usable. The correctness of this characterization is equivalent to Con(SRP) over WKL_0.

In section 3.4, we consider k-usability of isometries between pairs of line segments in $Q[0,k]^n$ with integral endpoints.

3.1. FOUR EQUIVALENCE RELATIONS

We now introduce five distinguished equivalence relations on the Q^k.

DEFINITION 3.1.1. triv[k] is the trivial equivalence relation on Q^k, where $x R y \leftrightarrow x = y$. $x \in Q^k$ is diagonal if and only if all of its coordinates are equal. DZ[k] is the equivalence relation on Q^k defined by x DZ[k] y if and only if $x = y \vee (x,y \in Z^k \wedge x,y$ are diagonal). DQ\Z[k] is the equivalence relation on Q^k defined by x DQ\Z[k] y if and only if $x = y \vee (x,y \in (Q\setminus Z)^k \wedge x,y$ are diagonal).

The fourth equivalence relation is crit[k], already defined in section 2.3 (Definition 2.3.1). The fifth equivalence relation is dual to crit[k]. We just change < to > in the definition of crit[k].

DEFINITION 3.1.2. x crit[k] y if and only if
i. x,y are order equivalent elements of Q^k.
ii. min(x) = min(y).
iii. If $x_i \neq y_i$ then every $x_j \geq x_i$ and every $y_j \geq y_i$ are integers.

DEFINITION 3.1.3. x -crit[k] y if and only if
i. x,y are order equivalent elements of Q^k.

ii. $\max(x) = \max(y)$.
iii. If $x_i \neq y_i$ then every $x_j \leq x_i$ and every $y_j \leq y_i$ are integers.
DEFINITION 3.1.4. The four equivalence relations DZ[k], DQ\Z[k], crit[k], -crit[k] are the k-principal relations.

Note that we leave out "equivalence" in this name for brevity; they are all equivalence relations on Q^k.

DEFINITION 3.1.5. The coordinate permutations of Q^k are the bijections $f:Q^k \to Q^k$, where for some bijection $\pi:\{1,\ldots,k\} \to \{1,\ldots,k\}$, $f(x_1,\ldots,x_k) = (x_{\pi 1},\ldots,x_{\pi k})$. $R \subseteq Q^{2k}$ is permutation invariant if and only if for all coordinate premutations $f:Q^k \to Q^k$ and $x,y \in Q^k$, $x\ R\ y \leftrightarrow f(x)\ R\ f(y)$. $R \subseteq Q^{2k}$ is permutation order,Z invariant if and only if it is permutation invariant and order,Z invariant.

We sometimes abuse notation and write πx for $f(x)$ here. We caution the reader that $R \subseteq Q^{2k}$ does not imply that $x \in R \to \pi x \in R$. For permutation invariance, the elements of R must be divided into first and second halves, with a common permutation applied to both halves.

THEOREM 3.1.1. (EFA) triv[k] and the k-principal relations are permutation order,Z invariant equivalence relations on Q^k.

Proof: We can easily give quantifier free definitions of these five over $(Q,<,Z)$ with no parameters, and so we can apply Theorem 3.2. For permutation invariance, let $\pi:\{1,\ldots,k\} \to \{1,\ldots,k\}$ be a bijection. Then $(p_1,\ldots,p_k)$ DZ[k] $(q_1,\ldots,q_k) \leftrightarrow (p_1,\ldots,p_k)$ triv[k] $(q_1,\ldots,q_k) \vee$ $(p_1,\ldots,p_k),(q_1,\ldots,q_k) \in Z^k$ are diagonal, if and only if $(p_{\pi 1},\ldots,p_{\pi k})$ triv[k] $(q_{\pi 1},\ldots,q_{\pi k})$ or $(p_{\pi 1},\ldots,p_{\pi k}),(q_{\pi 1},\ldots,q_{\pi k}) \in Z^k$ are diagonal. The same argument works for DQ\Z[k]. Now let $(p_1,\ldots,p_k)$ crit[k] $(q_1,\ldots,q_k)$. Then $(p_1,\ldots,p_k),(q_1,\ldots,q_k)$ are order equivalent and $\min(p_1,\ldots,p_k) = \min(q_1,\ldots,q_k)$. Hence $(p_{\pi 1},\ldots,p_{\pi k}),(q_{\pi 1},\ldots,q_{\pi k})$ are order equivalent and $\min(p_{\pi 1},\ldots,p_{\pi k}) = \min(q_{\pi 1},\ldots,q_{\pi k})$. Also if $p_i \neq q_i$ then every $p_j \geq p_i$ and every $q_j \geq q_i$ are integers. Hence if $p_{\pi i} = q_{\pi i}$ then every $p_{\pi j} \geq p_{\pi i}$ and every $q_{\pi j} \geq q_{\pi i}$ are integers. Therefore crit[k] is permutation invariant. By the same argument with sign changes, -crit[k] is permutation invariant. The last claim is trivial. QED

From a careful reading of the definitions of crit(k) and -crit(k), we see that crit(1) = -crit(1) = triv[1].
Note that there is an obvious kind of duality between crit[k] and -crit[k]. We see a robust kind of natural duality as follows.

DEFINITION 3.1.6. Let φ be a 0-definition in (Q,<,Z). We even allow quantifiers (this goes beyond our scope) but no parameters. dual(φ) is obtained by reversing all signs <. For $A \subseteq Q^k$ that are definable over (Q,<,Z), let dual(A) be obtained from A by using the dual of some 0-definition over (Q,<,Z) of A.

THEOREM 3.1.2. (RCA_0) Let $A \subseteq Q^k$ be 0-definable over (Q,<,Z). The definition of dual(A) in Definition 3.1.4 is well defined. I.e., you get the same dual(A) no matter what 0-definition of A that you use. dual(A) is also the image of A under any isomorphism from (Q,<,Z) onto (Q,>,Z), which do exist. Most naturally, the additive inverse -:Q → Q.

Proof: This is an application of elementary model theory that we leave to the reader. QED

THEOREM 3.1.3. (EFA) The following hold.
i. -crit[k] is the additive inverse of crit[k]. I.e., x -crit[k] y ↔ -x crit[k] -y.
ii. DZ[k] = -DZ[k] = dual(DZ[k]). DQ\Z[k] = -DQ\Z[k] = dual(DQ\Z[k]).
iii. Five of the six pairwise intersections of any two distinct k-principal relations is triv[k], the sixth being crit[k] ∩ -crit[k] = {(x,y) ∈ Z^{2k}: x,y are order equivalent ∧ min(x) = min(y) ∧ max(x) = max(y)} ∪ triv[k].
iv. If k ≤ 2 then all of these six intersections are triv[k].

Proof: i is by inspection. ii follows from Theorem 3.1.2 since -crit[k] was defined in Definition 3.2.2 as the dual of crit[k]. iii is by inspection. For iv, first note that crit[1] = -crit[1] = triv[1], DZ[1] = {(p,q): p,q ∈ Z} ∪ triv[1], DQ\Z[1] = {(p,q): p,q ∈ Q\Z} ∪ triv[1]. The six intersections are obviously triv[1]. Now let k ≥ 2. Clearly (p,...,p) ≠ (q,...,q) cannot be related by crit[k] or crit[k] because of the min (max) equality condition. Hence DZ[k] ∩ DQ\Z[k] = DZ[k] ∩ crit[k] = DZ[k] ∩ -crit[k] =

DQ\Z[k] ∩ crit[k] = DQ\Z[k] ∩ -crit[k] = triv[k]. Their remains only crit[k] ∩ -crit[k]. Let $(p_1,...,p_k,q_1,...,q_k)$ lie in this intersection, where $(p_1,...,p_k) \neq (q_1,...,q_k)$. Obviously $\min(p_1,...,p_k) = \min(q_1,...,q_k) \wedge \max(p_1,...,p_k) = \max(q_1,...,q_k)$. Now let $p_i \neq q_i$. Then every $p_j \geq p_i$ and $q_j \geq q_i$ is an integer, and every $p_j \leq p_i$ and $q_j \leq q_i$ is an integer. Hence $p_1,...,p_k,q_1,...,q_k \in Z$. Also, $(p_1,...,p_k),(q_1,...,q_k)$ are order equivalent. So $(p_1,...,p_k,q_1,...,q_k)$ is in the intersection displayed in iv. For the final claim, note that min(x) = min(y) ∧ max(x) = max(y) → x = y in dimensions 1,2. QED

THEOREM 3.1.4. The following holds.
i. RCA_0 proves that DZ[k], DQ\Z[k] are k-usable.
ii. RCA_0 proves that for all k, crit[k] is k-usable if and only if -crit[k] is k-usable.
iii. Z_3 proves that the 2-principal relations are 2-usable.
iv. The following is equivalent to Con(SRP) over WKL_0. For all k ≥ 1, the k-principal relations are k-usable.

