FROMAL STATEMENTS OF GODEL'S SECOND INCOMPLETENESS THEOREM

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Abstract. Informal statements of Gödel's Second Incompleteness Theorem, referred to here as Informal Second Incompleteness, are simple and dramatic. However, current versions of Formal Second Incompleteness are complicated and awkward. We present new versions of Formal Second Incompleteness that are simple, and informally imply Informal Second Incompleteness. These results rest on the isolation of simple formal properties shared by consistency statements. Here we do not address any issues concerning proofs of Second Incompleteness.

1. SECOND INCOMPLETENESS FOR PA.

We start with the most commonly quoted form of Gödel's Second Incompleteness Theorem - for the system PA = Peano Arithmetic.

PA can be formulated in a number of languages. Of these, L(prim) is the most suitable for supporting formalizations of the consistency of Peano Arithmetic.

We write L(prim) for the language based on 0,S and all primitive recursive function symbols. We let PA(prim) be the formulation of Peano Arithmetic for the language L(prim). I.e., the nonlogical axioms of PA(prim) consist of the
axioms for successor, primitive recursive defining equations, and the induction scheme applied to all formulas in L(prim).

INFORMAL SECOND INCOMPLETENESS (PA(prim)). Let A be a sentence in L(prim) that adequately formalizes the consistency of PA(prim), in the informal sense. Then PA(prim) does not prove A.

We have discovered the following result. We let PRA be the important subsystem of PA(prim), based on the same language L(prim), where we require that the induction scheme be applied only to quantifier free formulas of L(prim).

FORMAL SECOND INCOMPLETENESS (PA(prim)). Let A be a sentence in L(prim) such that every equation in L(prim) that is provable in PA(prim), is also provable in PRA + A. Then PA(prim) does not prove A.

NOTE: We are referring to equations which may contain variables (open equations).

We can think of the condition in the above Formal Second Incompleteness (PA(prim)) as a formal adequacy condition on a formalization in L(prim) of the consistency of PA(prim).

The following Thesis provides the needed connection between Formal and Informal Second Incompleteness (PA(prim)).

INFORMAL THESIS (PA(prim)). Let A be a sentence in L(prim) that adequately formalizes the consistency of PA(prim), in the informal sense. Then every equation in L(prim) that is provable in PR(prim), is also provable in PRA + A.

Informal Proof: Let Con(PA(prim)) be an adequate formalization in L(prim) of the consistency of PA(prim), in the informal sense. Assume

1) $s(x_1,\ldots,x_k) = t(x_1,\ldots,x_k)$ is provable in PA(prim)

where $s(x_1,\ldots,x_k),t(x_1,\ldots,x_k)$ are terms in L(prim) whose variables are among $x_1,\ldots,x_k$, $k \geq 0$.

We now argue in PRA + Con(PA(prim)). Let $x_1,\ldots,x_k$ in N, and assume not $s(x_1,\ldots,x_k) = t(x_1,\ldots,x_k)$. Let x_i^ be the numeral for x_i; i.e., S...S0, where there are x_i S's. Then, according to PRA,

2) PRA proves not $s(x_1\text{^},\ldots,x_k\text{^}) = t(x_1\text{^},\ldots,x_k\text{^})$.

In particular,

3) PA(prim) proves not $s(x_1\text{^},\ldots,x_k\text{^}) = t(x_1\text{^},\ldots,x_k\text{^})$.

From 1) and 3), we see that PA(prim) is inconsistent. This contradicts Con(PA(prim)). QED

From the Informal Thesis (PA(prim)), we immediately see that Formal Second Incompleteness (PR(prim)) implies Informal Second Incompleteness (PR(prim)).

Previous attempts to give a Formal Second Incompleteness (PR(prim)) through a formal adequacy condition were comparatively unsatisfactory. There were two existing approaches.

One is through the Hilbert Bernays derivability conditions and its variants.

These do not provide a direct condition on a formalization in L(prim) of the consistency of PA(prim). Instead, a proof predicate is introduced, as well as some auxiliary syntactic notions, and conditions are placed on all of these notions, simultaneously. A fully rigorous development is rather subtle
and involved, and uses the construction of auxiliary sentences.

Another is the approach of S. Feferman, which places conditions on the formalization of all the relevant syntactic notions that lead up to the formalization in L(prim) of the consistency of PA(prim).

These are more straightforward than the Hilbert Bernays conditions, but significantly more complicated than the Hilbert Bernays conditions. They are significantly less complicated and ad hoc than any direct formalization in L(prim) of the consistency of PA(prim).

However, they also remain unacceptably complicated
for use in the statement of this vitally important theorem.

The development presented here does NOT address any issues concerning the complications and subtleties present in PROOFS of the Second Incompleteness Theorem. It only relates to the STATEMENT of Second Incompleteness.

