MY FORTY YEARS ON HIS SHOULDERS

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1. GENERAL REMARKS.

Godel's legacy is still very much in evidence. It must be noted that a careful analysis reveals that this great insights raise more issues than they resolve. The Godel legacy practically begs for renewal and expansion at a fundamental level.

When I entered the field some forty years ago, I seized on one glaring opportunity for renewal and expansion. The independence results from ZFC and significant fragments lied in a very narrow range, and had systemic features that are glaringly unrepresentative of mathematics and mathematical subjects generally.

This state of affairs suggests obvious informal conjectures to the effect that there are severe systemic limitations to the incompleteness phenomena, culminating in informal conjectures to the effect that, in principle, there is no relevance of set theoretic methods to "genuine" mathematical activity.

Now, there is no question that this central aspect of Godel's legacy, incompleteness, will diminish over time if such informal conjectures are not addressed in a substantial way. I have devoted the major part of my efforts over forty years to this effort.

I view this effort as part of a perhaps slow but steady evolutionary process. I have every confidence that this process will steadily continue in a striking manner as long as mathematics remains a vibrant activity.

2. WQO THEORY.

Wqo theory is a branch of combinatorics which has proved to be a fertile source of deep metamathematical pheneomena.

A qo (quasi order) is a reflexive transitive relation $(A,\leq)$. A wqo (well quasi order) is a qo $(A,\leq)$ such that for all $x_1,x_2,\ldots$ from A, $\exists$ $i < j$ such that $x_i \leq x_j$.

Highlights of wqo theory: that certain qo's are wqo's.

J.B. Kruskal treats finite trees as finite posets, and studies the qo

$\exists$ an inf preserving embedding from T_1 into T_2.

THEOREM 2.1. (J.B. Kruskal). The above qo of finite trees as posets is a wqo.

We observed that the connect-ion between wqo's and well orderings can be combined with known proof theory to establish independence results.

The standard formalization of "predicative mathematics" is due to Feferman/Schutte = FS. Poincare, Weyl, and others railed against impredicative mathematics.

THEOREM 2.2. Kruskal's tree theorem cannot be proved in FS.

KT goes considerably beyond FS, and an exact measure of KT is known through work of Rathjen/Weiermann.

Kruskal actually considered finite trees whose vertices are labeled from a wqo $\leq^*$. The additional requirement on embeddings is that

$$\text{label}(v) \leq^* \text{label}(h(v)).$$

THEOREM 2.3. (J.B. Kruskal). The qo of finite trees as posets, with vertices labeled from any given wqo, is a wqo.

Labeled KT is considerably stronger, proof theoretically, than KT, even with only 2 labels, $0 < 1$. I have not seen a metamathematical analysis of labeled KT.

Note that KT is a Π^1_1 sentence and labeled KT is a Π^1_2 in the hyperarithmetic sets.

THEOREM 2.4. Labeled KT does not hold in the hyperarithmetic sets. In fact, RCA_0 + KT implies ATR_0.

It is natural to impose a growth rate in KT in terms of the number of vertices of T_i.

COROLALRY 2.5. (Linearly bounded KT). Let $T_1,T_2,\ldots$ be a linearly bounded sequence of finite trees. $\exists$ $i < j$ such that T_i is inf preserving embeddable into T_j.

COROLLARY 2.6. (Computational KT). Let $T_1,T_2,\ldots$ be a sequence of finite trees in a given complexity class. There exists $i < j$ such that T_i is inf preserving embeddable into T_j.
Note Corollary 2.6 is Π^0_2.

THEOREM 2.7. Corollary 2.5 cannot be proved in FS. This holds even for linear bounds with nonconstant coefficient 1.

THEOREM 2.8. Corollary 2.6 cannot be proved in FS, even for linear time, logarithmic space.

By an obvious application of weak Konig's lemma, Corollary 2.5 has very strong uniformities.

THEOREM 2.9. (Uniform linearly bounded KT). Let $T_1,T_2,\ldots$ be a linearly bounded seq-uence of finite trees. There exists $i < j \leq n$ such that T_i is inf preserving embeddable into T_j, where n depends only on the given linear bound, and not on $T_1,T_2,\ldots$

With this kind of strong uniformity, we can obviously strip the statement of infinite sequences of trees.

For nonconstant coefficient 1, we have:

THEOREM 2.10. (finite KT). Let $n \gg k$. For all finite trees $T_1,\ldots,T_n$ with each $|T_i| \leq i+k$, there exists $i < j$ such that T_i is inf preserving embeddable into T_j.