Proof: For i, let $B \subseteq Q[0,n]^k$ be finite. If B has no diagonal element then any maximal emulator of B will be completely DZ[k] invariant, as it will have no diagonal element. If B has exactly one diagonal element then build a maximal emulator S of $B \subseteq Q[0,n]^k$ containing (1/2,...,1/2). S will have exactly one diagonal element, namely (1/2,...,1/2), and so S is completely DZ[k] invariant. Finally, suppose B has at least two diagonal elements. Then build a maximal emulator of $B \subseteq Q[0,n]^k$ containing all diagonal elements of $Q[0,n]^k$. This will certainly be completely DZ[k] invariant. The proof that DQ\Z[k] is k-usable is the same except with (1/2,...,1/2) replaced by (0,...,0). These proofs are carried out in RCA_0.

For ii, note that we can readily express "crit[k] is k-usable" as the individual satisfiability of various first order sentences in the structures (Q[0,n],<,S,{0,1,...,n}), n ≥ 1, with binary relation symbol <, and k-ary relation symbol S, and {0,1,...,n} is a unary relation, and where there is a sentence using each n and finite external $B \subseteq Q[0,n]^k$. Obviously -crit[2] (using Definition 3.1.3) is 2-usable is expressed as the individual satisfiability of the corresponding first order sentences in the structures (Q[0,n],>,S,{0,1,...,n}), where the symbol < is merely replaced by the symbol >. However (Q[0,n],<,•,{0,1,...,n}) and (Q[0,n],>,•,{0,1,...,n}) are isomorphic (without the S's). Hence these satisfiability conditions are equivalent. So ii is established.

For iii and iv, merely apply Theorem 2.3.7. QED

We have already given two definitions of crit[k] using order equivalence. See Definition 2.3.1 and Appendix A. Here we give a definition using only a particularly vivid special case of order equivalence.

DEFINITION 3.1.7. $Q^{k\le}$ is the set of all $x \in Q^k$ that are increasing (≤) in the sense that $x_1 \le \ldots \le x_k$. We also analogously define $Q^{k<}$, $Q^{k\ge}$, $Q^{k>}$. $x,y \in Q^{k\le}$ ($Q^{k\ge}$) are repetition equivalent if and only if for all $1 \le i,j \le k$, $x_i = x_j \leftrightarrow y_i = y_j$.

Note that order equivalence on $Q^{k\le}$, $Q^{k\ge}$ is equivalent to repetition equivalence on $Q^{k\le}$, $Q^{k\ge}$, respectively.

DEFINITION 3.1.8. Let $x,y \in Q^k$. x crit[k,≤] y if and only if x = y or ($x,y \in Q^{k\le}$ are repetition equivalent, where their longest common initial segment is nonempty and is followed by integers only). x crit[k,≥] y if and only if x = y or ($x,y \in Q^{k\ge}$ are repetition equivalent, where their longest common initial segment is nonempty and is followed by integers only).

THEOREM 3.1.5. (RCA_0) crit[k], -crit[k]) is the least permutation invariant equivalence relation on Q^k containing crit[k,≤], crit[k,≥], respectively.

Proof: Suppose x crit[k,≤] y. If x = y then x crit[k] y. If $x,y \in Q^{k\le}$ are repetition equivalent then x,y are order equivalent. Evidently x crit[k] y by the definition of crit[k]. Now let R be a permutation invariant equivalence relation on Q^k containing crit[k,≤]. Suppose x crit[k] y. Then x,y are order equivalent. Let π be a coordinate permutation such that $\pi x \in Q^{k\le}$. Then $\pi y \in Q^{k\le}$, and $\pi x,\pi y$ are repetition equivalent. From the definition of crit[k] we see that x crit[k,≤] y. Note that x R y since R is permutation invariant. The argument for -crit[k] is the same with < replaced by >. QED

We now dispense with the 1-usability of order,Z invariant transitive relations on Q.

LEMMA 3.1.6. (RCA_0) DZ[1] = {(p,p): p ∈ Z} ∪ triv(1). DQ\Z[1] = {(p,p): p ∉ Z}. Every order,Z invariant equivalence relation on Q^2 is among triv[1], DZ[1], DQ\Z[1], DZ[1] ∪ DQ\Z[1], Q^2.

Proof: The first claim is by inspection. For the second claim, let R be as given. We claim that if (0,1/2) or (1/2,1) ∈ R, then EQR(R) = Q^2. From (0,1/2) we get (0,3/2), (1,3/2), (0,1), (1/2,3/2), (1/2,1), representing all of the nontrivial pairs up to order equivalence/Z. Hence EQR(R) = Q^2. Similarly, from (1/2,1) we get (1/2,2), (3/2,2), (1/2,3/2), (1,2), (1,3/2), also representing all of the nontrivial pairs up to order equivalence/Z.

So we can assume that (0,1/2), (1/2,1) ∉ R. The only other pairs up to order equivalence/Z are (0,0), (1/2,1/2). There are four cases of membership of these pairs in R, and they yield the four. QED

LEMMA 3.1.7. (RCA_0) DZ[1], DQ\Z[1] are 1-usable. DZ[1] ∪ DQ\Z[1] is not 1-usable.

Proof: Let B ⊆ Q[0,n] be finite. If |B| ≤ 1 then use maximal emulator {1/2} for DZ[1] and {0} for DQ\Z[1]. If }B| ≥ 2 then use maximal emulator Q[0,n] of {0} ⊆ Q[0,n]. Now let B ⊆ Q[0,1] be {0}. The maximal emulators are just the {p}, and none are completely invariant with respect to DZ[1] ∪ DQ\Z[1]. QED

THEOREM 3.1.8. (RCA_0) An order,Z invariant equivalence relation on Q is 1-usable if and only if it is contained in at least one of DZ[1], DQ\Z[1].

Proof: By Lemma 3.1.6, 3.1.7. QED

Suppose we enlarge the R's to be considered by looking at the order,Z invariant relations R on Q. Then there are more 1-uable relations.

THEOREM 3.1.9. (RCA_0) {(p,q): p < q ∧ p ∈ Z ∧ q ∉ Z} ⊆ Q^2 is order,Z invariant and 1-usable. It is not contained in DZ[1] or DQ\Z[1]. It is, moreover, transitive. The least equivalence relation in Q^2 containing ({(p,p,q,q): p < q ∧ p ∈ Z ∧ q ∉ Z}) is Q^2, and is not 1-usable.

Proof: Let B ⊆ Q[0,n] be finite. If |B| is 0 or ≥ 2 then use the maximal emulator Q[0,n] of B ⊆ Q[0,n]. Suppose |B| = 1. Use the maximal emulator {n} of B ⊆ Q[0,n], which is completely invariant with respect to {(p,p,q,q): p < q ∧ p ∈ Z ∧ q ∉ Z}. To see that EQR({(p,p,q,q): p < q ∧ p ∈ Z ∧ q ∉ Z}) = Q2, first let n,m ∈ Z. Let n,m < p ∉ Z, and note that (n,p),(m,p),(n,m) ∈ EQR({(p,p,q,q): p < q ∧ p ∈ Z ∧ q ∉ Z}). Now let p,q ∉ Z. Let n < p,q where n ∈ Z. Then (n,p),(n,q),(p,q) ∈ EQR({(p,p,q,q): p < q ∧ p ∈ Z ∧ q ∉ Z}). Let p < n where p ∉ Z and n ∈ Z. Let m < p < n, m ∈ Z. Then (m,n),(m,p),(p,n) ∈ EQR({(p,p,q,q): p < q ∧ p ∈ Z ∧ q ∉ Z}). Finally, Q^2 is not 1-usable since no emulator of {0} ⊆ Q[0,1] is completely Q^2 invariant. QED

Finally, we characterize the k-usable order invariant relations on Q^k (not necessarily equivalence relations).

THEOREM 3.1.10. (RCA_0) Let R ⊆ $Q^k \times Q^k = Q^{2k}$ be order invariant. Then R is k-usable if and only if R ⊆ triv[k].

Proof: Let R be as given, and be k-usable. Then R is order preserving. Let x R y ∧ x ≠ y. If x,y are diagonals then let S be a maximal emulator of {(0,...,0)} ⊆ Q[0,1]k. No maximal emulator of S is completely R invariant. So we assume that x,y are not diagonals. Below, we say that x surrounds y if and only if y lies strictly between min(x) and min(y).

case 1. x does not surround y and y does not surround x. Let z,w ∈ Q[0,1]k be order equivalent to x and z surrounds w. Let S be a maximal emulator of {z,w} ⊆ Q[0,1]k. Then S consists of two or more tuples order equivalent to x, and for each pair of them, one surrounds the other or vice versa. In particular, there exists u ∈ S from Q(0,1)k. Let (u,u') be order equivalent to (x,y), u' ∈ Q[0,1]k. Since u' ∉ S, we have violated the complete R invariance of S.

case 2. x surrounds y. Let S be a maximal emulator of {z} ⊆ Q[0,1]k, where z is order equivalent to x. Then S = {w}, w order equivalent to x. Let (w,w') be order equivalent to (x,y). Since w' $\notin$ S, we have violated the complete R invariance of S.

QED

3.2. ME[R,2,∞,2,*,*] CHARACTERIZATION

In this section we determine the

permutation order,Z invariant equivalence relations on Q^2

that are 2-usable. We show that they are the ones that are contained in at least one of the 2-principal relations.