2. MANY SORTED EXTENSIONS OF FULL INDUCTION WITH PRIM.

Let L(many) be many sorted predicate calculus. Here the sorts, constants, relations, and functions are indexed using nonnegative integers.

Let L be a fragment of L(many) that contains L(prim). For definiteness, we require that L(prim) lives in the first sort. We let the theory PA(L) in L consist of the axioms for successor, the defining equations for the primitive recursive function symbols of L(prim), and the induction scheme applied to all formulas of L.

INFORMAL SECOND INCOMPLETENESS (many sorted induction, prim). Let L be a fragment of L(many) containing L(prim). Let T be a consistent extension of PA(L) in L. Let A be a sentence in L that adequately formalizes the consistency of T, in the informal sense. Then T does not prove A.

FORMAL SECOND INCOMPLETENESS (many sorted induction, prim). Let L be a fragment of L(many) containing L(prim). Let T be a consistent extension of PA(L) in L. Let A be a sentence in L such that every equation in L(prim) that is provable in T, is also provable in PRA + A. Then T does not prove A.

INFORMAL THESIS (many sorted induction, prim). Let L be a fragment of L(many) containing L(prim). Let T be a consistent extension of PA(L) in L. Let A be a sentence in L that adequately formalizes the consistency of T, in the informal sense. Then every equation in L(prim) that is provable in T, is also provable in PRA + A.

We now address the question of just which sentences obey the conditions in Formal Incompleteness above.

THEOREM 2.1. Let L be a fragment of L(many) containing L(prim). Let T be an extension of PA(L) in L. Let A be a sentence in L. The following are equivalent.
i. Every equation in L(prim) that is provable in T, is also provable in PRA + A.
ii. PRA + A proves the formal consistency of every finite fragment of T.
Furthermore, if T is recursively axiomatized then the formal consistency of T, formulated on the basis of an algorithm for the recursive axiomatization, obeys i,ii. If T contains all true universally quantified equations in L(prim), then for all A obeying i,ii, A is refutable in PRA.

3. GENERAL MANY SORTED THEORIES WITH PRIM.

So far, all of our theorems apply only to contexts in which the induction scheme for all formulas in the language are present in the theory. In particular, we have not addressed finitely axiomatized T.

We say that a sentence is in Π_1(prim) if and only if it is a universally quantified bounded formula in L(prim).

For n ≥ 0, we write $I\Sigma_n$(prim) for the fragment of PA(prim) where the induction scheme is applied to Σ_n(prim) formulas only. I.e., formulas in L(prim) starting with at most n quantifiers, the first of which is existential, followed by a quantifier free formula.

The following refutes our previous Formal Second Incompleteness, if we do not insist on extending induction with respect to all formulas.

THEOREM 3.1. Let T be a fragment of $I\Sigma_n$(prim) containing PRA. There is a theorem A of T such that every theorem of T lying in Π_1(prim) is provable in PRA + A.

The key notion needed for General Second Incompleteness is the notion of relativization. Let S,T be two theories in L(many), where every symbol appearing in S also appears in T (with the same sort information). A relativization of S in T consists of a formula in L(T) which, provably in T, defines a nonempty set which contains the constants appearing in the axioms of S, and which is closed under the operation symbols appearing in S, and where the result of restricting the quantifiers present in S to this nonempty set, is provable in T.

INFORMAL SECOND INCOMPLETENESS (general many sorted, prim). Let L be a fragment of L(many) containing L(prim). Let T be a consistent extension of PRA in L. Let A be a sentence in L that adequately formalizes the consistency of T, in the informal sense. Then T does not prove A.

FORMAL SECOND INCOMPLETENESS (general many sorted, prim). Let L be a fragment of L(many) containing L(prim). Let T be a consistent extension of PRA in L. Let A be a sentence in L such that every universalized equation in L(prim) with a relativization in T, is provable in PRA + A. Then T does not prove A.

INFORMAL THESIS (general many sorted, prim). Let L be a fragment of L(many) containing L(prim). Let T be a consistent extension of PRA in L. Let A be a sentence in L that adequately formalizes the consistency of T, in the informal sense. Then every universalized equation in L(prim) with a relativization in T, is provable in PRA + A.

We now address the question of just which sentences obey the conditions in Formal Incompleteness above.

THEOREM 3.2. Let L be a fragment of L(many) containing L(prim). Let T be an extension of PRA in L. Let A be a sentence in L. The following are equivalent.
i. Every universalized equation in L(prim) with a relativization in T, is provable in PRA + A.
ii. PRA + A proves the formal consistency of every finite fragment of T.
If T is finitely axiomatized, then condition ii asserts that PRA + A proves the formal consistency of T.

4. MANY SORTED EXTENSIONS OF FULL INDUCTION WITH EXPARITH.

Let L(arith) be the single sorted language based on 0,S,+,dot. Let L(exparith) be the single sorted language based on 0,S,+,dot,exp, where exp is the binary exponential function (where exp(x,0) = 1).