Since Theorem 2.10 $\rightarrow$ Theorem 2.9 $\rightarrow$ Corollary 2.5 (nonconstant coefficient 1), we see that Theorem 2.10 is not provable in FS.

Other Π^0_2 forms of KT involv-ing only the internal structure of a single finite tree can be found in the Feferfest volume.

I proved analogous results for EKT = extended Kruskal theorem, which involves a finite label set and a gap embedding condition. Only here the strength jumps up to that of Π^1_1-CA_0.

I said that the gap condition was natural (i.e., EKT was natural). Many people were unconvinced.

Soon later, EKT became a tool in the proof of the famous graph minor theorem of Robertson/Seymour.

THEOREM 2.11. Let $G_1,G_2,...$ be finite graphs. There exists $i < j$ such that G_i is minor included in G_j.

I then asked Robertson/ Seymour to prove a form of EKT that I knew implied full EKT, just from GMT. They complied, and we wrote a triple paper.

The upshot is that GMT is not provable in Π^1_1-CA_0. Just where GMT is provable is unclear, and recent discussions with Robertson have not stabilized. I disavow remarks in the triple paper about where GMT can be proved.

An extremely interesting consequence of GMT is the subcubic graph theorem. A subcubic graph is a graph where every vertex has valence ≤ 3. (Loops and multiple edges are allowed).

THEOREM 2.12. Let $G_1,G_2,...$ be subcubic graphs. There exists $i < j$ such that G_i is embeddable into G_j as topological spaces (with vertices going to vertices).

Robertson/Seymour also claims to be able to use the subcubic graph theorem for linkage to EKT. Therefore the subcubic graph theorem (even in the plane) is not provable in Π^1_1-CA_0.

We have discovered lengths of proof phenomena in wqo theory. We use Π^0_1 sentences.

*) Let $T_1,...,T_n$ be a sufficiently long sequence of trees with vertices labeled from {1,2,3}, where each $|T_i| \leq i$.

There exists $i < j$ such that T_i is inf and label preserving embeddable into T_j.

**) Let $T_1,...,T_n$ be a sufficiently long sequence of subcubic graphs, where each $|T_i| \leq i+13$. There exists $i < j$ such that G_i is homeomorphically embeddable into G_j.

THEOREM 2.13. Every proof of *) in FS uses at least $2^{[1000]}$ symbols. Every proof of **) in Π^1_1-CA_0 uses at least $2^{[1000]}$ symbols.

3. BOREL SELECTION.

Let $S \subseteq \mathbb{R}^2$ and $E \subseteq \mathbb{R}$. A selection for A on E is a function $f:E \to \mathbb{R}$ whose graph is contained in S.

A selection for S is a selection for S on $\mathbb{R}$.

We say that S is symmetric if and only if $S(x,y) \leftrightarrow S(y,x)$.

THEOREM 3.1. Let $S \subseteq \mathbb{R}^2$ be a symmetric Borel set. Then S or $\mathbb{R}^2 \backslash S$ has a Borel selection.

My proof of Theorem 3.1 relied heavily on Borel determinacy, due to D.A. Martin.

THEOREM 3.2. Theorem 3.1 is provable in ZFC, but not without the axiom scheme of replacement.

There is another kind of Borel selection theorem that is implicit in work of Debs and Saint Raymond of Paris VII. They take the general form: if there is a nice selection for S on compact subsets of E, then there is a nice selection for S on E.

THEOREM 3.3. Let $S \subseteq \mathbb{R}^2$ be Borel and $E \subseteq \mathbb{R}$ be Borel with empty interior. If there is a continuous selection for S on every compact subset of E, then there is a continuous selection for S on E.

THEOREM 3.4. Let $S \subseteq \mathbb{R}^2$ be Borel and $E \subseteq \mathbb{R}$ be Borel. If there is a Borel selection for S on every compact subset of E, then there is a Borel selection for S on E.

THEROEM 3.5. Theorem 3.3 is provable in ZFC but not without the axiom scheme of replacement. Theorem 3.4 is neither provable nor refutable in ZFC.

We can say more.

THEOREM 3.6. The existence of the cumulative hierarchy up through every countable ordinal is sufficient to prove Theorems 3.1 and 3.3. However, the existence of the cumulative hierarchy up through any suitably defined countable ordinal is not sufficient to prove Theorem 3.1 or 3.3.

DOM: The f:N □ N constructible in any given x □ N are eventually dominated by some g:N □ N.

THEOREM 3.7. ZFC + Theorem 3.4 implies DOM (H. Friedman). ZFC + DOM implies Theorem 3.4 (Debs/Saint Raymond).