A very natural wider class consists of the permutation order invariant/Z transitive relations $R \subseteq Q^{2k}$. However, we have seen that already in dimension 1, this enlarges the usable relations beyond those contained in the 1-principal relations, and so we will steer clear of this in this paper. See Theorem 3.1.9.

We begin with a development in k dimensions that we use in section 3.3. Here we will only put it to use in dimension k = 2.

LEMMA 3.2.1. (RCA_0) If $R \subseteq Q^{2k}$ is k-usable then R is an order preserving relation on Q^k. I.e., $(\forall x,y \in Q^k)(x\ R\ y \rightarrow$ x,y are order equivalent).

Proof: By Theorem 2.2.15. QED

DEFINITION 3.2.1. OP[k] is the set of all order preserving $x \in Q^k \times Q^k = Q^{2k}$. I.e., $x \in$ OP[k] if and only if $(x_1,\ldots,x_k),(x_{k+1},\ldots,x_{2k})$ are order equivalent.

LEMMA 3.2.2. (RCA_0) If $R \subseteq Q^{2k}$ is k-usable then $R \subseteq$ OP[k].

Proof: Immediate from Lemma 3.2.1. QED

DEFINITION 3.2.2. R is k-good if and only if
i. $R \subseteq$ OP[k].
ii. R is an equivalence relation on Q^k.
iii. R is permutation invariant.
iv. R is order,Z invariant.

DEFINITION 3.2.3. x,y are k-similar if and only if x,y $\in$

OP[k] and there is a permutation π such that x is order,Z equivalent to $(y_{\pi 1},\ldots,y_{\pi k},y_{\pi k+1},\ldots y_{\pi 2k})$ or $(y_{\pi k+1},\ldots,y_{\pi 2k},y_{\pi 1},\ldots,y_{\pi k})$.

LEMMA 3.2.3. (EFA) k-similarity is an equivalence relation on Q^k. It has finitely many equivalence classes. x,y are k-similar if and only if -x,-y are k-similar.

Proof: Let GRP[k] be the set of coordinate permutations $f:Q^{2k} \to Q^{2k}$ which take one of the following two forms:

$$f(p_1,\ldots,p_{2k}) = (p_{\pi 1},\ldots,p_{\pi k},p_{\pi k+1},\ldots,p_{\pi 2k})$$
$$f(p_1,\ldots,p_{2k}) = (p_{\pi k+1},\ldots,p_{\pi 2k},p_{\pi 1},\ldots,p_{\pi k}).$$

These form a subgroup of the coordinate permutations on Q^{2k}. Also going back to the definition of order,Z equivalence, we see that two arguments are order,Z equivalent if and only if their values are order,Z equivalent, under any f ∈ GRP[k]. I.e., the application of any f ∈ GRP[k] preserves order,Z equivalence.

It is clear that x,y are k-similar if and only if x is order,Z equivalent to some f(y), f ∈ GRP[k]. Reflexivity follows using the identity. Suppose x is order,Z equivalent to some f(y), f ∈ GRP[k]. Then f-1(x) is order,Z equivalent to y. Now suppose x is order,Z equivalent to f(y) and y is order,Z equivalent to g(z). Then f(y) is order,Z equivalent to f(g(z)). Hence x is order,Z equivalent to f(g(z)), and therefore x,z are similar.

Now suppose x is order,Z equivalent to f(y), f ∈ GRP[k]. Then -x is order,Z equivalent to -f(y). But -f(y) = f(-y). So -x,-y are k-similar.

Note that k-similarity implies order,Z equivalence implies order equivalence, and so the last claim is immediate. QED

THEOREM 3.2.4. (RCA_0) If R is k-good then R is completely invariant with respect to k-similarity. R is k-good if and only if -R is k-good.

Proof: Let R be k-good, x ∈ R, and x,y k-similar. Let x is order,Z equivalent to y or $(y_{\pi k+1},\ldots,y_{\pi 2k},y_{\pi 1},\ldots,y_{\pi k})$. By order,Z invariance, y or $(y_{\pi 1},\ldots,y_{\pi k},y_{\pi k+1},\ldots y_{\pi 2k})$ is in R. In the first case we are done. In the second case, since R is symmetric, $(y_{\pi 1},...,y_{\pi k},y_{\pi k+1},...y_{\pi 2k}) \in R$, and since R permutation invariant, $y \in R$. QED

In this section, we show that if R is 2-good and 2-usable then R is contained in at least one of the 2-principal relations. The classification follows, as if R is 2-usable then $R \subseteq OP[2]$.

We now compile a complete list of representatives for 2-similarity. This is List[1], and it is organized into seven convenient groups. Each group consists of specific quadruples representing the items under the header, up to similarity, with the exception of the first two groups, which are simply left as sets of quadruples. Each group is meant to cover the quadruples falling under its title that do not appear in any earlier group.

We then determine some x in List[1] such that every 2-good 2-usable R avoids x. (It will become apparent that the remaining quadruples on List[1] are individually 2-usable). These are removed from List[1], thereby creating List[2]. List[2] is so much thinner than List[1], that we reorganize it for greater clarity.

Considerations in List[2] lead to a proof that any 2-good 2-usable R is a subset of at least one of the 2-principal relations. It follows immediately that any 2-usable permutation order,Z invariant equivalence relation is a subset of at least one of the 2-principal relations.

LIST[1]

quadruples $(p_1,p_2,q_1,q_2) \in OP[4]$

$p_1 \leq p_2 \wedge q_1 \leq q_2 \wedge$

$(p_1,p_2) \leq_{lex} (q_1,q_2)$

TRIV[2]

(p,q,p,q), $p,q \in Q$

NONREPEATING FRACTIONS

discard for List[2]

$(\exists i)(x_i \notin \{x_j: j \neq i\},Z)$.

TWO COORDINATES

(0,0,1,1)
(1/3,1/3,1/2,1/2)
(0,0,1/2,1/2) discard for List[2]
(1/2,1/2,1,1) discard for List[2]

TOP/BOTTOM INTEGER MOVERS

(0,1,0,2)
(0,2,1,2)
(1/2,1,1/2,2)
(0,3/2,1,3/2)

MIDDLE EQUALS
discard for List[2]

(0,1,1,2)
(0,1/2,1/2,1)

FOUR INTEGERS
discard for List[2]

(0,1,2,3)
(0,2,1,3)
(0,3,1,2)

THEOREM 3.2.5. (EFA) Every $x \in OP[4]$ is 2-similar to a quadruple on List[1]. It is unique if we count the two categories triv[2], Nonrepeating Fractions as one each.

Proof: Let $x \in OP[4]$. It is clear that x is similar to some (p_1,p_2,q_1,q_2) where $p_1 \leq p_2$, and hence to some (p_1,p_2,q_1,q_2) with

$$A)\ p_1 \leq p_2 \wedge q_1 \leq q_2 \wedge (p_1,p_2) \leq_{lex} (q_1,q_2).$$

This condition A) provides a unique permutation of x that is 2-similar to x. We therefore assume throughout that $x = (p_1,p_2,q_1,q_2) \in OP[4]$ satisfies A). It remains to show that x is order equivalent/Z to a quadruple on List[1]. We will abuse terminology and simply say that x is on List[1]. Clearly Nonrepeating Fractions is disjoint from triv[2].

Now assume x is not in triv[2] or Nonrepeating Fractions. Let x have exactly two distinct coordinates. Then $p_1 = p_2$ is impossible, and so $p_1 < p_2 \wedge q_1 < q_2$. Hence $p_1 = q_1 < p_2 = q_2$. There are four possibilities for membership in Z of p_1,p_2 here, and so x must be on Two Coordinates.

Now suppose x has exactly three distinct coordinates. There are six possibilities for where the unique equal pair of coordinates lie. $p_1 = p_2$ and $q_1 = q_2$ are impossible. If $p_1 = q_2$ then $p_1 \leq q_1 < q_2 = p_1$, which is impossible. So we are left with three cases.

case 1. $p_1 = q_1$. Then $p_1 = q_1 < p_2 < q_2$. Hence $p_2,q_2 \in Z$. Thus x is on Top/Bottom Integer Movers among the last two quadruples there.
case 2. $p_2 = q_2$. Then $p_1 < q_1 < p_2 = q_2$. Hence $p_1,q_1 \in Z$. Thus x is on Top/Bottom Integer Movers among the first two quadruples there.
case 3. $p_2 = q_1$. Then $p_1 < p_2 = q_1 < q_2$. Hence $p_1,q_2 \in Z$. So x appears in Middle Equals.

Finally, suppose x has exactly four distinct coordinates. Then $p_1,p_2,q_1,q_2 \in Z$. If $p_2 < q_1$ then obviously $p_1 < p_2 < q_1 < q_2$ and x appears on Four Integers as (0,1,2,3). Alternatively, if $q_1 < p_2$ then $p_1 < q_1 < p_2 < q_2$ or $p_1 < q_1 < q_2 < p_2$. In the first case, x appears as (0,2,1,3). In the second case, x appears as (0,3,1,2). QED

We now start the process of discarding items in List[1].

