FINITE TREES AND THE NECESSARY USE OF LARGE CARDINALS

by

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We introduce the basic concept of insertion rule that specifies the placement of new vertices into finite trees. We prove that every “good” insertion rule generates a tree with simple structural properties in the style of classical Ramsey theory. This is proved using large cardinal axioms going well beyond the usual axioms for mathematics.

And this result cannot be proved without these large cardinal axioms. We also prove that every insertion rule greedily generates a tree with these same structual properties. It is also nec-essary and sufficient to use the same large cardinals. The results suggest a new area of research - "greedy Ramsey theory."

A partial ordering is a pair (X,□), where X is a nonempty set, and □ is reflexive, transitive, and antisymmetric. The ancestors of x in X are just the y < x. A tree T = (V,□) is a partial ordering with a least element (root), where the the ancestors of any x in V form a finite linearly ordered set under ≤.

If x < y and for no z is x < z < y, then we say that y is a child of x and x is the parent of y. V = V(T) is the set of vertices of the tree T = (V,□). Every vertex other than the root has a unique parent; i.e., is a child.

For finite rooted trees, T_1 □ T_2 means that T_1,T_2 have the same root, and if x is the parent of y in T_1, then x is the parent of y in T_2. I.e., no parent/child bond is broken by going from T_1 to T_2.

Let N = {1,2,...}. A k-tree is a finite rooted tree whose root is   and whose children are k element subsets of N. The trivial k-tree is the k-tree with the single vertex . We write $[N]^k$ be the set of all k element subsets N and $[N]^{k+}$ for $[N]^k$ □ { }. We use □* for the standard reverse lexicographic ordering on $[N]^k$, in which sets are compared according to the lexicographic ordering on N^k after the sets are placed in strictly descending order. We extend this linear ordering to $[N]^{k+}$ by placing   at the top.

Let T be a k-tree and $x \in [N]^k$. We say that x dominates T if and only if for all $y \in Ch(T)$, we have $x >^* y$.

Insertion rules specify the placement of new, dominating, vertices into trees.

Formally, an insertion rule on TR(k) is a function f defined on exactly the pairs (T,x), where T is a k-tree and $x \in [N]^k$ dominates T, where each defined f(T,x) lies in V(T).

How do we use an insertion rule on TR(k)? Let $A \subseteq [N]^k$ be finite and write $A = v_1 <^* \ldots <^* v_n$. Construct k-trees $T_0 \subseteq \ldots \subseteq T_n$ as follows. T_0 is the trivial k-tree. T_{i+1} is the k-tree obtained by inserting v_{i+1} into T_i as a new child of the vertex $f(T_i,v_{i+1})$. We write $T_{i+1} = T_i/v_{i+1},f(T_i,v_{i+1})$. Clearly $Ch(T_n) = A$.

In this way, f generates a unique k-tree with any finite subset of $[N]^k$ as its children.

Often one concentrates on a subclass S of trees and inserts vertices only into trees in S. This leads to the more general notion of insertion rule in TR(k). This is a pair (S,f) such that S is a nonempty subset of TR(k) and f is a function defined on exactly the pairs (T,x), where $T \in S$ and $x \in [N]^k$ dominates T, where each defined T/x,f(T,x) lies in S.

Here S is called the field, and f is called the function.

We say that an insertion rule in TR(k) admits T if and only if T lies in its field. We say that (S,f) is initial if and only if it admits the trivial k-tree.

The study of trees generated by insertion rules on TR(k) is closely related to the study of trees admitted by insertion rules in TR(k). The latter is somewhat more general, as can be seen by the following utterly straightforward background theorem.

BACKGROUND. Let $k \geq 1$ and $S \subseteq TR(k)$. If S is the set of trees generated by some insertion rule on TR(k), then S is the set of trees admitted by some insertion rule in TR(k). However, the converse is false. On the other hand, if S is the set of trees admitted by some initial insertion rule in TR(k), then S is the set of trees generated by some insertion rule for TR(k).

Here is a template:

PROPOSITION 1. Let k,p ≥ 1. Every "good" insertion rule on (in) TR(k) generates (admits) a k-tree in which all k element subsets of some p element set are vertices that are "similar."