4. BOOLEAN RELATION THEORY.

We begin with two examples of statements in BRT of special importance for the theory.

THIN SET THEOREM. Let k ≥ 1 and f:N^k □ N. There exists an infinite set A □ N such that f[A^k] ≠ N.

COMPLEMENTATION THEOREM. Let k ≥ 1 and f:N^k □ N. Suppose that for all x □ N^k, f(x) > max(x). There exists an infinite set A □ N such that f[A^k] = N\A.

These two theorems are official statements in BRT. In the complementation theorem, A is unique.

We now write them in BRT form.

THIN SET THEOREM. For all f □ MF there exists A □ INF such that fA ≠ N.

COMPLEMENTATION THEOREM. For all f □ SD there exists A □ INF such that fA = N\A.

The thin set theorem lives in IBRT in A,fA. There are only $2^{2^\wedge 2}$ = 16 statements in IBRT in A,fA. These are easily handled.

The complementation theorem lives in EBRT in A,fA. There are only $2^{2^\wedge 2}$ = 16 statements in IBRT in A,fA. These are easily handled.

For EBRT/IBRT in A,B,C,fA,fB, fC,gA,gB,gC, we have $2^{2^9} = 2^{512}$ statements. This is entirely unmanageable. It would take several major new ideas to make this manageable.

DISCOVERY. There is a statement in EBRT in A,B,C,fA,fB, fC,gA,gB,gC that is independent of ZFC. It can be proved in MAH+ but not in MAH, even with the axiom of constructibility.

The particular example is far nicer than any "typical" statement in EBRT in A,B,C,fA,fB,fC,gA,gB,gC. However, it is not nice enough to be regarded as suitably natural.

Showing that all such statements can be decided in MAH+ seems to be too hard.

What to do? Look for a natural fragment of full EBRT in A,B,C,fA,fB,fC,gA,gB,gC that includes the example, where I can decide all statements in the fragment within MAH+.

Also look for a bonus: a striking feature of the classification that is itself independent of ZFC.

Then we have a single natural statement independent of ZFC.

In order to carry this off, we use somewhat different functions.

We use ELG = expansive linear growth.

These are functions $f:N^k \to N$ such that there exist constants c,d > 1 such that

$$c|x| \leq f(x) \leq d|x|$$

holds for all but finitely many $x \in N^k$.

TEMPLATE. For all $f,g \in$ ELG there exist $A,B,C \subseteq$ INF such that

$$X \cup. fY \subseteq V \cup. gW$$
$$P \cup. fR \subseteq S \cup. gT.$$

Here X,Y,V,W,P,R,S,T are among the three letters A,B,C.

Note that there are 6561 such statements. We have shown that all of these statements are provable or refutable in RCA_0, with exactly 12 exceptions.

These 12 exceptions are really exactly one exception up to the obvious symmetry: permuting A,B,C, and switching the two clauses.

The single exception is the **exotic case**

PROPOSITION A. For all f,g □ ELG there exist A,B,C □ INF such that

A □. fA □ C □. gB
A □. fB □ C □. gC.

This statement is provably equivalent to the 1-consistency of MAH, over ACA'.

If we replace "infinite" by "arbitrarily large finite" then we can carry out this second classification entirely within RCA_0.

Inspection shows that all of the nonexotic cases come out with the same truth value in the two classifications, and that is of course provable in RCA_0.

Furthermore, the exotic case comes out true in the second classification.

THEOREM 4.1. The following is provable in MAH+ but not in MAH, even with the axiom of constructibility. An instance of the Template holds if and only if in that instance, "infinite" is replaced by "arbitrarily large finite".

5. FINITE GRAPH THEORY.

THEOREM 1. For all strictly dominating order invariant R □ N^k x N^k there exists A □ N^k such that RA = A□.

THEOREM 2. For all strictly dominating order invariant R □ $[1,n]^k$ x $[1,n]^k$ there exists A □ $[1,n]^k$ such that RA = A□.

THEOREM 3. Every strictly dominating order invariant R □ $[1,n]^k$ x $[1,n]^k$ has an antichain A such that RA = A□.

THEOREM 4. Every strictly dominating order invariant R □ $[1,n]^k$ x $[1,n]^k$ has an antichain A such that $(RA)^3$, $(A□)^3$, $(A□)^3 \backslash 2^{8k^2}-1$ have the same elements up to order equivalence relative to the powers of 2.

This is a purely universal sentence provably equivalent to the consistency of MAH.