LEMMA 3.2.6. (RCA_0) Every 2-good R containing a Nonrepeating Fraction contains at least one of the following quadruples.
(1,3/2,1,4/3).
(1/2,3/2,1/2,4/3).
(1/3,1,1/2,1)
(1/3,3/2,1/2,3/2).

Proof. Let x be on Nonrepeating Fraction lying in the 2-good R. Let x_i be the non repeating fractional coordinate of x. Let x' be the result of strictly raising x_i so that x,x' are order,Z equivalent, and in particular x' $\in$ R. If i = 1,2 then x,x' have the same last two coordinates and if i = 3,4 then x,x' have the same first two coordinates. Hence we can apply the transitivity of R to obtain that at least one of (x_1,x_2,x_1',x_2), (x_1,x_2,x_1,x_2'), (x_3,x_4,x_3',x_4), (x_3,x_4,x_3,x_4') lies in R. Since in each of these quadruples, the moved coordinate, before and after, are fractions, and the moved coordinates are not the same as the ostensibly repeated coordinate, these quadruples are 2-similar to the specific ones displayed, establishing the Lemma using Theorem 3.2.4. QED

LEMMA 3.2.7. (RCA_0) Any 2-good R containing (0,0,1/2,1/2) or (1/2,1/2,1,1) contains all (p,p,q,q), p,q ∈ Q.

Proof: Let R be as given. For the first case, in R we have

(0,0,1/2,1/2)
(0,0,3/2,3/2)
(1/2,1/2,3/2,3/2)
(1,1,3/2,3/2)
(1/2,1/2,1,1)

For the second case, in R we have

(1/2,1/2,1,1)
(1/2,1/2,2,2)
(1,1,2,2)
(3/2,3/2,2,2)
(1/2,1/2,3/2,3/2)
(1,1,3/2,3/2)

QED

LEMMA 3.2.8. (RCA_0) Let R be 2-good and contain any of (0,1,2,3), (0,1,1,2), (0,2,1,3), (0,3,1,2).
i. (0,1,0,2), (0,2,1,2) ∈ R.
ii. (0,1,2,3) ∈ R.
iii. $Z^{2<} \times Z^{2<} \subseteq R$.

Proof: Let R be as given. For i, we make four derivations in R.

(0,1,2,3)
(0,1,3,4)
(0,2,3,4)
(0,1,0,2)
(1,2,3,4)
(0,2,1,2)

(0,1,1,2)
(0,1,1,3)
(-1,1,1,2)

(-1,1,0,1)
(0,1,0,2)
(0,2,1,2)

(0,2,1,3)
(0,2,1,4)
(1,3,1,4)
(0,1,0,2)
(-1,2,1,3)
(-1,2,0,2)
(0,2,1,2)

(0,3,1,2)
(0,4,1,2)
(0,3,0,4)
(0,1,0,2)
(-1,3,1,2)
(-1,3,0,3)
(0,2,1,2)

For ii, we make one derivation in R using the fruits of i.

(0,1,0,2)
(0,2,1,2)
(0,1,0,3)
(0,3,2,3)
(0,1,2,3)

For iii, let $n_1 < n_2 \wedge n_3 < n_4$ be integers. By ii,

$(n_1,n_2,|n_1|+|n_2|+|n_3|+|n_4|+1,|n_1|+|n_2|+|n_3|+|n_4|+2) \in R$.
$(n_3,n_4,|n_1|+|n_2|+|n_3|+|n_4|+1,|n_1|+|n_2|+|n_3|+|n_4|+2) \in R$.
$(n_1,n_2,n_3,n_s) \in R$.

QED

We now establish a lot of non 2-usability.

LEMMA 3.2.9. (RCA_0) Suppose 2-good R contains all (p,p,q,q), $p,q \in Q$. Then R is not 2-usable.

Proof: Let $B = \{(0,0)\} \subseteq Q[0,1]^2$. The maximal emulators of B are the $\{(p,p)\}$, $p \in Q$, which are evidently not completely R invariant. QED

THEOREM 3.2.10. (RCA_0) Suppose 2-good R contains any of the

following quadruples. Then R is not 2-usable.
(1,3/2,1,4/3).
(1/2,3/2,1/2,4/3).
(1/3,1,1/2,1)
(1/3,3/2,1/2,3/2).

Proof: Let R be as given, where R is 2-usable. Then MCres(R,2,∞,2,*,*). Let G be the order invariant graph on $Q[0,1]^2$, where $x,y \in Q[0,1]^2$ are nonadjacent if and only if $x = y$ or

$$x_1 = x_2 \vee y_1 = y_2 \vee$$
$$(x_1 < x_2 \wedge y_1 < y_2 \wedge x_1 = y_1 \wedge x_2 \neq y_2) \vee$$
$$(x_1 > x_2 \wedge y_1 > y_2 \wedge x_2 = y_2 \wedge x_1 \neq y_1) \vee$$
$$(x_1 < x_2 \wedge y_1 > y_2 \wedge x_1 = y_2 \wedge x_2 = y_1)$$

If $\max(x_1,x_2) < \min(y_1,y_2)$ then none of these four displayed disjuncts can hold. Hence G is a restricted order invariant graph on $Q[0,1]^2$.

Note that $S \subseteq Q[0,1]^2$ is a clique in G if and only if S is a set of pairs $(p,q) \in Q[0,1]^2$, $p \neq q$, where

i. For all p there is at most one $q > p$ such that $(p,q) \in S$.
ii. For all p there is at most one $q' > p$ such that $(q',p) \in S$.
iii. For all p these q,q' cannot be the same (if they are both exist).

Let S be an R invariant maximal clique in G. Then for all $p \in Q[0,1)$, there exist unique $q_1 \neq q_2$ such that $q_1,q_2 > p \wedge (p,q_1),(q_2,p) \in S$.

We claim $(0,1/2,0,1/3) \notin R$. Suppose $(0,1/2,0,1/3) \in R$. Let $(0,q_1),(q_2,0) \in S$. Since $q_1 \neq q_2$ are nonzero, clearly $q_1 \notin Z \vee q_2 \notin Z$. Assume $q_1 \notin Z$. Then complete R invariance is violated from $(0,q_1) \in S$ because of the uniqueness of q_1. Assume $q_2 \notin Z$. Then complete R invariance is violated from $(q_2,0) \in S$ because of the uniqueness of q_2.

We claim $(1/4,1/2,1/4,1/3) \notin R$. Suppose $(1/4,1/2,1/4,1/3) \in R$. Let $(1/4,q_1),(q_2,1/4) \in S$. Again, $q_1 \notin Z \vee q_2 \notin Z$, and $q_1,q_2 > 1/4$. We also violate the complete R invariance of S.

For establishing (1/2,1,1/3,1), (1/2,3/2,1/2,5/2) ∉ R, we can either repeat the argument using the "opposite" graph defined with the signs reversed (so that properties i,ii hold with > replaced by <). Or we can argue more abstractly that the statement is preserved under any reverse automorphism of Q[0,1] such as h(p) = 1-p. QED

LEMMA 3.2.11. (RCA_0) Let 2-good R contain any of these four sets of two quadruples. Then R is not 2-usable.
i. [(0,3,1,3), (3,0,2,0)}.
ii. {(0,3,1,3), (3,1/2,2,1/2)}.
iii. [(0,3/2,1,3/2), (3,0,2,0)}.
iv. {(0,5/2,1,5/2), (3,1/2,2,1/2)}.

Proof: Let R be as given. By Theorem 2.2.9, if R is 2-usable then ME(res)[R,2,∞,2,*,*]. So it suffices to build an order invariant graph on $Q[0,3]^k$ such that no maximal clique is completely invariant with respect to any of these two element relations. Let G be the order invariant graph on $Q[0,3]^2$, where x,y ∈ $Q[0,3]^2$ are nonadjacent if and only if

$$*) \; x = y \vee y_2 < x_1 < y_1 < x_2 \vee x_2 < y_1 < x_1 < y_2.$$

Note that *) is reflexive and symmetric, and so G is a graph. Obviously * is order invariant. Also if max(x) < min(y) then $x_1,x_2 < y_1,y_2$, violating *). Hence G is restricted. Let S be a completely R invariant maximal clique in G.

We claim that for all 0 < p ≤ 3, (0,p) ∈ S. Suppose (0,p) ∉ S. Then S ∪. {(0,p)} is not a clique in G. Let (0,p),y be non adjacent in G, y ∈ S. Since y ≠ (0,p), the displayed disjunction must hold for (0,p),y, and in particular, the first disjunct must hold with (0,p) = x. However, we have y_2 < x1 = 0, which is impossible. This establishes (0,p) ∈ S for 0 < p ≤ 3.

We claim that for all 0 ≤ q < 3, (3,q) ∈ S. Suppose (3,q) ∉ S. Then S ∪. {(3,q)} is not a clique in G. Let x,(3,q) be nonadjacent in G, x ∈ S. Since x ≠ (3,q), the displayed disjunction holds for x,(3,q), and in particular, the first disjunct must hold with y = (3,q). But we have 3 = $y_1 < x_1$, which is impossible. Hence (3,q) ∈ S.