Ex: For 1-trees T and x □ N dominating T, let f(T,x) = max(Ch(T)) if T is nontrivial;   otherwise. The trees generated by f consist of just the 1-trees with no splitting, where parents (other than the root) are strictly smaller than children. No two vertices in such 1-trees are "similar" in any appropriate sense. So this f is not "good."

INFINITE RAMSEY THEOREM. Let k,p ≥ 1. In every function on $[N]^k$ with finite range, all k element subsets of some p element set are "similar."

INFINITE RAMSEY THEOREM. Let k,p ≥ 1. In every function on $[N]^k$ with finite range, all k element subsets of some p element set have the same value.

Here we use two related concepts of "similar" between vertices in k-trees. The ancestors of a vertex v in a tree T = (V(T),□) are the vertices w such that w < v (in the sense of the tree partial order □).

The first notion of similarity is that x,y have the same number of ancestors. In this case, for various notions of "good" we have the following theorems.

THEOREM 2. Let k,p ≥ 1. Every "good" insertion rule on (in) TR(k) generates (admits) a k-tree in which all k element subsets of some p element set are vertices with the same number of ancestors.

We use some combina-torial set theory to derive Theorem 2. Let X be any nonempty set and k ≥ 1. A k,X-tree is a tree whose set of vertices is exactly $[X]^k$ = the set of all k element subsets of X. Recall that set of all ancestors of every vertex is finite.

THEOREM 3. Let X have power $\square_{\square}$. Let $k,p \geq 1$. In every k,X-tree, all k element subsets of some p element set are vertices with the same number of ancestors.

We use Theorem 3 to derive Theorem 2 (under various notions of "good"). This provides a proof of Theorem 2 in Zermelo set theory. We prove that the use of infinitely many uncountable cardin-als cannot be removed.

Second notion of similarity: Let x,y □ $[N]^k$. We say that x is enti-rely lower than y if and only if every element of x is strictly smaller than every element of y.

The second notion of similarity is that x,y have the same entirely lower ancestors. I.e., every entirely lower ancestor of x is an entirely lower ancestor of y, and vice versa.

PROPOSITION 4. Let $k,p \geq 1$. Every "good" insertion rule on (in) TR(k) generates (admits) a k-tree in which all k element subsets of some p element set are vertices with the same entirely lower ancestors.

We prove Proposition 4 using large cardinal axioms that go well beyond the usual axioms for mathematics - the subtle cardinals of finite order. And we prove that they are necessary.

We now present the simplest of our arsenal of notions of "good" insertion rules on (in) TR(k).

A decreasing insertion rule on TR(k) is an insertion rule f on TR(k) such that for all k-trees T_1 □ T_2 and x □ $[N]^k$ dominating T_2, $f(T_1,x) \geq^* f(T_2,x)$.

A decreasing insertion rule in TR(k) is an insertion rule (S,f) in TR(k) such that for all T_1 □ T_2 from S and x □ $[N]^k$ dominating T_2, $f(T_1,x) \geq^* f(T_2,x)$.

THEOREM 5. Let $k,p \geq 1$. Every decreasing insertion rule on (in) TR(k) generates (admits) a k-tree in which all k element subsets of some p element set are vertices with the same number of ancestors.

PROPOSITION 6. Let $k,p \geq 1$. Every decreasing insertion rule on (in) TR(k) generates (admits) a k-tree in which all k element subsets of some p element set are vertices with the same entirely lower ancestors.

We prove Theorem 5 in Z = Zermelo set theory. We show that it cannot be proved in Z with full separation replaced by bounded separaton.

We prove Proposition 6 in ZFC + (□k)(there exists a subtle cardinal of order k). We prove that these large cardinals are necessary by showing that Proposition 6 cannot be proved in ZFC + $\{$there exists a subtle cardinal of order k$\}_k$.

We now give a formulation of these necessary uses of set theory in terms of "greedy" constructions.

The term "greedy" comes from "greedy" algorithms, whereby optimal finite objects are sought in various context. In certain important contexts, the standard efficient algorithms proceed by building up the desired optimized object in sequentially, where at each stage the construction extends the object built thus far in a relatively optimal way. This kind of construction is used for the standard efficient algorithms in such diverse contexts as minimal spanning trees, Huffman codes, task-scheduling, shortest paths, etc.

Our results definitely reflect this "greedy" idea. Moreover, we expect that further results will be obtained with yet closer connections with greedy algorithms of the kind that figure so prominently in the theory of algorithms.

