EXOTIC PREFIX THEORY

abstract

by

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(This corrects an earlier version with the same date that has been removed from my website).

The state of the art on Order Calculus and the variants of the Π^0_1 independent statements that are associated with the subtle cardinal hierarchy (not the Mahlo cardinal hierarchy), is now contained in

Search for Consequences, under downloadable manuscripts, Lecture Notes, http://www.math.ohio-state.edu/%7Efriedman/

Note that the independent statements in Search for Consequences go forward and backward. The ones in posting #299 only go forward, and we are no longer confident that we can get away with this in #299.

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The Order Calculus development now appears as key Lemmas in Exotic Prefix Theory.

Exotic Prefix Theory is the study of basic prefix classes, comprising finitely many sentences, at least one of which is independent of ZFC. Such a prefix class is said to be "exotic".

The goal is to show that various exotic prefix classes can be "tamed by large cardinals". I.e., every statement in the class is either provable or refutable using presently formulated large cardinals. Some of these exotic prefix classes consist entirely of explicitly Π^0_1 sentences.

At the moment, this goal is out of reach for any basic exotic prefix class. However, we expect a long series of partial results, that various subclasses can be tamed by various large cardinals.

In both Search for Consequences, and Forty Years on his Shoulders, under downloadable manuscripts, http://www.math.ohio-state.edu/%7Efriedman/ we made the following conjecture:

EXOTIC CONJECTURE

Every interesting mathematical theorem can be recast as one among a natural finite set of statements, all of which can be decided using well studied extensions of ZFC, but not within ZFC itself.

We now present some prefix classes that comprise finitely many statements, at least one of which is independent of ZFC (exotic prefix classes). Some of the prefix classes consist entirely of explicitly Π^0_1 sentences.

This suggests the following:

EXOTIC PREFIX CONJECTURE

Every interesting mathematical theorem is an element of a natural exotic prefix class. In other words, every interesting mathematical theorem can be recast as an element of a natural prefix class, with finitely many elements, all of which can be decided using well studied extensions of ZFC, but not within ZFC itself.

As we have done many times before, we start with what we call the complementation theorem for relations.

Lower case letters stand for positive integers, unless otherwise indicated. We write [inf] for the set of all positive integers, and [n] for {1,...,n}.

For $A \subseteq [n]^k$, we write $A' = [n]^k \setminus A$.

Let $R \subseteq [n]^k \times [n]^k = [n]^{2k}$. We write $RA = \{b: (\exists a \in A)(R(a,b))\}$.

We say that R is strictly dominating iff for all a,b, if R(a,b) then max(a) < max(b).

COMPLEMENTATION THEOREM (finite relations). For all $n,k \geq 1$ and strictly dominating $R \subseteq [n]^{2k}$, there exists $A \subseteq [n]^k$ such that RA = A'. In fact, A is unique.

Here we will also consider the obvious infinite form. Let [inf] be the set of all positive integers.

COMPLEMENTATION THEOREM (infinite relations). For all k ≥ 1 and strictly dominating R □ [inf]^{2k}, there exists A □ [inf]^{k} such that RA = A′. In fact, A is unique.

The above is a special case of a well known and well studied theorem in digraph theory - for any dag G, there is a (unique) A □ V(G) such that GA = V(G)\A. (Here GA is the set of heads coming out from A).

We can put the complementation theorem in the following successive logical forms.

If R is strictly dominating then there exists A such that RA = A′.

For all R there exists A such that if R is strictly dominating then RA = A′.

For all R there exists A such that if (□a,b)(R(a,b) □ a < b) then RA □ A′ and A′ □ RA.

For all R there exists A such that if (□a,b)(R(a,b) □ a < b) then (□a,b □ A)(□R(a,b) □ (□a □ A′)(□b □ A)(R(b,a)).

For all R there exists A such that □□ or (□□ □ □□).

For all R there exists A such that □□□□.

In the above, we have used < for

a < b iff max(a) < max(b).

Thus we have put the Complementation Theorem (relations), without uniqueness, into the following prefix forms:

Let k ≥ 1. (□R □ [inf]^{2k})(□A □ [inf]^{k})(□a,b □ [inf]^{k})(□c,d □ [inf]^{k})(□(R,A,a,b,c,d)).

Let k,n ≥ 1. (□R □ [n]^{2k})(□A □ [n]^{k})(□a,b □ [n]^{k})(□c,d □ [n]^{k})(□(R,A,a,b,c,d)).

where □ is a propositional combination of atomic formulas

x < y.
x □ A.
R(x,y).

Here x,y are among the variables a,b,c,d.

However, we can give obviously equivalent forms of the Complementation Theorem (relations) which need one less quantifier to state:

COMPLEMENTATION THEOREM (infinite relations). For all k ≥ 1 and R □ $[\text{inf}]^{2k}$, there exists A □ $[\text{inf}]^{k}$ such that $R_<A$ = A′. In fact, A is unique.

COMPLEMENTATION THEOREM (finite relations). For all n,k ≥ 1 and R □ $[n]^{2k}$, there exists A □ $[n]^{k}$ such that $R_<A$ = A′. In fact, A is unique.