We can now show that S is not completely invariant with respect to these specific pairs of quadruples. In each of the four cases, assume S is completely invariant with respect to the set consisting of the pairs, and then get a violation of S being a clique.

(0,3,1,3), (3,0,2,0). Since (0,3),(3,0) ∈ S, we have (1,3),(2,0) ∈ S. But (1,3),(2,0) obeys the first disjunct of *), and therefore (1,3),(2,0) are nonadjacent.

(0,3,1,3), (3,1/2,2,1/2). Since (0,3),(3,1/2) ∈ S, we have (1,3),(2,1/2) ∈ S. But they are not adjacent.

(0,3/2,1,3/2), (3,0,2,0). Since (0,3/2),(3,0) ∈ S, we have (1,3/2),(2,0) ∈ S. But they are not adjacent.

(0,5/2,1,5/2), (3,1/2,2,1/2). Since (0,5/2),(3,1/2) ∈ S, we have (1,5/2),(2,1/2) ∈ S. But they are not adjacent. QED

LEMMA 3.2.12. (RCA_0) If 2-good R contains (0,1/2,1/2,1) then R is not 2-usable.

Proof: Let R be as given. We obtain the following in R.

(0,1/2,1/2,1)
(0,1/2,1/2,2)
(1/2,1,1/2,2)
(-1,1/2,1/2,1)
(-1,1/2,0,1/2)

Now apply Lemma 3.2.11 iv to the third and fifth in the above list of five. QED

We have succeeded in actually discarding all of the entries in List[1] that were marked for discarding. This is documented below. We are left with the following.

LIST[2]

TRIV[2]

(p,p,q,q), p,q ∈ Q

TWO LIKE COORDINATES

(0,0,1,1)
(1/3,1/3,1/2,1/2)

TOP MOVERS

(0,2,1,2)
(1/2,1,1/2,2)

BOTTOM MOVERS

(0,1,0,2)
(0,3/2,1,3/2)

LEMMA 3.2.13. (RCA_0) Let 2-good R be 2-usable. Every element of R is 2-similar to a quadruple on List[2]. The quadruple is unique if you count triv[2] as one.

Proof: Let R be as given. and x ∈ R. By Theorem 3.2.6, x is 2-similar to a unique quadruple on List[1] (counting triv[2] and Nonrepeating Fractions as one each). To show that x is 2-similar to some quadruple on List[2], we need to show that x is not 2-similar to any of the quadruples on List[1] that didn't make it to List[2]. Thee are nine such x in addition to the Nonrepeating Fractions. We argue by cases.

Nonrepeating Fractions. By Lemma 3.2.7, R contains at least one of the quadruples displayed in Lemma 3.2.6. By Theorem 3.2.10, R is not 2-usable.

(0,0,1/2,1/2), (1/2,1/2,1,1). By Lemma 3.2.7, R contains all (p,p,q,q), p,q, ∈ Q. By Lemma 3.2.9, R is not 2-usable.

(0,1,1,2), (0,1,2,3), (0,2,1,3), (0,3,1,2). By Lemma 3.2.8, $Z^{2<} \times Z^{2<} \subseteq R$. In particular, (0,3,1,3), (0,3,0,2), (3,0,2,0) ∈ R. By Lemma 3.2.11, R is not 2-usable.

(0,1/2,1/2,1). By Lemma 3.2.12, R is not 2-usable.

QED

LEMMA 3.2.14. (RCA_0) Let 2-good R contain a quadruple from Top Movers and a quadruple from Bottom Movers. Then R is not 2-usable.

Proof: There are four possibilities for these two quadruples. Each of them are represented up to 2-similarity by i-iv of Lemma 3.2.11. QED

LEMMA 3.2.15. (RCA_0) Let 2-good R contain (0,0,1,1) and (1/3,1/3,1/2,1/2). Then R is not 2-usable.

Proof: Let R be as given. Let S be a maximal emulator of $\{(0,0)\} \subseteq Q[0,1]2$. Then $S = \{(p,p)\}$, $p \in Q[0,1]$. Now clearly R contains some (p,p,q,q), $q \neq p$. Hence S cannot be completely invariant with respect to R. QED

LEMMA 3.2.16. (RCA_0) Let 2-good R contain (0,0,1,1), and also contain at least one of (0,1,0,2), (1/2,1,1/2,2), (0,2,1,2), (0,3/2,1,3/2). Then R is not 2-usable.

Proof: Let R be 2-good and contain (0,0,1,1). Assume R is 2-usable. Let $B \subseteq Q[0,3]^2 = \{x \in \{0,1,2,3\}^2: x_1 = x_2 \rightarrow x_1 \neq 2,3\}$. Let S be a completely R invariant maximal emulator of $B \subseteq Q[0,3]^2$. We claim $(0,0) \in S$. It suffices to verify that $S \cup \{(0,0)\}$ is an emulator of B. In fact, every (0,0,x), $x \in Q[0,3]^2$, is order equivalent to an element of B^2. This is because $(0,0) \in B$ and if x is a diagonal we can move x to (0,0) or (1,1) from B, and if x is not diagonal, we can move x to an element of $\{0,1,2\}^2$. Hence $(0,0) \in S$. Therefore $(3,3) \in S$.

We claim that for all $p < 1$, $(p,3) \in S$. Fix $p < 1$. It suffices to prove $S \cup \{(p,3)\}$ is an emulator of B. We claim that every (p,3,x), $x \in Q[0,3]^2$, is order equivalent to an element of B^2. Suppose x is not a diagonal. If $\min(x) \geq p$ then we can move p to 0 and x into $\{0,1,2,3\}^2$. If $\min(x) < p$ then we can move p to 1 and x into $\{0,1,2,3\}^2$. Suppose x is a diagonal. If $\min(x) = p$ then move p to 0 and x to (0,0). If $\min(x) < p$ then move p to 1 and x to (0,0). If $p < \min(x) < 3$, move p to 0 and x to (1,1). If $\min(x) = 3$, move p to 0, 3 to 1 and x to (1,1). Thus $(p,3) \in S$.

Firstly suppose R contains (0,1,0,2) or (1/2,1,1/2,2). Since $(0,3),(1/2,3) \in S$, we have $(0,1) \in S \vee (1/2,1) \in S$.

Now look at $(p,1),(3,3) \in S$, $p \in \{0,1/2\}$. (p,1,3,3) cannot be order equivalent to any element of B^2 because $p < 1 < 3$, and the only diagonals in B are at the two lowest levels, (0,0),(1,1). Thus $(p,1),(3,3) \in S$ violates that S is an emulator of $B \subseteq Q[0,3]^2$. Thus R is not 2-usable.

Secondly suppose R contains (0,2,1,2) or (0,3/2,1,3/2). We have shown that R is not 2-usable if R contains (0,1,0,2) or (1/2,1,1/2,2). By duality, -R is 2-good and contains (0,0,-1,-1) and therefore (0,0,1,1) as they are 2-similar. Also -R contains (0,-2,-1,-2) or (0,-3/2,-1,-3/2), and hence (0,1,0,1) or (1/2,1,1/2,2), also by 2-similarity. By the above, -R is not 2-usable, and therefore R is not 2-usable. QED

LEMMA 3.2.17. (RCA_0) Let 2-good R contain (1/3,1/3,1/2,1/2), and at least one of (0,1,0,2), (1/2,1,1/2,2), (0,2,1,2), (0,3/2,1,3/2). Then R is not 2-usable.

Proof: Let R be 2-good and contain (1/3,1/3,1/2,1/2). Assume R is 2-usable. Let $B \subseteq Q[0,2]^2$ be $\{(0,0),(0,2),(2,2)\} \cup \{1/2,1,3/2\}^2$. Let S be a complete R invariant maximal emulator of $B \subseteq Q[0,2]^2$. We claim that there exists $p < q$ such that $S = \{(p,p),(p,q),(q,p),(q,q)\} \cup Q(p,q)^2$.

Clearly there exists $p' \notin Z$ such that $(p',p') \in S$. Now if $(p,q) \neq (0,2)$ then there exists $q' \notin Z$ such that $(q',q') \notin S$, contradicting that R is 2-usable. Therefore $p = 0 \wedge q = 2$. Hence $S = \{(0,0),(0,2),(2,2)\} \cup Q(0,2)^2$.

We can now read off a lot of challenges to 2-usability.

$(0,2) \in S$, $(0,1) \notin S$ giving $(0,1,0,2) \notin R$.
$(0,2) \in S$, $(1,2) \notin S$ giving $(0,2,1,2) \notin R$.
$(1/2,1) \in S$, $(1/2,2) \notin S$ giving $(1/2,1,1/2,2) \notin R$.
$(1,3/2) \in S$, $(0,3/2) \notin S$ giving $(0,3/2,1,3/2) \notin R$.

QED

LEMMA 3.2.18. (RCA_0) Let 2-good R contain a subset A of List[2], where R is 2-usable. Then there exists $1 \leq i \leq 4$ such that $A \subseteq W_i$.
W1. $\{(0,0,1,1)\} \cup$ triv[2].
W2. $\{(1/3,1/3,1/2,1/2)\} \cup$ triv[2].
W3. $\{(0,2,1,2),(0,3/2,1,3/2)\} \cup$ triv[2].
W4. $\{(0,1,0,2),(1/2,1,1/2,2)\} \cup$ triv[2].