Let $I\Sigma_0$(exparith) be the fragment of PA(exparith) where induction is applied to bounded formulas in L(exparith), only.

Let $I\Sigma_0$(arith) be the fragment of PA(arith) where induction is applied to bounded formulas in L(arith), only.

INFORMAL SECOND INCOMPLETENESS (many sorted induction, exparith). Let L be a fragment of L(many) containing L(exparith). Let T be a consistent extension of PA(L) in L. Let A be a sentence in L that adequately formalizes the consistency of T, in the informal sense. Then T does not prove A.

FORMAL SECOND INCOMPLETENESS (many sorted induction, exparith). Let L be a fragment of L(many) containing L(exparith). Let T be a consistent extension of PA(L) in L. Let A be a sentence in L such that every inequation in L(exparith) that is provable in T, is also provable in $I\Sigma_0$(exparith) + A. Then T does not prove A.

INFORMAL THESIS (many sorted induction, exparith). Let L be a fragment of L(many) containing L(exparith). Let T be a consistent extension of PA(L) in L. Let A be a sentence in L that adequately formalizes the consistency of T, in the informal sense. Then every inequation in L(prim) that is provable in T, is also provable in $I\Sigma_0$(exparith) + A.

THEOREM 4.1. Let L be a fragment of L(many) containing L(exparith). Let T be an extension of PA(L) in L. Let A be a sentence in L. The following are equivalent.
i. Every inequation in L(exparith) that is provable in T, is also provable in $I\Sigma_0$(exparith) + A.
ii. $I\Sigma_0$(exparith) + A proves the formal consistency of every finite fragment of T.
Furthermore, if T is recursively axiomatized then the formal consistency of T, formulated on the basis of an algorithm for the recursive axiomatization, obeys i,ii. If T contains all true universally quantified equations in L(exparith), then for all such A, $I\Sigma_0$(exparith) + A is inconsistent.

5. GENERAL MANY SORTED THEORIES WITH EXPARITH.

INFORMAL SECOND INCOMPLETENESS (general many sorted, exparith). Let L be a fragment of L(many) containing L(exparith). Let T be a consistent extension of $I\Sigma_0$(exparith) in L. Let A be a sentence in L that adequately formalizes the consistency of T, in the informal sense. Then T does not prove A.

FORMAL SECOND INCOMPLETENESS (general many sorted, exparith). Let L be a fragment of L(many) containing L(exparith). Let T be a consistent extension of $I\Sigma_0$(exparith) in L. Let A be a sentence in L such that every universalized inequation in L(exparith) with a relativization in T, is provable in $I\Sigma_0$(exparith) + A. Then T does not prove A.

INFORMAL THESIS (general many sorted, exparith). Let L be a fragment of L(many) containing L(exparith). Let T be an extension of $I\Sigma_0$(exparith) in L. Let A be a sentence in L that adequately formalizes the consistency of T, in the informal sense. Then every universalized inequation in L(exparith) with a relativization in T, is provable in $I\Sigma_0$(exparith) + A.

THEOREM 5.1. Let L be a fragment of L(many) containing L(prim). Let T be an extension of $I\Sigma_0$(exparith). Let A be as sentence in L. The following are equivalent.
i. Every inequation in L(exparith) with a relativization in T, is also provable in $I\Sigma_0$(exparith) + A.

ii. $I\Sigma_0$(exparith) + A proves the formal consistency of every finite fragment of T.
If T is finitely axiomatized, then condition ii asserts that
$I\Sigma_0$(exparith) + A proves the formal consistency of T.

6. GENERAL MANY SORTED THEORIES WITH ARITH.

Here we need to consider Π_1(arith) sentences. These are the result of placing zero or more universal quantifiers in front of a bounded formula of L(arith).

INFORMAL SECOND INCOMPLETENESS (general many sorted, arith). Let L be a fragment of L(many) containing L(arith). Let T be a consistent extension of $I\Sigma_0$(arith) in L. Let A be a sentence in L that adequately formalizes the consistency of T, in the informal sense. Then T does not prove A.

FORMAL SECOND INCOMPLETENESS (general many sorted, arith). Let L be a fragment of L(many) containing L(arith). Let T be a consistent extension of $I\Sigma_0$(arith) in L. Let A be a sentence in L such that every sentence in Π_1(arith) with a relativization in T, is provable in $I\Sigma_0$(arith) + A. Then T does not prove A.

INFORMAL THESIS (general many sorted, arith). Let L be a fragment of L(many) containing L(arith). Let T be an extension of $I\Sigma_0$(arith) in L. Let A be a sentence in L that adequately formalizes the consistency of T, in the informal sense. Then every sentence in Π_1(arith) with a relativization in T, is provable in $I\Sigma_0$(arith) + A.