Here $R_<A$ is taken to be {b: (□a □ A)(a < b □ R(a,b)}.

If R is strictly dominating then there exists A such that $R_<A$ = A′.

For all R there exists A such that $R_<A$ = A′.

For all R there exists A such that if $R_<A$ □ A′ and A′ □ $R_<A$.

For all R there exists A such that (□a,b □ A)(□R(a,b) □ a < b) and (□a □ A′)(□b □ A)(R(b,a) □ b < a).

For all R there exists A such that □□ and □□.

For all R there exists A such that □□□.

Thus we have put the Complementation Theorem (relations), without uniqueness, into the following form:

Let k ≥ 1. (□R □ $[\text{inf}]^{2k}$)(□A □ $[\text{inf}]^{k}$)(□a,b □ $[\text{inf}]^{k}$)(□c □ $[\text{inf}]^{k}$)(□(R,A,a,b,c)).

Let k,n ≥ 1. (□R □ $[n]^{2k}$)(□A □ $[n]^{k}$)(□a,b □ $[n]^{k}$)(□c □ $[n]^{k}$)(□(R,A,a,b,c)).

where □ is a propositional combination of atomic formulas of the forms

x < y.

x □ A.
R(x,y).

Here x,y are among the variables a,b,c.

We can now ask for a determination of the truth values of all instances of

Let k ≥ 1. (□R □ [inf]^{2k})(□A □ [inf]^{k})(□a,b □ [inf]^{k})(□c □ [inf]^{k})(□(R,A,a,b,c)).

Let k,n ≥ 1. (□R □ [n]^{2k})(□A □ [n]^{k})(□a,b □ [n]^{k})(□c □ [n]^{k})(□(R,A,a,b,c)).

where □ is as given above.

We strongly believe that these classification problems are manageable, with some real effort, and also that these particular prefix classes are not exotic. I.e., these sets of statements are not comprehensive enough to force us out of ZFC.

We have been able to extend these Templates above that incorporate the Order Calculus development. In particular, consider the following four Templates:

Let k ≥ 1. (□R □ [inf]^{2k})(□S □ [inf]^{2k})(□a,b,c,d □ [inf]^{k})(□e,f,g,h □ [inf]^{k})(□(R,S,a,b,c,d,e,f,g,h)).

Let n >> k. (□R □ [n]^{2k})(□S □ [n]^{2k})(□a,b,c,d □ [n]^{k})(□e,f,g,h □ [n]^{k})(□(R,S,a,b,c,d,e,f,g,h)).

For all k there exists n such that the following holds. (□R □ [n]^{2k})(□S □ [n]^{2k})(□a,b,c,d □ [inf]^{k})(□e,f,g,h □ [n]^{k})(□(R,S,a,b,c,d,e,f,g,h)).

Let k ≥ 1. (□R □ [(8k)!]^{2k})(□S □ [(8k)!]^{2k})(□a,b,c,d □ [(8k)!]^{k})(□e,f,g,h □ [(8k)!]^{k})(□(R,S,a,b,c,d,e,f,g,h)).

where □ is a propositional combination of formulas of the form

x < y.
R(x,y).
S(x,y).
(x,y) ~ (z,w).

Here x,y,z,w are among the variables a,b,c,d,e,f,g,h. Also, ~ denotes order equivalence between the concatenation of x,y, and the concatenation of z,w.

Using the Order Calculus development, we can show that all four of these prefix classes are exotic. In fact, we find a single □(R,S,a,b,c,d,e,f,g,h) using only the atomic formulas above, such that all four of the above Templates are provable in ZFC + "there exists a 2-subtle cardinal", but not in ZFC, or even in ZFC + "there exists a subtle cardinal kappa such that there are kappa many subtle cardinals below kappa".

Note that these Templates are, explicitly, in

□12, □03, □02, □01

form, respectively. For our exotic example, all four statements imply consistency of the weaker theory above and follow from the stronger theory above. RCA$_0$ suffices for the first form, and EFA (exponential function arithmetic) suffices for the remaining three forms.

We conjecture that these four exotic prefix can all be tamed by large cardinals. In fact, that all instances of these four Templates can be decided in ZFC + {there exists an n-subtle cardinal}$_n$.

We can refer to these prefix forms as

(□□;$□^4□^4$).

Using (□□;$□^5□^5$), we can get past 2-subtle cardinals, (□□;$□^6□^6$), past 3-subtle cardinals, etcetera. Of course, at some point, we should run into the possibility of rampant coding, so that the prefix classes becomes "completely wild" - whatever that means. But we are convinced that (□□;$□^4□^4$) is far below the "completely wild" level – since we claim that it can be tamed using presently formulated large cardinals.

Some partial results will probably involve restrictions on the atomic formulas that appear in the instances.

We have no idea whether these three classes remain exotic if we use only x ~ y instead of the more powerful (x,y) ~ (z,w).

Coming back to the Exotic Prefix Conjecture, we can obviously view the Complementation Theorem (relations) as an instance of the exotic classes presented above. Of course, the challenge that awaits us is to eventually show that these exotic classes can be tamed by large cardinals.