Proof: The first case that applies is operative.

case 1. $(0,0,1,1) \in A$. By Lemmas 3.2.16, 3.2.17, we have $(1/3,1/3,1/2,1/2)$, $(0,1,0,2)$, $(1/2,1,1/2,2)$, $(0,2,1,2)$, $(0,3/2,1,3/2) \notin A$. So $A \subseteq W_1$.

case 2. $(1/3,1/3,1/2,1/2) \in A$. By Lemmas 3.2.16, 3.2.18, we have that $(0,0,1,1)$, $(0,1,0,2)$, $(1/2,1,1/2,2)$, $(0,2,1,2)$, $(0,3/2,1,3/2) \notin A$. So $A \subseteq W_2$.

case 3. $(0,2,1,2) \in A$. By Lemma 3.2.15, $(0,2,1,2)$, $(1/2,1,1/2,2) \notin A$. By Lemmas 3.2.17, 3.2.18, $(0,0,1,1)$, $(1/3,1/3,1/2,1/2) \notin A$. So $A \subseteq W_3$.

case 4. $(0,1,0,2) \in A$. By Lemma 3.2.15, $(0,1,0,2)$, $(0,3/2,1,3/2) \notin A$. By Lemmas 3.2.17, 3.2.18, $(0,0,1,1)$, $(1/3,1/3,2/3,2/3) \notin A$. So $A \subseteq W_4$.

case 5. $(1/2,1,1/2,2) \in A$. By Lemma 3.2.15, $(0,1,0,2)$, $(0,3/2,1,3/2) \notin A$. By Lemmas 3.2.17, 3.2.18, $(0,0,1,1)$, $(1/3,1/3,1/2,1/2) \notin A$. So $A \subseteq W_4$.

case 6. $(0,3/2,1.3/2) \in A$. By Lemma 3.2.15, $(0,2,1,2)$, $(1/2,1,1/2,2) \notin A$. By Lemmas 3.2.17, 3.2.18, $(0,0,1,1)$, $(1/3,1/3,1/2,1/2) \notin A$. So $A \subseteq W_3$.

case 7. Otherwise. Since List[2] only contains the above six quadruples together with triv[2], clearly $A \subseteq$ triv[2].

QED

LEMMA 3.2.19. (EFA) In Lemma 3.2.18, every element of DZ[2], DQ\Z[2], crit[2], -crit[2] is 2-similar to an element of W_1, W_2, W_3, W_4, respectively, and vice versa.

Proof: For W_1,W_2, this is obvious. For sets W_3, W_4, clearly W3, W4 is contained in -crit[2], crit[2], respectively. It remains to show that

i. Every $(p_1,p_2,q_1,q_2) \in$ crit[2] is 2-similar to an element of W4. Let $(p_1,p_2,q_1,q_2) \in$ crit[Z]\triv[2]. Then $p_1 \neq q_1 \vee p_2 \neq q_2$.

case 1. $p_1 \neq q_1$. Now $\min(p_1,p_2) = \min(q_1,q_2)$, and $(p_1,q_1),(p_2,q_2)$ are order equivalent, thus taking their min's at the same position. Hence the mins are taken at position 2 with $p_2 = q_2 < p_1,p_2$. Since $p_1,q_1 \in Z$, (p_1,p_2,q_1,q_2) is 2-similar to an element of W_4.

case 2. $p_2 \neq q_2$. The same argument interchanging positions 1 and 2.

ii. Every $(p_1,p_2,q_1,q_2) \in$ -crit[2] is similar to an element of W_4. Let $(p_1,p_2,q_1,q_2) \in$ crit[Z]\triv[2]. Then $p_1 \neq q_1 \vee p_2 \neq q_s$. The same argument as given for i above, with min's replaced by max's and W_3.

QED

LEMMA 3.2.20. (RCA_0) Every 2-usable 2-good R is contained in at least one of DZ[2], DQ\Z[2], crit[2], -crit(2). Every 2-usable 2-good R is contained in triv[2] or is contained in exactly one of DZ[2], DQ\Z[2], crit[2], -crit[2].

Proof. Let R be 2-good and 2-usable. Let $x \in R$. By Lemma 3.2.13, x is 2-similar to an element of List[2]. Hence every element of R is 2-similar to an element of List[2]. For each $x \in R$ let x' be an element of List[2] 2-similar to x. Let A' be the set of all x' so arising. Then A' is a subset of List[2]. Since $A' \subseteq R$, we know that by Lemma 3.2.19, there exists $1 \leq i \leq 4$ such that $A' \subseteq W_i$. Hence every element of R is similar to an element of W_i. By Lemma 3.2.19, R is included in the corresponding 2-principal set. For the second claim, let R be not contained in triv[2]. If it is contained in more than one of the 2-principal relations, then it meets their intersection, which is triv[2] by Theorem 3.1.4 iv. QED

We are now ready to state our Classification Theorem for k = 2.

THEOREM 3.2.21. Let R be a coordinate order,Z invariant equivalence relation on Q^2. The following are equivalent.
i. Every finite subset of $Q[0,n]^2$ has a completely R invariant maximal emulator.
ii. R is included in at least one of DZ[2], DQ\Z[2], crit[2], -crit[2].
Moreover, any $R \neg\subseteq$ triv[2] whatsoever is included in at most one of DZ[2], DQ\Z[2], crit[2], -crit[2].
All of these results are provable in Z_3. i $\rightarrow$ ii and the "moreover" are provable in RCA_0.

Proof: Let R be as given. Assume i, which means that R is 2-usable. By Lemma 3.2.2, R is order preserving, and hence R ⊆ OP[4]. Therefore R is 2-good and 2-usable. Now apply Lemma 3.2.20 to obtain ii. For ii → i, see Theorem 3.1.4 iii. QED

In Theorem 3.2.21, we can obviously modify i to any of the stronger forms considered in section 2.2, as long as n is still universally quantified (i.e., with ∞ right after k = 2). This is because i' → ii comes for free, and ii → i' is by Theorem 2.3.7.

3.3. ME[R,k,∞,2,*,*] CHARACTERIZATION

We now address the characterization of the k-usable

1) permutation order,Z invariant equivalence relations on Q^k.

The case k = 2 is handled by Theorem 3.2.21, and the trivial case k = 1 was dispensed with by Theorem 3.1.8.

We have run into difficulties analyzing 1). In particular, we have discovered examples of such k-usable relations that are not contained in any of the k-principal relations, for small k, necessarily somewhat larger than 2. However, we have found a modified form of 1) so that we have been able to characterize k-usability. In particular, we show that any

2) permutation order,Z invariant initially fractional equivalence relations on Q^k

is k-usable if and only if it is included in crit[k]. We will also see that this characterization is provably equivalent to Con(SRP) over WKL_0 (relying on the promised reversal in [Fr19]).

DEFINITION 3.3.1. An equivalence relation on Q^k is initially fractional if and only if for all distinct x R y, the one or more fractional coordinates of (x,y) are less than the one or more integer coordinates of (x,y).

LEMMA 3.3.1. (EFA) Let x,y ∈ Q^k be distinct and order equivalent, where (x,y) is initially fractional. Then
i. (∃i)(x_i ∉ Z ∧ x_i not a coordinate of y); or

ii. $(\exists i)(y_i \notin Z \wedge y_i$ not a coordinate of x); or
iii. $(\exists i)((x_1,\ldots,x_i) = (y_1,\ldots,y_i) \wedge x_{i+1},\ldots,x_k,y_{i+1},\ldots,y_k \in Z)$.

Proof: Let x,y be as given. Let i be least such that $x_i \neq y_i$.

case 1. $x_i,y_i \in Z$. If $i = 1$ then the least coordinate of (x,y) is an integer, contrary to being initially fractional. Hence $i \geq 2$. Also by initially fractional, all coordinates of (x,y) that are at least $\min(x_i,y_i)$ are integers. Since $i \geq 2$, we can use this i-1 for iii.

case 2. $x_i,y_i \notin Z$. If $x_i < y_i$ then clearly x_i is not a coordinate of y, and so i holds. If $x_i > y_i$ then clearly y_i is not a coordinate of x, and so ii holds.

case 3. $x_i \notin Z \wedge y_i \in Z$. Clearly $x_i < y_i$. Obviously x_i is not a coordinate of y, and so i holds.

QED

DEFINITION 3.3.2. Let $x \in Q^{k_s}$. x is (i,j)-diagonal if and only if $1 \leq i \leq j \leq k$, $(i,j) \neq (1,k)$, and $x_{i-1} < x_i = \ldots = x_j < x_{j+1}$. The (i,j)-diagonal of x is $(x_i,\ldots,x_j)$.

DEFINITION 3.3.3. A fractional mover is an $(x,y) \in Q^{k_s} \times Q^{k_s}$ where
i. x,y are order equivalent.
ii. There exists $1 \leq i \leq j \leq k$ and y is the result of replacing the (i,j)-diagonal of x with a different (i,j)-diagonal, both of which are fractional.