Let f be an arbitrary insertion rule on TR(k). We now consider the k-trees greedy generated by f.

In a greedy generation of a k-tree using f, we start with a nonempty finite set of vertices $v_1 <^* \ldots <^* v_n$ from $[N]^k$, $n \geq 1$. We are free to pick any such set of vertices, but from now on the construc-tion is entirely deterministic.

We begin the construction at stage 0 with the trivial k-tree. At stage $0 \leq i < n$, we have a k-tree T_i with vertices $v_1,\ldots,v_i$, and at stage i+1 we seek to insert v_{i+1} into T_i. Recall that in tree generation, we place v_{i+1} into T_i as a new child of $f(T_i,v_{i+1})$.

In greedy generation, we instead consider the placement of v_{i+1} in all subtrees T' □ T and pick one where $f(T',v_{i+1})$ is <*

least. We continue this greedy process until we construct T_n, which has vertices $v_1,...,v_n$.

We say that T is greedy generated by f iff T results from such a greedy construction; i.e., $T = T_n$, where we use the vertices $v_1 <^* ... <^* v_n$ of T. For any finite $A \subseteq [N]^k$ there is a unique k-tree with children A which is greedy generated by f.

The following obvious result supplies us with the needed link:

THEOREM 7. Let $k \geq 1$ and f be a decreasing insertion rule on TR(k). The trees generated by f are exactly the trees greedy generated by f. And let g be an insertion rule on TR(k). Then there exists a decreasing insertion rule h on TR(k) such that the trees greedy generated by g are exactly the trees generated by h.

Now consider the following analogs to Theorem 5 and Proposition 6.

THEOREM 8. Let $k,p \geq 1$. Every insertion rule on TR(k) greedy generates a k-tree in which all k element subsets of some p element set are vertices with the same number of ancestors.

PROPOSITION 9. Let $k,p \geq 1$. Every insertion rule on TR(k) greedy generates a k-tree in which all k element subsets of some p element set are vertices with the same entirely lower ancestors.

By Theorem 7, we see that Theorem 8 is equivalent to Theorem 5 for on/generates, and Proposition 9 is equivalent to Proposition 6 for on/generates.

Thus Theorem 8 and Proposition 9 share these same metamathematical properties as Theorem 5 and Proposition 6:

a) Theorems 5 and 8 can be proved in Z but not in Z with separation replaced by bounded separation;
b) Propositions 6 and 9 can be proved in ZFC + $(\forall k)$(there exists a subtle cardinal of order k), but not in ZFC + $\{$there exists a subtle cardinal of order k$\}_k$.

We now present straightforward finite forms of the above results. We begin with Ramsey's finite form of the Infinite Ramsey theorem. Let $[n]^k = \{1,...,n\}^k$.

FINITE RAMSEY THEOREM. Let n >> k,r,p ≥ 1. For every f:$[n]^k$ □ [r], all k element subsets of some p element set have the same value.

We find it illuminating to use the >> notation.

Here we can replace the first sentence by: "for all k,p ≥ 1 there exists n such that the following holds."

For k,n ≥ 1, a k,n-tree is a k-tree all of whose vertices lie in $[n]^k$. Let TR(k,n) be the set of all k,n-trees.

The concepts of insertion rule on (in) $[n]^k$, generation, greedy generation, admitting, decreasing, are carried over straighforwardly.

Here are the straightforward finite forms of 5,6,8,9. They are provably equivalent to 5,6,8,9 respectively by a standard finitely branching tree or compactness argument.

THEOREM 5'. Let n >> k,p ≥ 1. Every decreasing insertion rule on (in) TR(k,n) generates (admits) a k,n-tree in which all k element subsets of some p element set are vertices with the same # of ancestors.

PROPOSITION 6'. Let n >> k,p ≥ 1. Every decreasing insertion rule on (in) TR(k,n) generates (admits) a k,n-tree in which all k element subsets of some p element set are vertices with the same entirely lower ancestors.

THEOREM 8'. Let n >> k,p ≥ 1. Every insertion rule on TR(k,n) greedy generates a k,n-tree in which all k element subsets of some p element set are vertices with the same number of ancestors.

PROPOSITION 9'. Let n >> k,p ≥ 1. Every insertion rule on TR(k,n) greedy generates a k,n-tree in which all k element subsets of some p element set are vertices with the same entirely lower ancestors.