LEMMA 3.3.2. (RCA_0) Let R be k-good and contain $(x,y) \in Q^{k_s} \times Q^{k_s}$, $x \neq y$. Assume i or ii holds from Lemma 3.3.1. Then there is a rational mover in R.

Proof: Let R,x,y be as given. By i, let $xi \notin Z$ and x_i not be a coordinate of y. Form x* by replacing all copies of xi by fractional $x_i' > x_i$, so that no coordinate of x,y lies strictly between x_i and x_i'. Since x_i is not a coordinate of y, we see that (x,y) and (x*,y) are order,Z equivalent, and so $(x^*,y) \in R$. Since R is an equivalence relation, $(x,x^*) \in R$ is a rational mover. The alternative case of ii in Lemma 3.3.1 is symmetric. QED

LEMMA 3.3.3. (RCA_0) Any k-good R containing a fractional mover is not k-usable.

Proof: Let R be k-good and contain the fractional mover $x = (p_1,\ldots,p_k,q_1,\ldots,q_k)$. Let (i,j) witness that x is a fractional mover. We have

1. $(p_1,\ldots,p_{i-1},p_{j+1},\ldots,p_k) = (q_1,\ldots,q_{i-1},q_{j+1},\ldots,q_k)$.
2. $p_i = \ldots = p_j \wedge q_i = \ldots = q_j \wedge p_i,q_i \notin Z$.
3. $(i,j) \neq (1,k)$.

Since x has an integer shift so that $p_1,q_1 \geq 0$, and integer shifts are order,Z equivalent, we can assume that $p_1,q_1 \geq 0$. We can also push the copies of p_{i-1} up toward p_i so that it has the same floor as p_i and also pull the copies of p_{j+1} down toward $p_j = p_i$ so that it has the same ceiling as $p_i = p_j$. This again maintains order,Z equivalence. Let x* be the result. Then $x^* \in R$. We abuse notation by still writing $x^* = (p_1,\ldots,p_{i-1},p,\ldots,p,p_{j+1},\ldots,p_k,p_1,\ldots,p_{i-1},q,\ldots,q,p_{j+1},\ldots,p_k)$, $p < q$.

case 1. $1 < i \leq j < k$. Since $p_1,q_1 \geq 0$, we fix n such that $p_1,q_1,p_k,q_k \in Q[0,n]^k$. Let G be the order invariant graph on $Q[0,n]^2$, where $v,w \in Q[0,n]^k$ are nonadjacent if and only if x = y or

$$v_1 \leq \ldots \leq v_{i-1} < v_i = \ldots = v_j < v_{j+1} \leq \ldots \leq v_k \wedge$$
$$w_1 \leq \ldots \leq w_{i-1} < w_i = \ldots = w_j < w_{j+1} \leq \ldots \leq w_k \wedge$$
$$(v_1,\ldots,v_{i-1},v_{j+1},\ldots,v_k) = (w_1,\ldots,w_{i-1},w_{j+1},\ldots,w_k) \wedge$$
$$v_i \neq w_i.$$

Clearly this nonadjacency relation is reflexive and symmetric and order invariant, and so defines an order invariant graph on $Q[0,n]^k$. If $\max(v) < \min(w)$ then obviously v,w are adjacent. So G is also restrictive. Let S be a maximal clique in G. Consider $p_1,\ldots,p_{i-1},p_{j+1},\ldots,p_k$. It is clear from the nonadjacency relation that there is at most one $p_{i-1} < p < p_{j+1}$ such that $(p_1,\ldots,p_{i-1},p,\ldots,p,p_{j+1},\ldots,p_k) \in S$. If there is no such p then we can extend the maximal clique S by using any $p_{i-1} < p < p_{j+1}$. Furthermore, this unique p must be fractional since there are no integers in $Q(p_{i-1},p_{j+1})$. This violates the complete invariance of S with respect to R.

case 2. $1 < i \leq j = k$. Thus $x^* = (p_1,\ldots,p_{i-1},p,\ldots,p,p_1,\ldots,p_{i-1},q,\ldots,q)$, where $p < q$ are fractional with the same floor, and there is no integer strictly between p_{i-1},p. We fix n to be the least positive integer greater than p. Let G' be the order invariant graph on $Q[0,n]^k$, where $v,w \in Q[0,n]^k$ are nonadjacent if and only if $x = y$ or

$$v_1 \leq \ldots \leq v_{i-1} < v_i = \ldots = v_k \wedge$$
$$w_1 \leq \ldots \leq w_{i-1} < w_i = \ldots = w_k \wedge$$
$$(v_1,\ldots,v_{i-1}) = (w_1,\ldots,w_{i-1}) \wedge$$
$$v_k \neq w_k$$
$$\vee$$
$$v_1 = \ldots = v_{k-i+1} > v_{k-i+2} \geq \ldots \geq v_k \wedge$$
$$w_1 = \ldots = w_{k-i+1} > w_{k-i+2} \geq \ldots \geq w_k \wedge$$
$$(v_{k-i},\ldots,v_k) = (w_{k-i},\ldots,w_k) \wedge$$
$$v_1 \neq w_1$$
$$\vee$$
$$v_1 \leq \ldots \leq v_{i-1} < v_i = \ldots = v_k \wedge$$
$$w_1 = \ldots = w_{k-i+1} > w_{k-i+2} \geq \ldots \geq w_k \wedge$$
$$(v_1,\ldots,v_{i-1}) = (w_k,\ldots,w_{k-i}) \wedge$$
$$v_k \neq w_1$$
$$\vee$$
$$w_1 \leq \ldots \leq w_{i-1} < w_i = \ldots = w_k \wedge$$
$$v_1 = \ldots = v_{k-i+1} > v_{k-i+2} \geq \ldots \geq v_k \wedge$$
$$(w_1,\ldots,w_{i-1}) = (v_k,\ldots,w_{k-i}) \wedge$$
$$w_k \neq v_1$$

Clearly this nonadjacency relation is reflexive and symmetric and order invariant, and so defines an order invariant graph on $Q[0,n]^k$. If $\max(v) < \min(w)$ then obviously v,w are adjacent. So G' is also restrictive. Let S be a maximal clique in G'. Consider $p_1 \leq \ldots \leq p_{i-1}$ and also $p_{i-1} \geq \ldots \geq p_1$. It is clear from the nonadjacency relation that there is

1) at most one $p > p_{i-1}$ such that $(p_1,\ldots,p_{i-1},p,\ldots,p) \in S$.
2) at most one $p' > p_{i-1}$ such that $(p',\ldots,p',p_{i-1},\ldots,p_1) \in S$.
3) any such p is not equal to any such p'.

If there is no such p for 1) then we can enlarge the maximal clique S by using any $p > p_{i-1}$ which is different from any p' for 2). Hence we have uniqueness for 1).

Arguing the same way, we have uniqueness for 2). Therefore there is

4) a unique $p > p_{i-1}$ such that $(p_1,...,p_{i-1},p,...,p) \in S$.
5) a unique $p' > p_{i-1}$ such that $(p,...,p,p_{i-1},...,p_1) \in S$.
6) $p \neq p'$.

We cannot argue as in case 1 that p is fractional and therefore S is not completely invariant with respect to R. Nor can we argue that p' is fractional. However, by the choice of the right endpoint n, and the choice of p_{i-1}, we know that either p or p' is fractional. If p is fractional then 4) clearly gives us a violation of the complete invariance of S with respect to R. Now suppose p' is fractional. Then we also get a violation of the complete invariance of S with respect to R using that R is permutation invariant.

case 3. $1 = i \leq j < k$. Thus $x^* = (p,...,p,p_{j+1},...,p_k,p',...,p',p_{j+1},...,p_k)$, where $p < p'$ are fractional. We can assume also that p,p' have floor 0 as integer shifts only create order,Z equivalent copies. We then repeat the treatment for case 2 by reversing signs. Before, there was no integer strictly between p,p' and the right endpoint n. Here there is no integer strictly between p,p' and the left endpoint 0.

QED

LEMMA 3.3.4. (RCA_0) Let R be k-good and k-usable. Every initially fractional element of $Q^{k}_{*} \times Q^{k}_{*}$ lies in crit[k].

Proof: Let R be as given. Let $x \in Q^{k}_{*} \times Q^{k}_{*}$ be initially fractional, x = (y,z). If y = z then obviously $x \in$ crit[k]. Suppose $y \neq z$. Since R is usable, x,y are order equivalent. By Lemma 3.3.1, we have i or ii or iii there. By Lemma 3.3.2, if i or ii holds, then there is a rational mover in R, which is impossible by Lemma 3.3.3. Hence iii holds, and so obviously $x \in$ crit[k]. QED

THEOREM 3.3.5. The following is provable in SRP^+ but not in ZFC (assuming ZFC is consistent), and not in SRP (assuming SRP is consistent). In fact, it is equivalent to Con(SRP) over WKL_0. Let R be a permutation order,Z invariant initially fractional equivalence relation in Q^k. The following are equivalent.

i. Every finite subset of $Q[0,n]^k$ has a completely R invariant maximal emulator.
ii. R is included in crit[k].
i → ii is provable in RCA_0.

Proof: Let R be as given. Assume i, which means that R is k-usable. Then R is order preserving. Let x R y, x ≠ y. Let π be a coordinate permutation such that πx R πy and πx,πy ∈ $Q^{k}{}_{\leq}$. Then (πx,πy) ∈ R, πx ≠ πy, and so by Lemma 3.3.4, (πx,πy) ∈ crit[k]. Hence (x,y) ∈ crit[k]. We already have ii → i by Theorem 2.3.7.

It remains to see that i → ii yields Con(SRP) over WKl_0. {(x,y) ∈ crit[k]: (x,y) is initially fractional} is a permutation order,Z invariant initially fractional equivalence relation on Q^k. It differs from crit[k] in such a minor way that the reversal of its k-usability is handled easily in the development promised in [Fr19]. QED

In Theorem 3.3.5, we can obviously modify i to any of the stronger forms considered in section 2.2, as long as n is still universally quantified (i.e., with ∞ right after k). This is because i' → ii comes for free, and ii → i' is by Theorem 2.3.7.

3.4. LINE SEGMENT ISOMETRIES

TO BE ADDRESSED IN A LATER DRAFT.

We define the directed line segment [(x,y)] for x ≠ y from $Q[0,n]^k$ as a subset of $Q[0,n]^k$. The directed isometry between two directed line segments exist and are unique if and only if the line segments are alike and have the same length. Alike refers to the endpoint situation. The default stability and the related ones with a change of coordinate, can be expressed as complete invariance with respect to the directed isometry between certain two directed line segments. But this leads to other kinds of stability of a different character for other choices of alike directed line segments of equal length. Also, sets of isometries are considered.

4. FURTHER INVESTIGATIONS

We are currently engaging in many directions. Above all, there is the promised reversals for [Fr19]. The main reversal promised, MCS(res) → Con(SRP) over RCA_0, is just the beginning of presumably a long story where we try to minimize the k and n in $Q[0,n]^k$ in order to achieve independence from ZFC.

We are also engaged in the development of Invariant Maximality where we look for completely invariant maximal objects which are not necessarily emulators. This loses the special tangible appeal of Emulation Theory but is of illuminating generality.

Superficially, it appears that if we are given an even slightly nontrivial small finite set from, e.g., $Q[0,4]^4$, then mathematicians would be very hard pressed to prove the existence of a fully stable maximal emulator without going beyond ZFC. We conjecture that for all such small data, ZFC is enough. However, if we are right about the difficulties here, then we have a plethora of specific concrete mathematical challenges no one knows how to meet staying within the usual ZFC axioms for mathematics. These are all implicitly Π^0_1. We would not have to design these challenges specifically to make them difficult.

Every aspect of the usability characterizations in section 3 suggest a massive number of modified classification projects, which appear daunting even under small modifications. And we have only scratched the surface with the usability of geometrically motivated functions as discussed in section 3.4.

All of these usability projects and small k,n projects are amplified in depth when they are expanded to incorporate the more general frameworks in section 2.2.

In addition, there is ongoing research on Finite Emulation Theory and Algorithmic Emulation Theory. Some developments include the following.

A. Nondeterministic algorithms for building stable maximal cliques in stages. These are being fine tuned for actual implementation in order to confirm (or possibly refute) the consistency of ZFC and fragments of SRP. Implementation is always at the level of modest finite lengths, producing finite cliques with stability properties. See [Fr18c].

B. Nondeterministic algorithms for building stable maximal cliques in stages. These are also being fine tuned for a different purpose. Namely for intellectual implementation, where the mathematical simplicity and naturalness of the procedures is paramount. This give a plethora of explicitly Π^0_1 sentences of various logical strengths ranging from weak fragments of arithmetic to various large cardinals. [Fr18d].

5. APPENDIX A. STABILITY NOTIONS

In this paper, we start with a particular default notion of stability (Definition 2.1.5), and move to using complete invariance with respect to a relation (Definition 2.2.1), often but not necessarily an equivalence relation, as the notion of stability. We single out two notions of stability with special names. The first is just stability (the default used for MES, MDS, MCS), and later full stability (Definition 2.3.2). Full stability is complete invariance with respect to the equivalence relation crit[k], Definition 2.3.1, which is the strongest notion of stability and strongest equivalence relation that we use for our independent statements. Theoretically, full stability and the associated crit[k] have a special theoretical place via Theorem 3.2.21 and 3.3.5, as a kind of essentially strongest choice (along with the dual -crit[k]) indicated by Definition 3.1.3, Theorem 3.1.3, and Theorem 3.2.21).

1. $S \subseteq Q[0,k]^k$ is stable (the default) if and only if for all $p < 1$, $(p,1,...,k-1) \in S \leftrightarrow (p,2,...,k) \in S$.
2. $S \subseteq Q[0,n]^k$ is fully stable if and only if $(m_1,...,m_i) \in S \leftrightarrow (r_1,...,r_i) \in S$ holds after the same rationals $p_1,...,p_{k-i}$ are placed in the same positions on both sides, provided $1 \leq i < k$, and $m_1,...,m_i$ and $r_1,...,r_i$ are order equivalent.

Here 2 is an alternative definition of fully stable which has some conceptual advantages. If < is replaced by > then 2 becomes the same as the complete invariance of $S \subseteq Q[0,n]^k$ with respect to the equivalence relation -crit[k] on Q^k.

Note that fully stable uses order equivalence of tuples whereas stable is more basic. There are many interesting choices intermediate between 1 and 2 that are also basic. Here are some choices. Below, k,i,n,m,r, with and without subscripts, are positive integers, and p,q, with and without subscripts are rationals.

3. $S \subseteq Q[0,k]^k$ is stable/1 if and only if $(1,...,k-1) \in S \leftrightarrow (2,...,k) \in S$ holds after $p < 1$ is inserted in one of the k positions (same position on both sides).
4. $S \subseteq Q[0,k]^k$ is stable/2 if and only if for all $p_1,...,p_i < i < k$, $(p_1,...,p_i,i,i+1,...,k-1) \in S \leftrightarrow (p_1,...,p_i,i+1,i+2,...,k) \in S$
5. $S \subseteq Q[0,n]^k$ is stable/3 if and only if $(1,....,k-i) \in S \leftrightarrow (2,...,k-i+1) \in S$ holds, $i < n$, after $p_1,...,p_i < 1$ are inserted in various positions (same position on both sides).
6. $S \subseteq Q[0,n]^k$ is stable/4 if and only if for all $p < m_1,...,m_{k-1} < n$, $(p,m_1,...,m_{k-1}) \in S \leftrightarrow (p,m_1+1,...,m_{k-1}+1) \in S$.
7. $S \subseteq Q[0,n]^k$ is stable/5 if and only if for all $p < m_1 < ... < m_{k-1} \leq n$ and $p < r_1 < ... < r_{k-1} \leq n$, $(p,m_1,...,m_{k-1}) \in S \leftrightarrow (p,r_1,...,r_{k-1}) \in S$.
8. $S \subseteq Q[0,n]^k$ is stable/6 if and only if for all $p_1,...,p_i < m_1 < ... < m_{k-i} \leq n$ and $p_1,...,p_i < r_1 < ... < r_{k-i} \leq n$, $(p_1,...,p_i,m_1,...,m_{k-i}) \in S \leftrightarrow (p_1,...,p_i,r_1,...,r_{k-i}) \in S$.
9. $S \subseteq Q[0,n]^k$ is stable/7 if and only if $(m_1,...,m_{k-i}) \in S \leftrightarrow (r_1,...,r_{k-i}) \in S$ holds after $p_1,...,p_i$ are inserted in various positions (same positions on both sides), provided $p_1,...,p_i < m_1 < ... < m_{k-i} \leq n$ and $p_1,...,p_i < r_1 < ... < r_{k-i} \leq n$.

All of these notions of stability are included in fully stable (2 above). Of these, only fully stable and 9 amount to complete invariance with respect to a permutation order,Z invariant equivalence relation on Q^k. In fact, a single such relation - crit[k] - that does not depend on the choice of n. The promised reversal in [Fr19] works for all of these notions of stability.

6. APPENDIX B. FORMAL SYSTEMS USED

EFA Exponential function arithmetic. Based on 0, successor, addition, multiplication, exponentiation and bounded induction. Same as $I\Sigma_0$(exp), [HP93], p. 37, 405.

RCA_0 Recursive comprehension axiom naught. Our base theory for Reverse Mathematics. [Si99,09].

ACA'

WKL_0 Weak Konig's Lemma naught. Our second level theory for Reverse Mathematics. [Si99,09].

Z_2 Second order arithmetic as a two sorted first order theory. [Si99,09].

Z_3 Third order arithmetic as a three sorted first order theory. Extends Z_2 with a new sort for sets of subsets of ω.

ZF(C) Zermelo Frankel set theory (with the axiom of choice). ZFC is the official theoretical gold standard for mathematical proofs. [Ka94].

SRP ZFC + $(\exists\lambda)(\lambda$ has the k-SRP), as a scheme in k. [Fr01].

SRP^+ ZFC + $(\forall k)(\exists\lambda)(\lambda$ has the k-SRP). [Fr01].

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